Sobolev Spaces and Partial Differential Equations
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Extension theorems - Part 1

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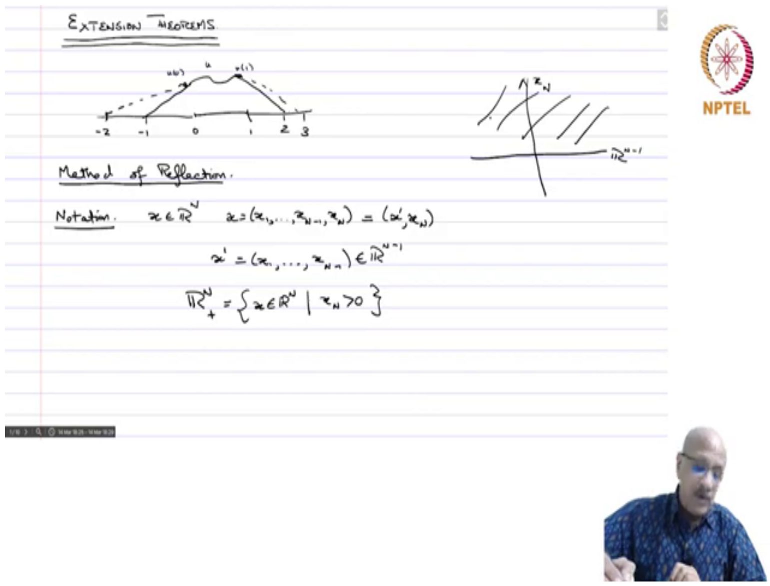

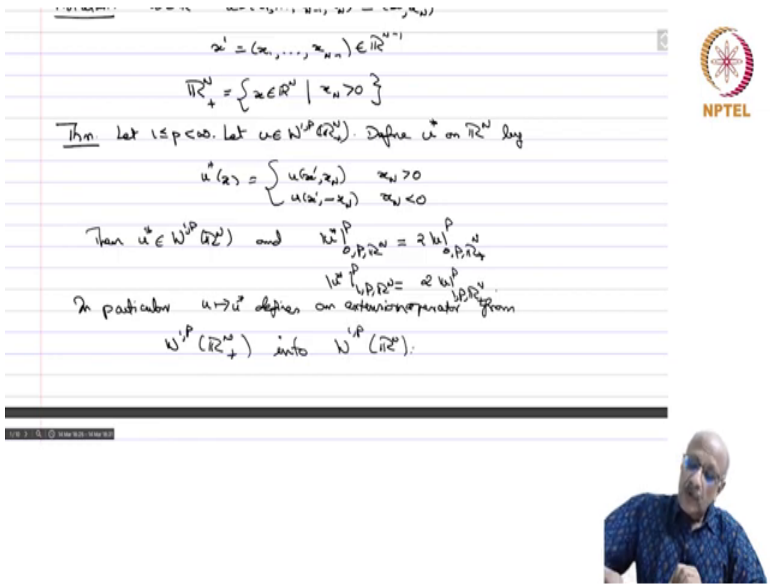

We will now look at Extension Theorems so, we already saw that if you, if Ω admitted, a extension operators for $W^{1,p}(\Omega)$ then we can improve Friedrich's theorem by having the conversions of the derivatives in $L^{p}(\Omega)$ rather than only on relatively compact sets. And also as I already remarked earlier, if you have many results using calculus can we prove in $\mathbb{R}^{N}$ because you are not constrict, constrained by any boundary and therefore, and you have tools like convolution and so on.

And then it is easier to restrict it, so having a prolongation operator is a good thing, for instance if you are having a function on 0, 1 for instance so, any $W^{1,p}(\Omega)$ function is going to be an absolutely continuous function as we saw. That means these values at the end points are well defined.

And now, you will simply connect it by means of a straight line and you can prove so, if you write down the formulae for these things. So, this is u so this will be u of 0 and this will be u of 1 and then you can easily write out what is this new function in R. And you can show that it is, in fact, an extension operator.

