Sobolev Spaces and Partial Differential Equations
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Extension theorems - Part 2

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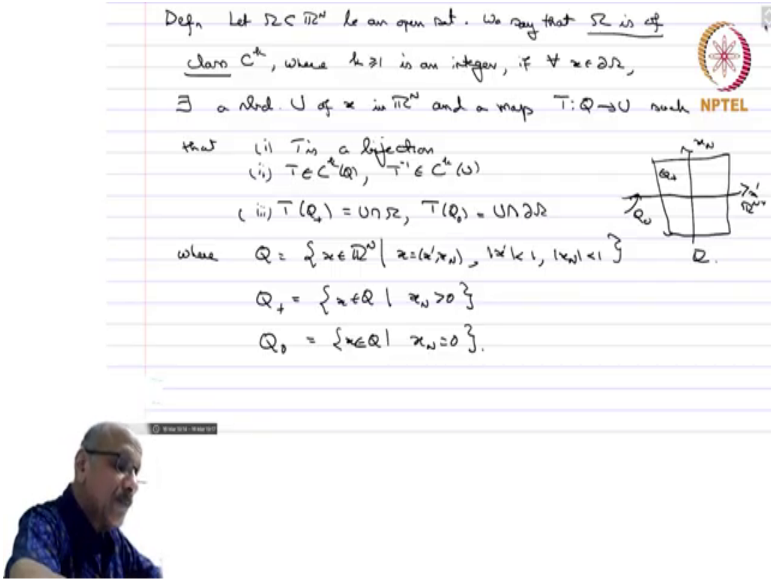

So, we will study the method of reflection for extension of functions, we will now see how it can be used for general domains. We start with the definition.

Definition: Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that Ω is of class C^k, where $k \geq 1$ is an integer, if $\forall x \in \partial\Omega$, $\exists\, nbd.\ U$ of $x\ in\ \mathbb{R}^N$ and a map $T: Q \to U$ such that

(i) T is a bijection.

(ii) $T \in C^k(Q)$, $T^{-1} \in C^k(Q)$.

(iii) $T(Q_+) = U \cap \Omega$, $T(Q_0) = U \cap \partial\Omega$, where

$$Q = \{x \in \mathbb{R}^N : x = (x', x_N),\ |x'| < 1,\ |x_N| < 1\},$$

$$Q_+ = \{x \in Q : x_N > 0\},\ \ Q_0 = \{x \in Q : x_N = 0\}.$$

So, remember this is the picture Q is this cylindrical domain so, this is x dash which is Rn minus 1 and here you have xn. So, Q plus is here this is Q0 and the whole cube is called Q. so, what do we mean by this so, if you have.

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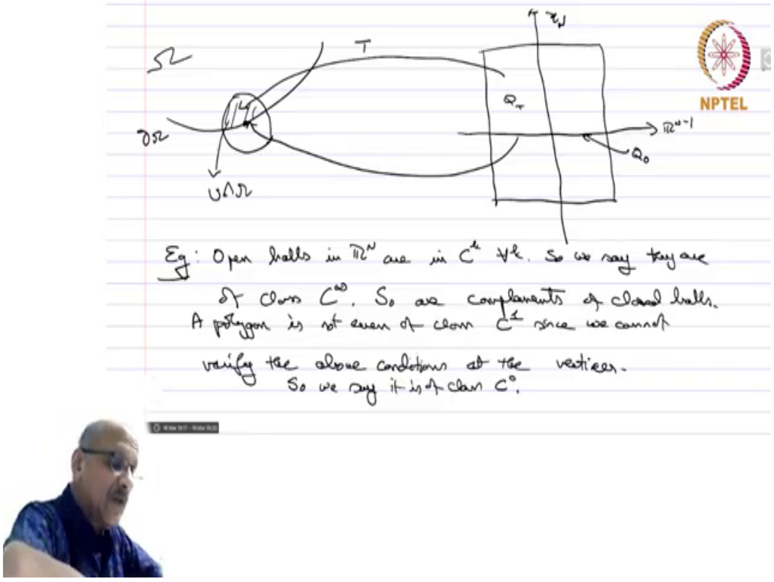

This is the boundary of Omega here d Omega and Omega is some set which is on this side so, for every x in d Omega you have a neighborhood u and you have here Q. And so, you have here Q_+ and this is Q_0 so, you have a mapping T here and you have, it takes Q_0 to the boundary. So, that is what it does and this is the intersection Omega the upper part here. So, you have this mapping so, example.

