Sobolev Spaces and Partial Differential Equations
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Poincare's inequality

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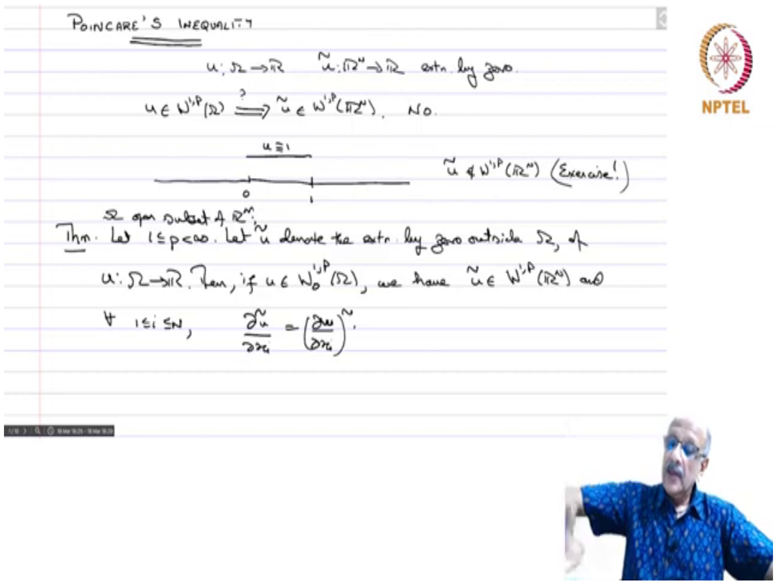

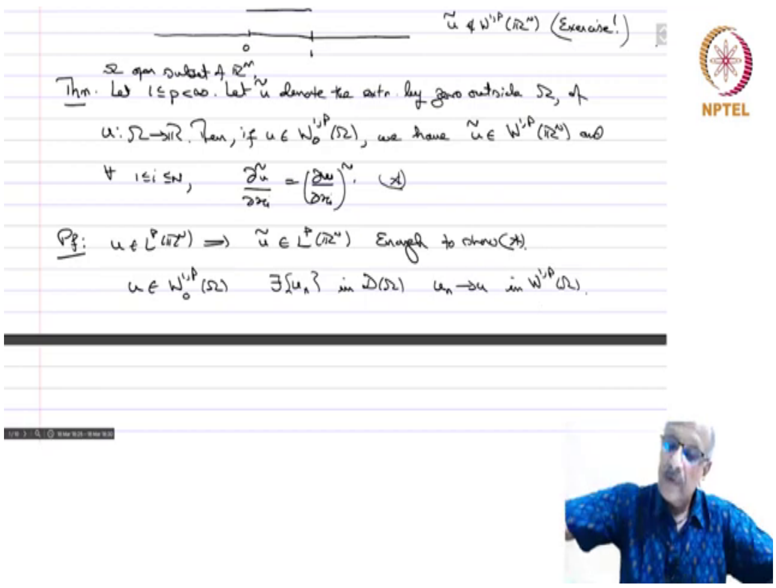

We will now discuss a very important result, this is called Poincare's inequality, this will play a very key role in the study of the Dirichlet boundary value problems for elliptic equations which we will under, other PDEs, as we will see later on. To start with, up to now we have been looking at extension theorems and we saw fairly elaborate conditions which are needed on the boundary or the domain so that there exists an extension operator.

One can always ask why cannot we just simply extend by 0? So,

$$u: \Omega \rightarrow \mathbb{R}, \ \tilde{u}: \mathbb{R}^N \rightarrow \mathbb{R} \text{ extension by 0.}$$

So, $u \in W^{1,p}(\Omega)$, does it imply that $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$? answer is no for instance, if you took the function take the interval 0, 1 in R and you take the function u identically equal to 1 and then you show so, u will belong to $W^{1,p}(\Omega)$ for any $1 \leq p < \infty$, $\tilde{u} \notin W^{1,p}(\mathbb{R}^N)$. so, I will leave you so, exercise, you will check why. So, you should be able to see this very easily.