But, you can, there are many of them you can also do say, another one like this, this will also be an extension operator. So, you can write down the formulae, you can check it for yourself, and will probably see it in exercises on the assignments. Now, we are going to describe the method of reflection, which gives you a very general method for when, especially when the flat portions of the boundary give you an extension.

Method of reflection: so first some notation.

Notations: so, we take $x \in \mathbb{R}^N$, $x = (x_1,..., x_{N-1}, x_N)$, $x = (x', x_N)$,

$x_N = (x_1,..., x_{N-1}) \in \mathbb{R}^{N-1}, \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$.

So, now we have the following theorem.

Theorem: Let $1 \leq p < \infty$, $u \in W^{1,p}(\mathbb{R}^N_+)$. Define u^* on $\mathbb{R}^N$ by

$$u^*(x) = u(x', x_N),\ if\ x_N > 0,$$

$$= u(x', -x_N),\ if\ x_N < 0.$$

Then $u^* \in W^{1,p}(\mathbb{R}^N)$ *and* $|u^*|^p_{0,p,\mathbb{R}^N} = 2\,|u|^p_{0,p,\mathbb{R}^N_+}$, $|u^*|^p_{1,p,\mathbb{R}^N} = 2\,|u|^p_{1,p,\mathbb{R}^N_+}$. In particular $u \to u^*$ defines an extension operator from $W^{1,p}(\mathbb{R}^N_+)$ into $W^{1,p}(\mathbb{R}^N)$.

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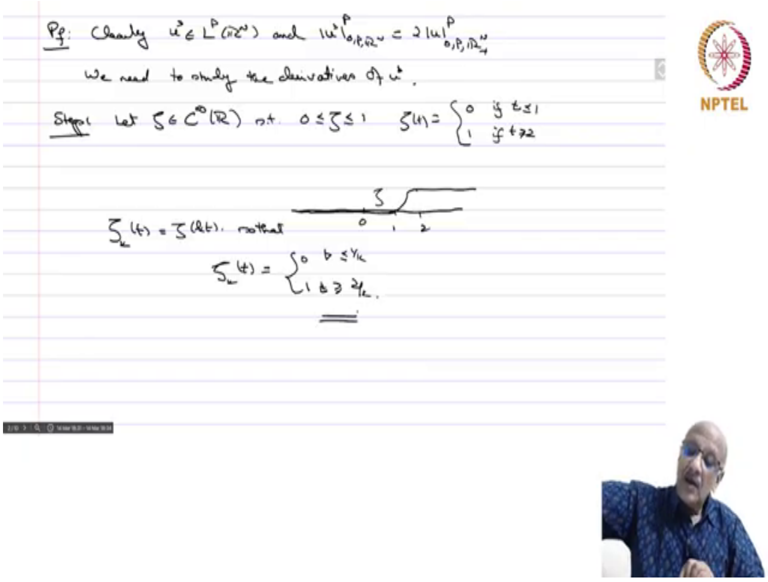