Examples: open balls in $\mathbb{R}^N$ are in C^k , for all k so, we say they are of class C^∞**.** So, are complements of closed balls, a polygon is not even of class C^1 since, we cannot verify the above conditions at the vertices so, we say it is of class C^0 so, C^∞ means it is in C^k for every k and this is of class C^0.

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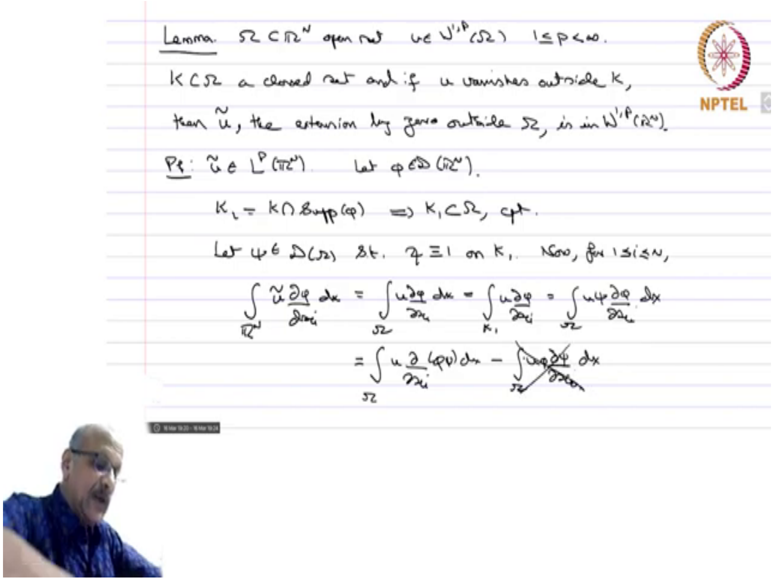

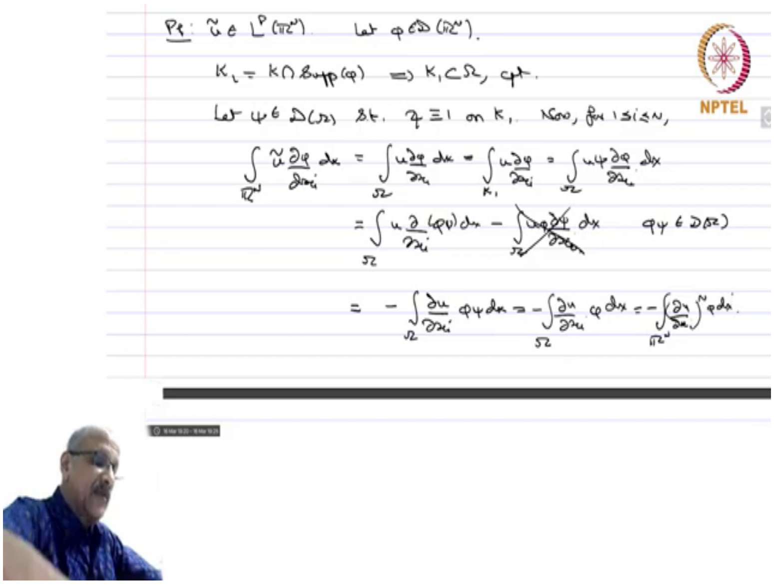

So, now we are going to prove an extension theorem so, before that we need the following technical Lanma.

Lemma: Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. $K \subset \Omega$ is a closed set and if u vanishes outside K, then $\tilde{u}$, the extension by 0 outside K, is in $W^{1,p}(\mathbb{R}^N)$.

proof: $\tilde{u} \in L^p(\mathbb{R}^N)$. $\phi \in D(\mathbb{R}^N)$.

$$K_1 = K \cap supp(\phi) \Rightarrow K_1 \subset \Omega \text{ - cpt.}$$

Let $\psi \in D(\Omega)$ s.t. $\psi \equiv 1$ *on* K. Now for $1 \leq i \leq N$,

$$\int_{\mathbb{R}^N} \tilde{u}\frac{\partial\phi}{\partial x_i}dx = \int_{\Omega} u\frac{\partial\phi}{\partial x_i}dx = \int_{k_1} u\frac{\partial\phi}{\partial x_i}dx = \int_{\Omega} u\psi\frac{\partial\phi}{\partial x_i}dx$$

$$= \int_{\Omega} u\frac{\partial(\phi\psi)}{\partial x_i}dx - \int_{\Omega} u\phi\frac{\partial\psi}{\partial x_i}dx$$

$$= -\int_{\Omega} \psi\phi\frac{\partial u}{\partial x_i}dx = -\int_{\Omega} \phi\frac{\partial u}{\partial x_i}dx = -\int_{\mathbb{R}^N} \left(\frac{\partial u}{\partial x_i}\right)^N \phi dx.$$

$$\Rightarrow \frac{\partial\tilde{u}}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^N \in L^P(\mathbb{R}^N).$$