Theorem: Let Ω open set in $\mathbb{R}^N$ and $1 \leq p < \infty$. Let $\tilde{u}$ denote the extension by 0 outside of Ω, $u: \Omega \rightarrow \mathbb{R}$. Then, if $u \in W_0^{1,p}(\Omega)$, $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$ and

$$\forall\, 1 \leq i \leq N, \ \frac{\partial \tilde{u}}{\partial x_i} = \widetilde{\frac{\partial u}{\partial x_i}} \quad \text{---------------(*)}$$

proof: So, $u \in L^p(\mathbb{R}^N) \Rightarrow \tilde{u} \in L^p(\mathbb{R}^N)$. So, enough to show (*).

So, we just have to show the derivatives so, $u \in W_0^{1,p}(\Omega)$, so, that means there exists

$\{u_n\} \in D(\Omega)\ s.t.\ u_n \rightarrow u \ in\ D(\Omega)$.

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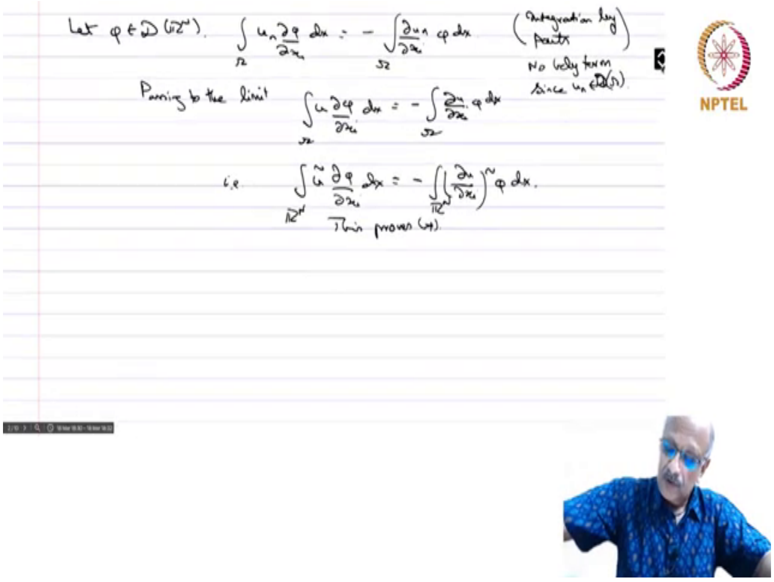

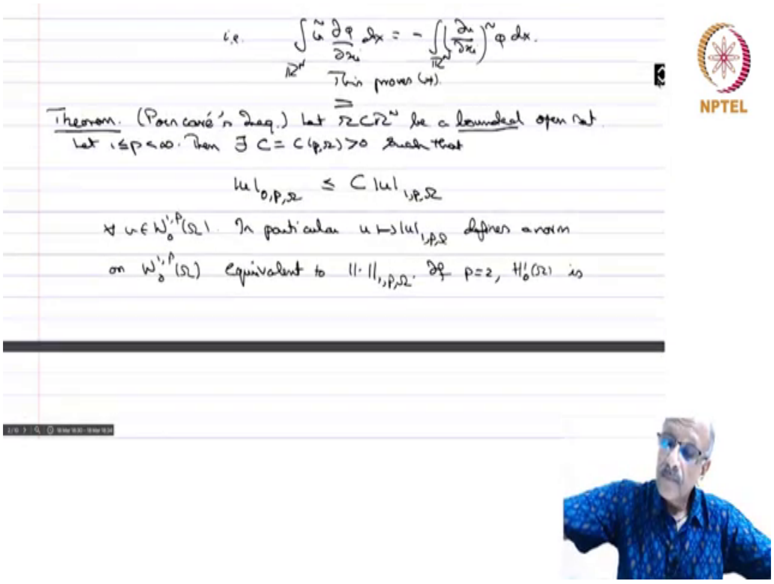

Let $\phi \in \mathbb{R}^N$. Then

$$\int_{\Omega} u_n \frac{\partial \phi}{\partial x_n} dx = -\int_{\Omega} \phi_n \frac{\partial u}{\partial x_n} dx.$$