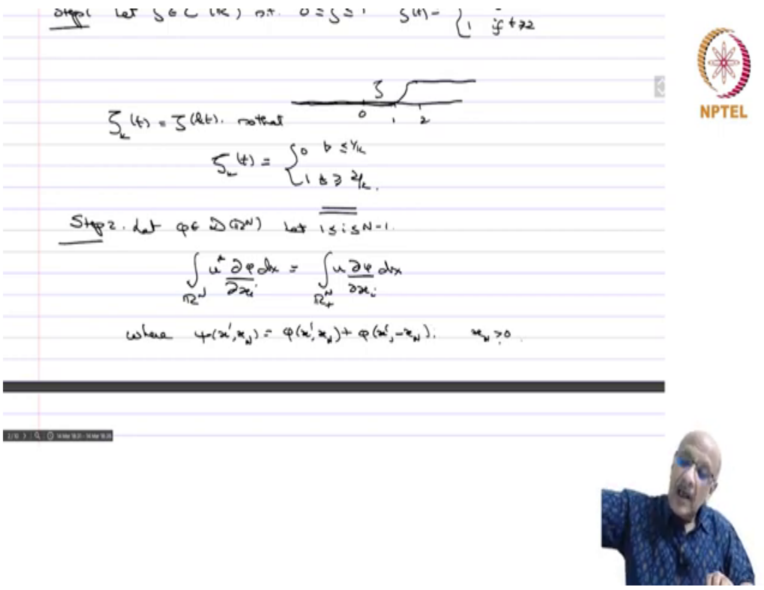

proof: clearly $u^{*} \in L^{p}(\mathbb{R}^{N}) \; and \; |u^{*}|^{p}_{0,p,\mathbb{R}^{N}} = 2\,|u\;|^{p}_{0,p,\mathbb{R}^{N}_{+}}$.

There is nothing to see here, it is just an integral that is repeated twice, once in the upper half plane once in the lower half plane by change of variable xn going to minus xn, you can convert it to that integral so you get twice.

So, we need to study the derivatives of u^{*}.

first step : we take $\zeta \in C^{\infty}(\mathbb{R}),\; 0 \le \zeta \le 1,\;\; \zeta(t) = 0,\; if\; t \le 1$

$$= 1,\; if\; t \ge 2.$$

And you define $\zeta_k(t) = \zeta(kt),\ so\ that\ \zeta_k(t) = 0,\ if\ t \le \frac{1}{k}$

$$= 1,\ if\ t \ge \frac{2}{k}.$$

step 2 : let $\phi \in D(\mathbb{R}^N),\ 1 \le i \le N - 1$

$$\int_{\mathbb{R}^N} u^* \frac{\partial \phi}{\partial x_i} dx = \int_{\mathbb{R}^N_+} u \frac{\partial \psi}{\partial x_i} dx.$$

where $\psi(x', x_N) = \phi(x', x_N) + \phi(x', -x_N),\ for\ x_N > 0.$

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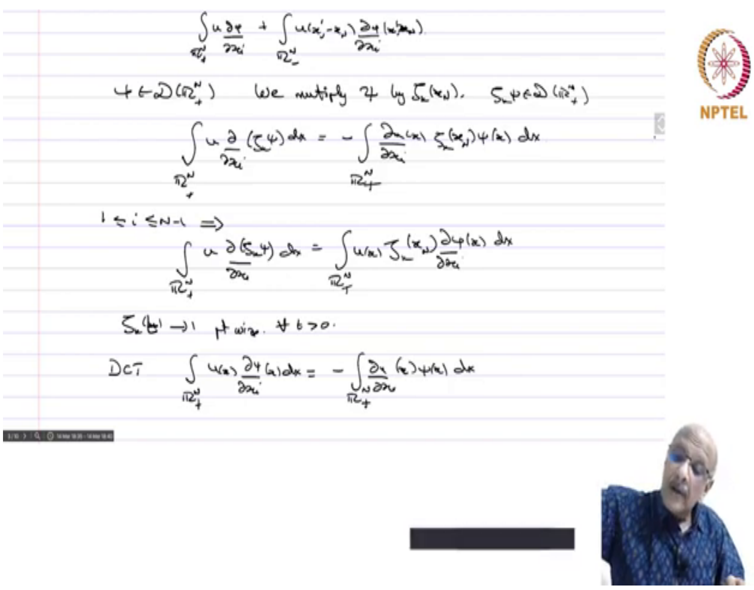