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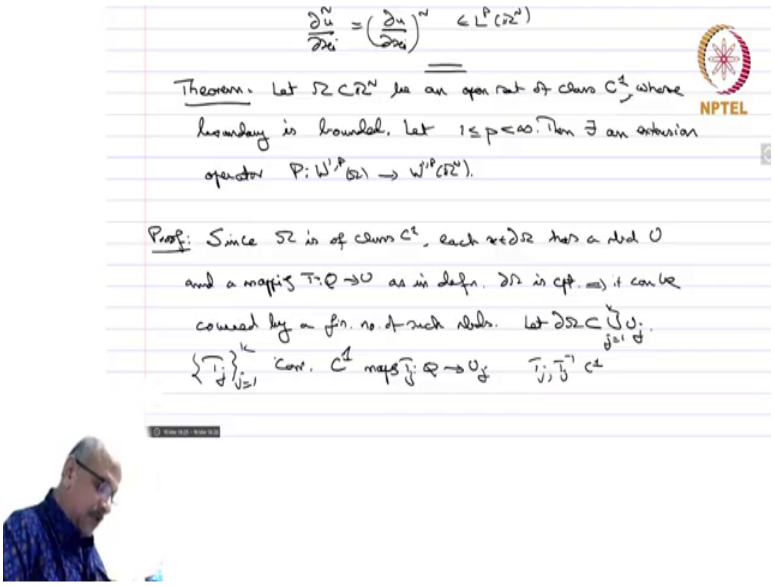

So, now we have the following theorem.

Theorem: Let $\Omega \subset \mathbb{R}^N$ be an open set of class C^1 whose boundary is bounded. Let $1 \leq p < \infty$. Then there exists an extension operator P: $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$.

proof: Since, Omega is of class C^1 , each $x \in \partial\Omega$ has a neighborhood U mapping T from Q to U as in definition. But $\partial\Omega$ is compact, which implies it can be covered by a finite number of such neighborhoods. Let us take, $\partial\Omega = \cap_{j=1}^{k} U_j$, $\{T_j\}_{j=1}^{k}$ corresponding C^1 maps T_j: $Q \to U_j$,T_j, $T_j^{-1} - C^{1.}$

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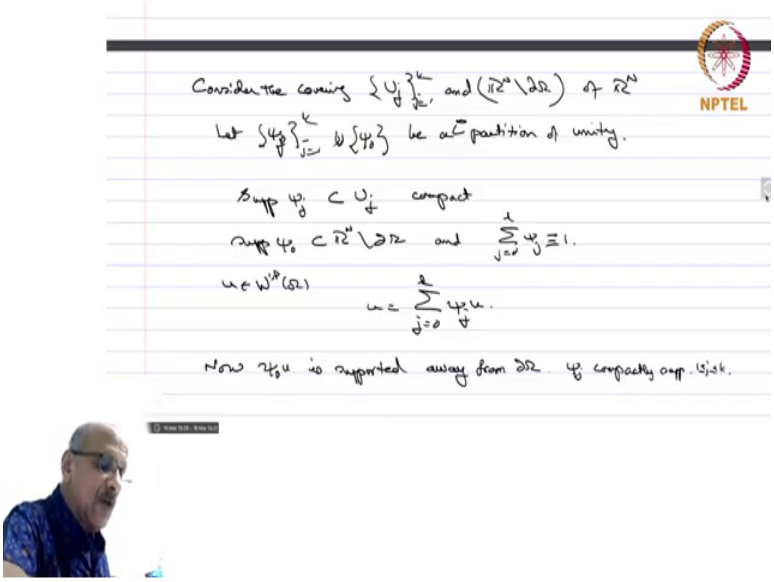