Passing to the limit

$$\int_{\Omega} u \frac{\partial \phi}{\partial x} dx = -\int_{\Omega} \phi \frac{\partial u}{\partial x} dx.$$

$$i.e., \int_{\mathbb{R}^N} \tilde{u} \frac{\partial \phi}{\partial x} dx = -\int_{\mathbb{R}^N} \phi \widetilde{\left(\frac{\partial u}{\partial x}\right)} dx.$$

So, now we have the important theorem called Poincare's inequality.

Theorem: Let Ω open set in $\mathbb{R}^N$ and $1 \leq p < \infty$. Then there exist a constant $C = C(p, \Omega)$ s.t.

$$|u|_{0,p,\Omega} \leq C|u|_{1,p,\Omega},\ u \in W_0^{1,p}(\Omega).$$

In particular , $u: \rightarrow |u|_{1,p,\Omega}$ defines a Norm on $W_0^{1,p}(\Omega)$ equivalent to $||.||_{1,p,\Omega}$. If $p = 2$,

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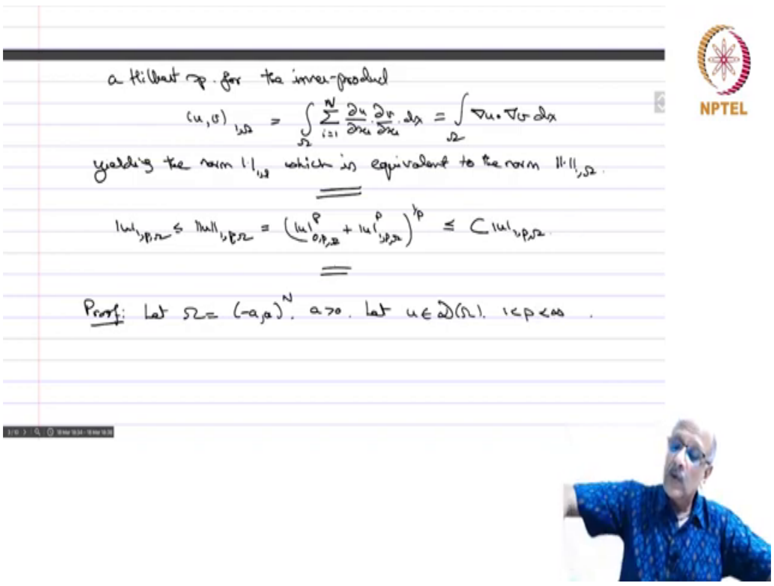