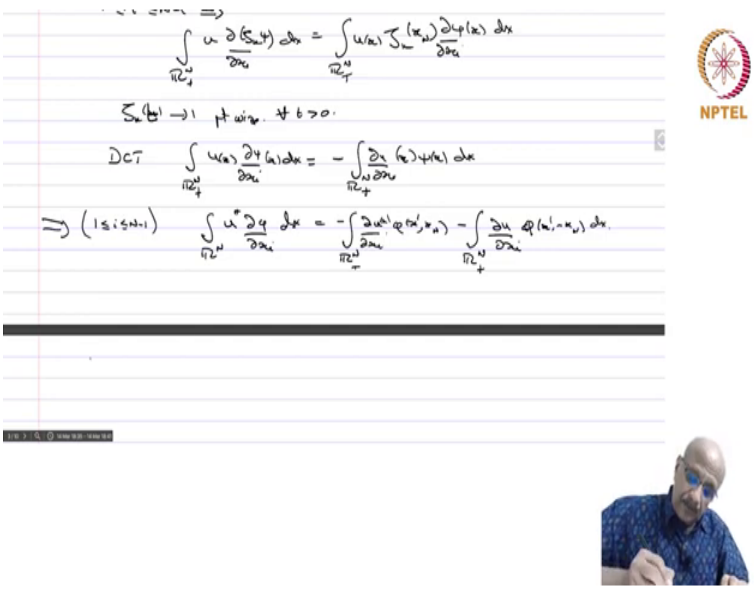

Integral of u over Rn plus d Phi by dxi plus integral over Rn minus of u of x dash minus xn, d Phi by dxi x dash minus, x dash xn sorry and now if you make a change of variable. Take minus x equals y so that y will become in Rn plus and then u this will become minus y and that is why you have this second term here in the thing so, this is just a change of variable, simple change of variable which we are doing.

So, now unfortunately Psi does not belong to d of Rn plus so, we because we do not have, we are just taking a Phi could be a function whose support is something like this and therefore it need not vanish here and I am just taking the reflection. This Phi of x dash minus xn, this reflection of this path over here and therefore it need not vanished anywhere near the boundary. So, therefore we are going to multiply it by Zeta k so, we multiply Psi by Zeta k xn. Then we have the Zeta k Psi belongs to d of Rn plus because Zeta will be 0, look at Zeta k, Zeta k will be 0, if xn will be 0 if 1 over k.

So, this small layer here, where it will be 0 and therefore when you multiply it by Zeta, Phi of Zeta k, a Psi times Zeta k belongs to d of Rn plus. Therefore, you have by the definition of the distribution of derivative, d of integral over Rn plus u d by dxi, Zeta k Psi dx is equal to minus integral over Rn plus du by dxi of x Zeta k which depends only on xn and you have Psi of x dx. So, now let us now 1 is less than equal to i is less than equal to n minus 1 and that implies that integral over Rn plus of u d Zeta k Psi by dxi dx is equal to integral over Rn plus u of x and then Zeta k of xn because that is like a constant, it will not (())(13:07) to d Psi by dxi at x.

So, now let us what happens, we have that Zeta k of x goes to 1 point wise because Zeta k of x will be 1 if x is, xn is bigger than 2 by k so, ultimately as k tends to infinity is almost all of Rn plus will slowly get covered and therefore point wise this converges to a 1. And for all point wise, for all t greater equal to 0, due to k of t let us say.

So, we can now apply the dominated conversions to other meanings. each of these integrals you have two fixed functions and Zeta k which converges to 1. And the modulus is bounded by Mod Psi into du by dxi and here or d Psi by dxi and u and they are integrable functions therefore, you have no problems at all.

So, by the dominated convergence theorem, you can pass to the limit so, the left-hand side when you go to, we will give you integral over Rn plus of u x d Psi by dxi at x dx equal to minus integral over Rn plus du dy dx at x Psi x dx. So, now going back to the definition of Psi so, what is the definition of Psi?

Psi if this function you go back and then you write this back as an integral over Rn instead of Rn plus and use again the fact the 1 is less than equal to i is less than equal to n minus 1 and therefore this gives you that integral over Rn of u star, d Phi by dxi dx equal to minus integral over Rn du by dxi Phi x, Phi x, du by dxi Phi. So, I am going to like the x dash xn minus so, this is integral Rn plus minus integral En plus du by dxi. Phi x dash minus xn dx.