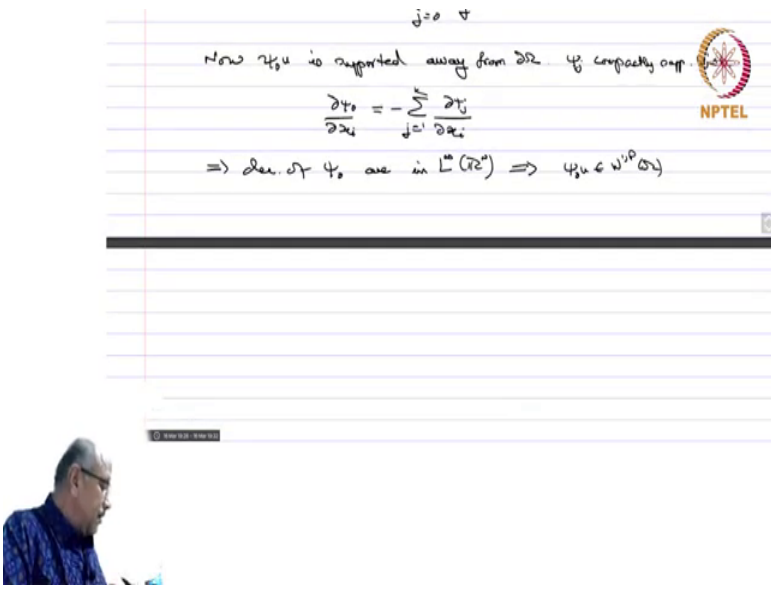

So, consider the covering $\{U_j\}_{j=1}^{k}$ and $\mathbb{R}^N\backslash\partial\Omega$ of $\mathbb{R}^N$ so, this is only a finite cover of $\mathbb{R}^N$ and then you do not have to worry about local finiteness, let Psi Naught Psi j, j equals 1 to k union Psi Naught be a partition of unity, C^{∞} partition of unity. I usually have to put the word locally finite but, now we only have a finite cover and therefore it does not matter, we do not have to use that word at all.

So, you have that the support of $\psi_j \subset U_j$ and U_j is a bounded set, because it is an image of Q and therefore this is compact. So, these are all compactly supported and support of ψ_0 is contained in Rn minus d Omega and finally we have Sigma j equals 0 to k ψ_j is identically equal to 1. So, if u is in $W^{1,p}(\Omega)$, you can write u equals Sigma j equals 0 to k Psi j u. So, now Psi 0 of you is supported away from d Omega and Psi j are compactly supported.

1 less than equal to j less than equal to k, that we already saw so, d ψ_0 by dxi is equal to minus sigma j equals 1 to k d ψ_j by dxi. So, implied derivatives of ψ_j are bounded, are in L infinity of Rn. Because all derivatives can be written in terms of ψ_j, ψ_j are compactly supported functions so they have bounded derivatives.

So, this implies that ψ_0 of u belongs to $W^{1,p}(\Omega)$, because its derivatives are all in the L infinity so, by the product rule d by dxi of this will be d ψ_0 by dxi into u which is in L infinity plus Psi Naught du by dxi that is also in L infinity. And therefore, you have a, that is true. Because sigma ψ_j is equal to 1 and all the ψ_j of course are 0 less than equal to ψ_j less than equal to 1 for all j. So, we also have that.

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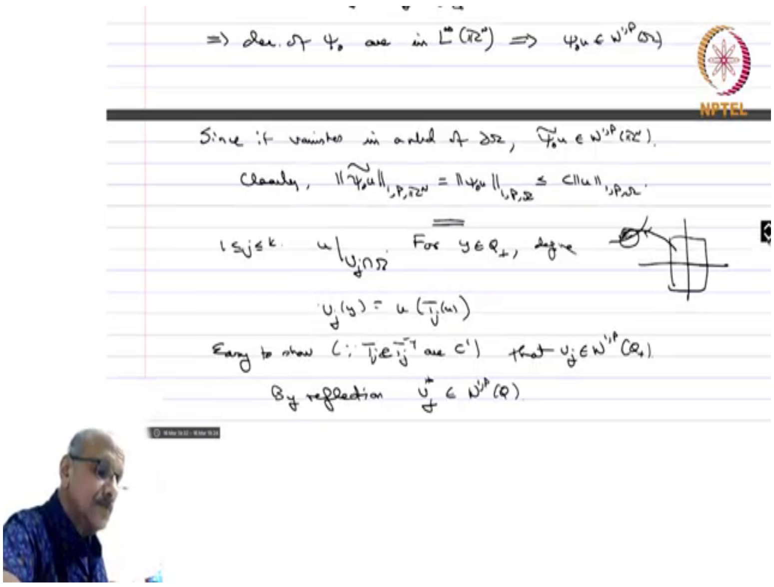