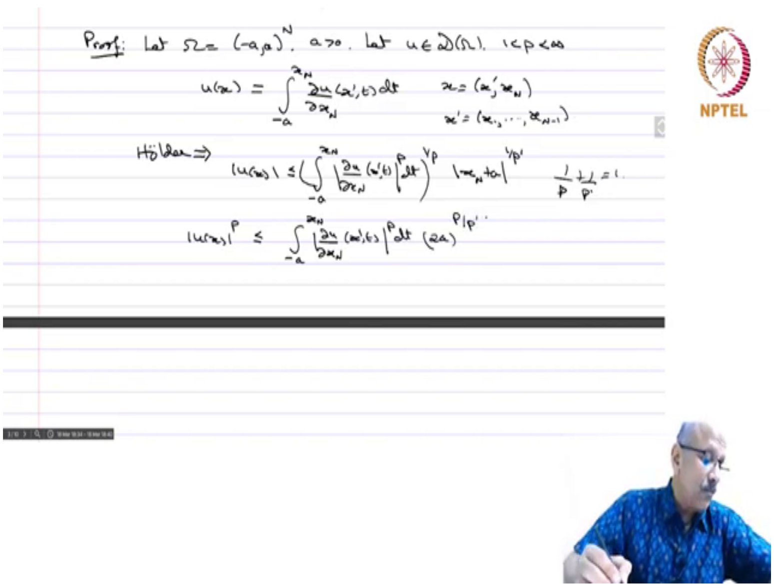

$H^1_0(\Omega)$ is a Hilbert space for the inner product

$$(u, v)_{1,\Omega} = \int_{\Omega} \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_{\Omega} \nabla u . \nabla v dx$$

yielding the Norm $|.|_{1,\Omega}$ which is equivalent to the Norm $||.||_{1,\Omega}$.

Mod 1 Omega which is equivalent to the Norm, Norm 1 Omega. So, this is the theorem so, what is this saying so, normally Mod u 1p Omega is only a semi norm, this is what we have saying. Why is it a semi norm, because if Mod u 1p of Omega that means du by dxi are all 0.

Then, that only says u is a constant, it does not imply that u equal to 0, now if you knew that u vanishes somewhere, then of course it is a constant vanishing somewhere if it has to be identically 0. And I have been hinting throughout that W1p0 of Omega are in fact functions which in some sense vanish on the boundary.

Therefore, if it is a constant vanishing on the boundary, it is better to be 0. So, this is the idea, we have not proved that of course so, but we can prove that, we can prove this result that in fact it is a Norm in a, because u equal to 0.

Why is it equivalent? Of course, you have that

$$|u|_{1,p,\Omega} \le ||u||_{1,p,\Omega} = (|u|^p_{0,p,\Omega} + |u|^p_{1,p,\Omega})^{\frac{1}{p}} \le C|u|_{1,p,\Omega}.$$

proof: Let $\Omega = (-a, a)^N$ *and* $\phi \in D(\Omega)$, $1 < p < \infty$.

So, now by the fundamental theorem of calculus

$$u(x) = \int_{-a}^{x_N} \frac{\partial u}{\partial x_N}(x', t)dt, \quad x = (x', x_N), \ x_N = (x_1, x_2, ..., x_{N-1}).$$

So, by Holder's inequality,

$$|u(x)| \le (\int_{-a}^{x_N} |\frac{\partial u}{\partial x_N}(x', t)|^p dt)^{\frac{1}{p}} |x_N + a|^{\frac{1}{p'}}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

$$|u(x)|^p \le \int_{-a}^{x_N} |\frac{\partial u}{\partial x_N}(x', t)|^p dt \quad (2a)^{\frac{p}{p'}}.$$

So, if p equals 1, I will just simply say this is less than equal to integral you do not have to put anything else after that.

You will just get Mod this that is all, so your Mod ux is less than a t integral Mod udu by dxn, that is all, that is the standard. So, p equals to 1 is easy to cover, everything else will now follow from this. So, 1 by p plus 1 by p dash, this is the conjugate exponent equal to 1, so Mod ux power p, I am going to take the pth power gets less than equal to integral minus a to xn Mod du by dxn x dash t, power p dt into Mod xn plus a I will just write it as 2a less than or equal to 2a because, Mod xn is the xn is, Mod x less than a, and therefore, then I get p by p dash.

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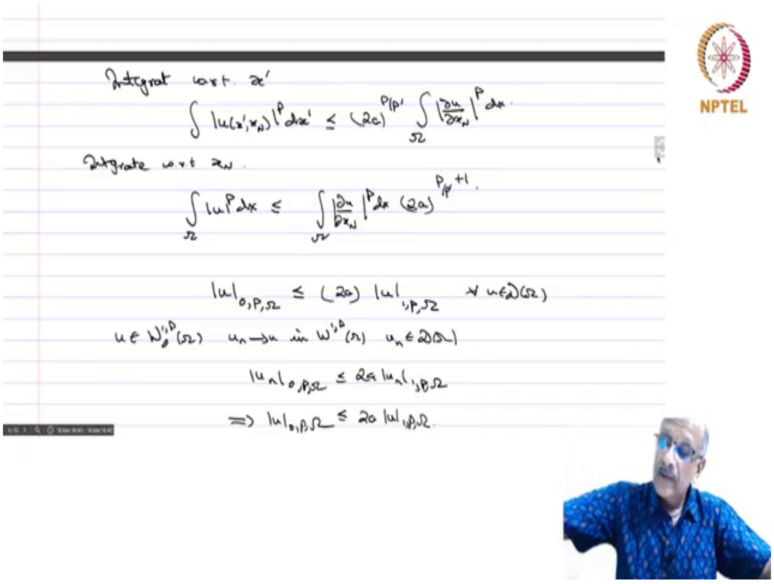

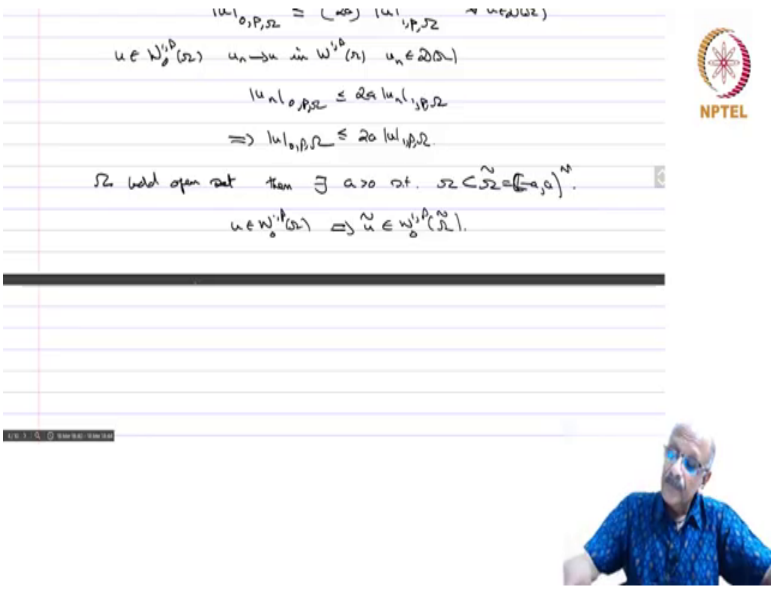

So, now integrate with respect to x'

$$\int_{\Omega} |u(x', x_N)|^p dx' \leq (2a)^{\frac{p}{p'}} \int_{\Omega} |\frac{\partial u}{\partial x_N}|^p dx.$$

Now, you integrate, integrate with respect x_N

$$\int_{\Omega} |u|^p dx_N \leq (2a)^{\frac{p}{p'}+1} \int_{\Omega} |\frac{\partial u}{\partial x_N}|^p dx.$$

So, now you take the pth root on both sides, you get

$$|u|_{0,p,\Omega} \leq 2a|u|_{1,p,\Omega}, \ \forall u \in D(\Omega).$$

If $u \in W_0^{1,p}(\Omega)$, so, that means there exists $\{u_n\} \in D(\Omega)\ s.t.\ u_n \to u\ in\ \in W^{1,p}(\Omega)$.

$$|u_n|_{0,p,\Omega} \leq 2a|u_n|_{1,p,\Omega} \ .$$

So, this proves the theorem when Omega is a box. So, if Omega is a bounded open set, then there exists $a > 0\ s.t.\ \Omega \subset \tilde{\Omega} = [-a, a]^N$.

$$\Rightarrow u \in W^{1,p}{}_0(\Omega) \Rightarrow \tilde{u} \in W^{1,p}{}_0(\tilde{\Omega}).$$

by the previous theorem extension by 0, gives you in fact W1p Rn function and then u can be approximated by un in d Omega which is also in d of Omega Tilde. And in Lp Norm and u Tilde nothing happens outside Omega so, everything will be okay, so you can check this.