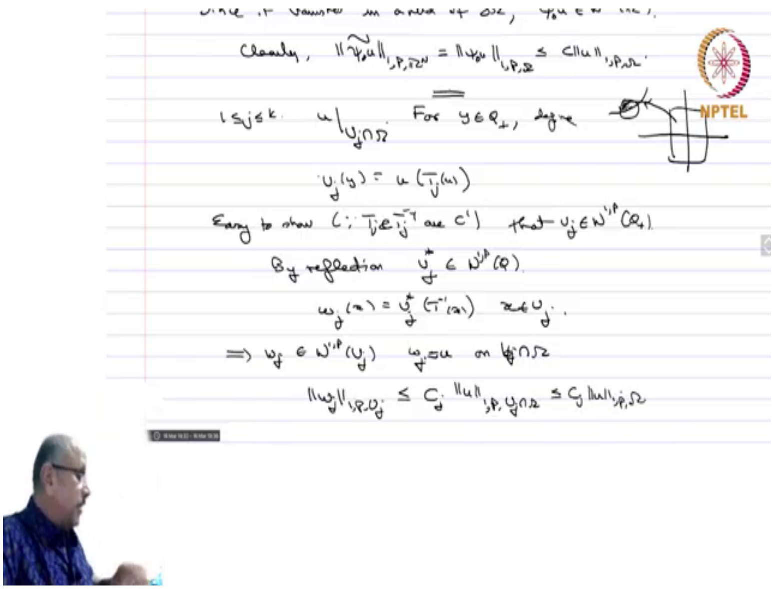

And since, it vanishes in a neighborhood of $\partial\Omega$, $\widetilde{\psi_0 u} \in W^{1,p}(\mathbb{R}^N)$.

Clearly, $||\widetilde{\psi_0 u}||_{1,p,\mathbb{R}^N} = ||\psi_0 u||_{1,p,\Omega} \leq c||u||_{1,p,\Omega}$.

Therefore, you have this.

So, now you let $1 \leq j \leq k$, $u|_{U_j \cap \Omega}$. For $y \in Q_1$, $define\ U_i(y) = u(T_j(u))$.

Enough to show (as T_j, $T_j^{-1} - C^1$) that $U_j \in W^{1,p}(Q_+)$.

By reflection $U_j^* \in W^{1,p}(Q)$. $w_j(x) = U_j^*(T^{-1}(x))$, $x \in U_j$.

$$\Rightarrow \ w_j \in W^{1,p}(U_j), \ \ w_j = u \ on \ \Omega \cap U_j.$$

$$||w_j||_{1,p,U_j} \le c_j ||u||_{1,p,U_j \cap \Omega} \le c_j ||u||_{1,p,\Omega}.$$

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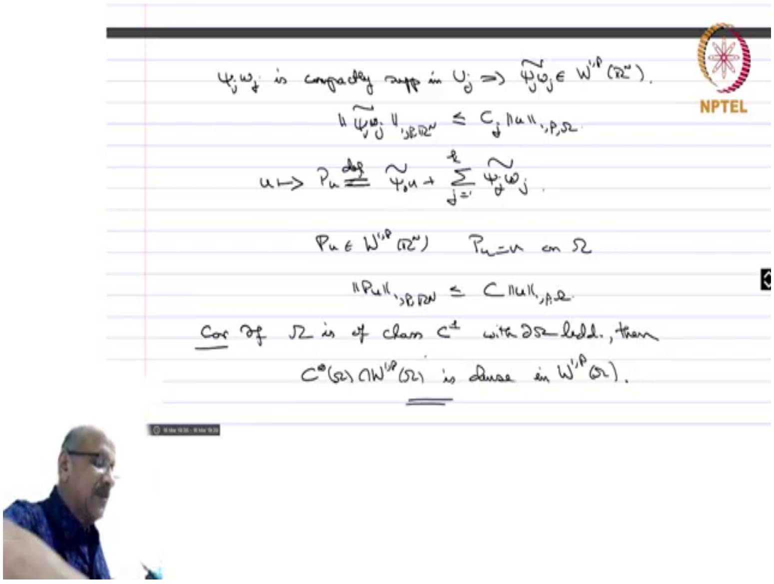

ψ_j, w_j is compactly supported in $U_j \ \Rightarrow \ w_j \widetilde{\psi_j} \ \in W^{1,p}(\mathbb{R}^N)$.

$$||\widetilde{\psi_j w_j}||_{1,p,\mathbb{R}^N} \le c_j ||u||_{1,p,\Omega}.$$

$$u \to P_u = \widetilde{\psi_0 u} + \sum_{j=1}^{k} \widetilde{\psi_j w_j}, \ \ P_u \in W^{1,p}(\mathbb{R}^N), \ \ P_u = u \ on \ \Omega.$$

$$||Pu||_{1,p,\mathbb{R}^N} \le c \ \ ||u||_{1,p,\Omega}.$$

Corollary: if Ω is of class C^1, with $\partial\Omega$ bounded, then as usual $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$. Because, you take the prolongation then you know that it can be approximated by d of Rn functions and then u restricted to Ω because you have a prolongation operator you do not have to use Friedrich's Theorem you have directly this thing.

So, this completes the extension theorem.