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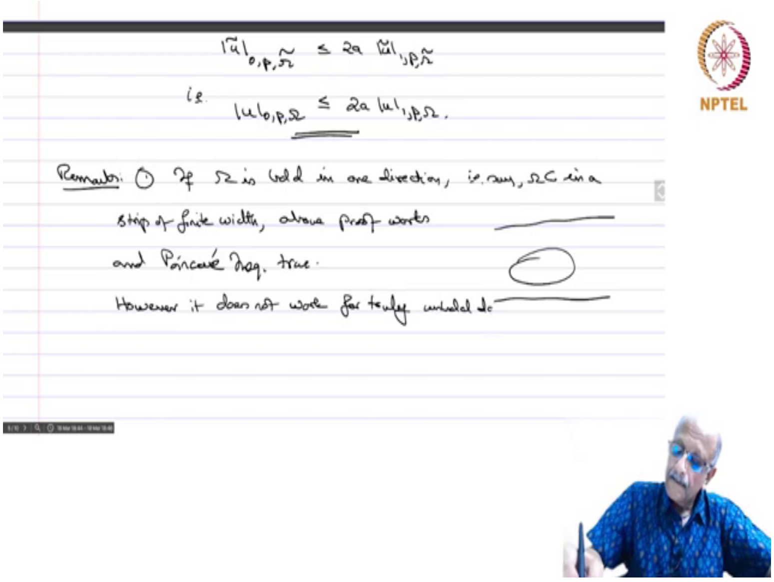

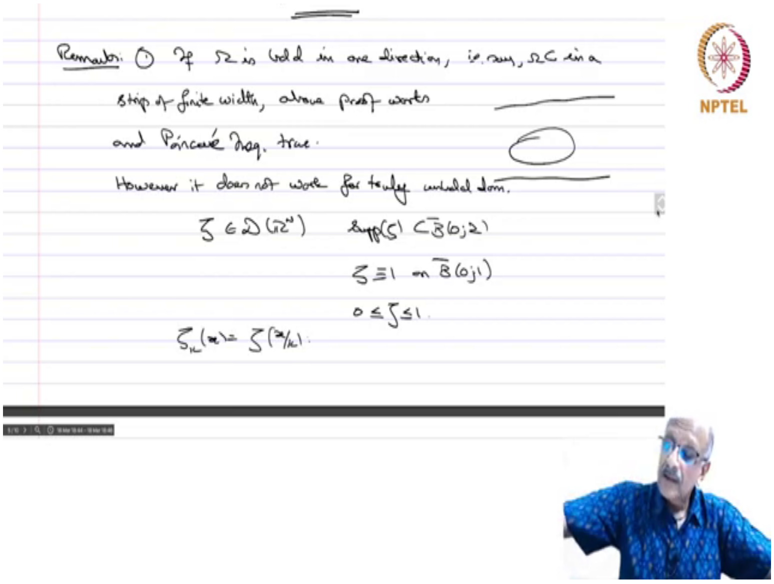

So, this implies that

$$|\widetilde{u}|_{0,p,\widetilde{\Omega}} \leq 2a|\widetilde{u}|_{1,p,\widetilde{\Omega}} \ .$$

$$\text{i.e., } |u|_{0,p,\Omega} \leq 2a|u|_{1,p,\Omega} \ .$$

So, this completes the proof of Poincare's inequality. So, we made a few nice remarks.

Remark 1: if Ω is bounded in one direction that is say Ω is contained in a strip of finite width so, you have some two parallel lines and then Omega is somewhere in between, it may be an unbounded domain and it need not be bounded.Now, since our proof is integrated only in one direction namely the xn so, you can do the same thing in this direction and therefore, above proof works. And Poincare's inequality is true, however it does not work for truly unbounded domains. For instance, if you took

$$\zeta \in D(\mathbb{R}^N),\ supp(\zeta) \subset \overline{B(0;2)},\ \zeta \equiv 1\ on\ \overline{B(0;1)},\ 0 \leq \zeta \leq 1.$$

and you take $\zeta_k(x) = \zeta(\frac{x}{k})$.

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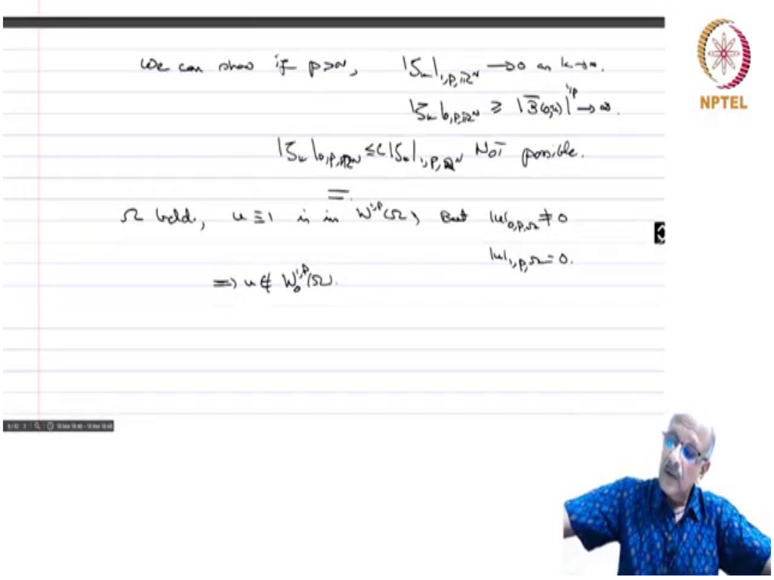

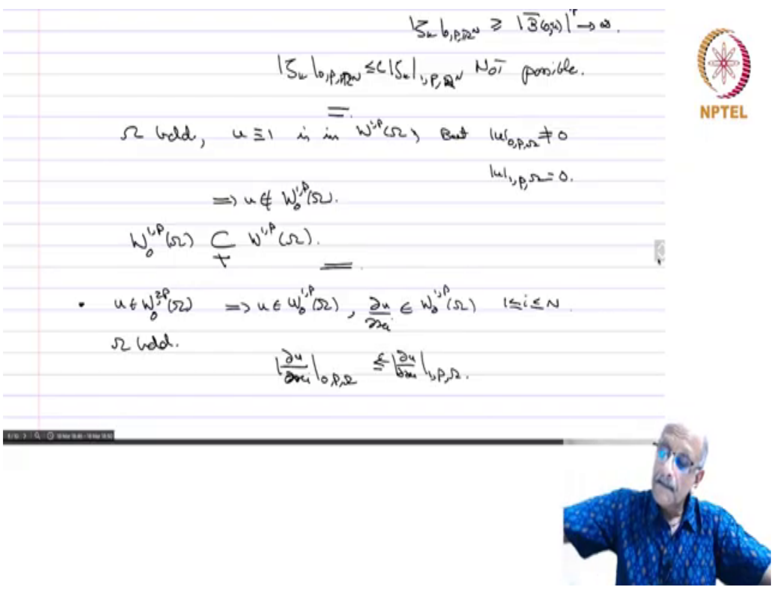

Then we can show if $p > N$, $|\zeta_k|_{1,p,\mathbb{R}^N} \to 0\ as\ k \to \infty$.

So, $|\zeta_k|_{0,p,\mathbb{R}^N} \geq |\overline{B(0;k)}| \to \infty$. Thus $|\zeta_k|_{0,p,\mathbb{R}^N} \leq c|\zeta_k|_{1,p,\mathbb{R}^N}$ - NOT POSSIBLE.

So, it cannot have Poincare's inequality in truly unbounded domains.

The other thing is if Ω is bounded, then $u \equiv 1$ is in $W^{1,p}(\Omega)$, *but* $|u|_{0,p,\Omega} \neq 0$, $|u|_{1,p,\Omega} = 0$.

$$\Rightarrow u \notin W_0^{1,p}(\Omega).$$

$$\Rightarrow W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega) \quad \text{(Proper subset).}$$

Recall in Rn we showed that both these spaces were the same but, if you have a bounded domain or even the domain which is bounded in one direction where Poincare's inequality is true, you can show that if Poincare's inequality is true, then this is always a different subspace.

Remark: Let us take $u \in W_0^{2,p}(\Omega) \Rightarrow u \in W_0^{1,p}(\Omega)$, $\frac{\partial u}{\partial x_i} \in W_0^{1,p}(\Omega)$, $\forall 1 \leq i \leq N$.

So, by Poincare's inequality you have

$$|\frac{\partial u}{\partial x_i}|_{0,p,\Omega} \leq |\frac{\partial u}{\partial x_i}|_{1,p,\Omega} .$$

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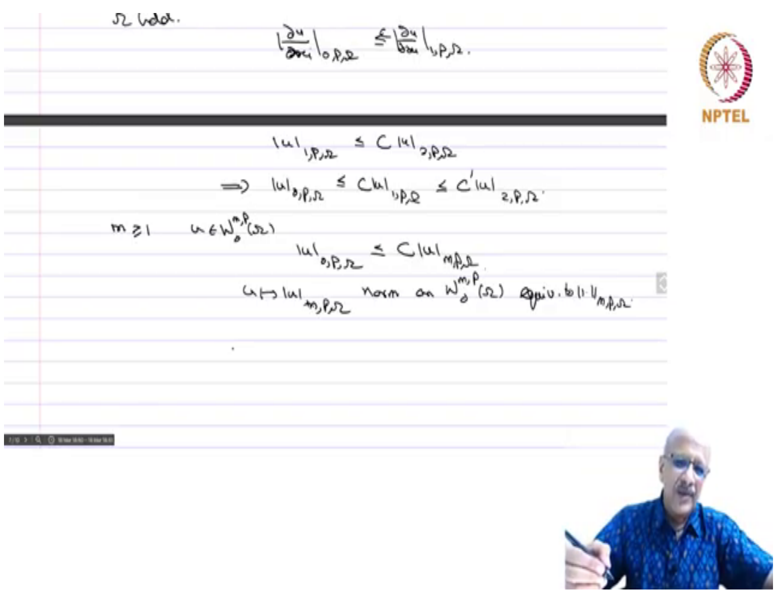

That is $|u|_{1,p,\Omega} \le c|u|_{2,p,\Omega} \Rightarrow |u|_{0,p,\Omega} \le c|u|_{1,p,\Omega} \le c'|u|_{2,p,\Omega}$.

More generally, if $m \ge 1$, $u \in W_0^{1,p}(\Omega) \Rightarrow |u|_{0,p,\Omega} \le c|u|_{m,p,\Omega}$.

And $u \to |u|_{0,p,\Omega}$ Norm on $W_0^{m,p}(\Omega)$ equivalent to the Norm $||.||_{m,p,\Omega}$.

Because, all in between derivatives can also be bounded by $|u|_{m,p,\Omega}$ so, this is another thing.

Now, the constant which we got is constant C is something like the diameter of Ω because, if you took a box then you saw that, 2a was the a constant and if you do if we, otherwise you put it inside the box if the smallest possible box will again have 2a as like the diameter of Ω.

So, something like diameter of Ω will be the constant but, that is not the best constant, the best constant has a very interesting connection to what are called eigenvalue problems and we will see that later.