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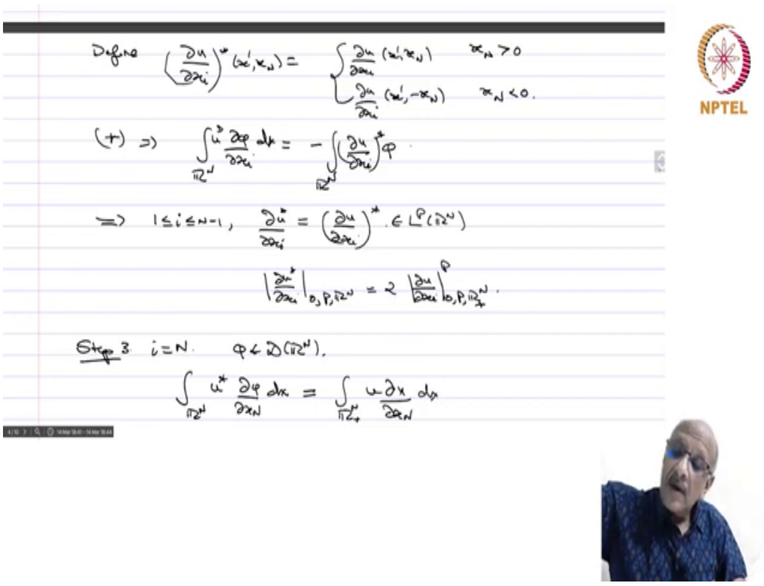

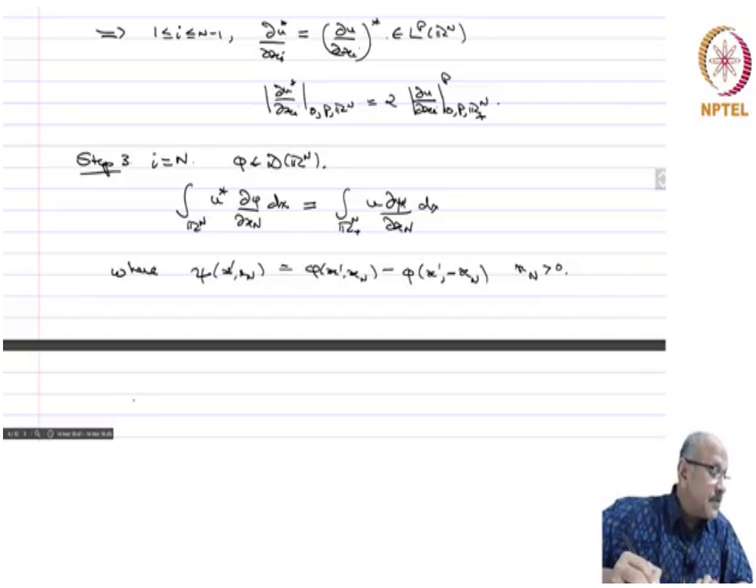

And therefore, you now define, du by dxi star is the same function of reflection so, x dash xn equal to du by dxi, x dash xn if xn is positive and du by dxi, x dash minus xn if xn is negative. Then star implies that the integral over Rn du, u star d Phi by dxi dx is equal to minus integral du by dxi star Phi over Rn.

So, this implies let for 1 less than equal to i less than equal to n minus 1 we have du star by dx Psi is nothing but, du by dxi star and therefore this belongs to Lp of Rn and in fact if you take the Lp Norm of this, you will precisely get 2 times the so, du by, du star by dxi 0 p Rn will be in fact twice integral, twice du by dxi over p 0 pRn plus. So, this is proved.

So, now we have to consider step 3 where we are taking i equals n, the derivative in the nth direction. So, once again you take Phi and d of Rn and now you take integral over Rn, of u star d Phi by dxN dx and then if you expand it, the definition of u star again En plus En minus, you break the integral and on Rn minus you make a change of variable.

But now since we have a derivative in the nth direction, when you differentiate with respect to xn, you pick up a minus sign. And therefore, this will give you the integral over Rn plus of u d Psi by dxN dx. Where, Psi of x dash xn is equal to Phi of x dash xn minus Phi of x dash minus xn. This is for xn positive.

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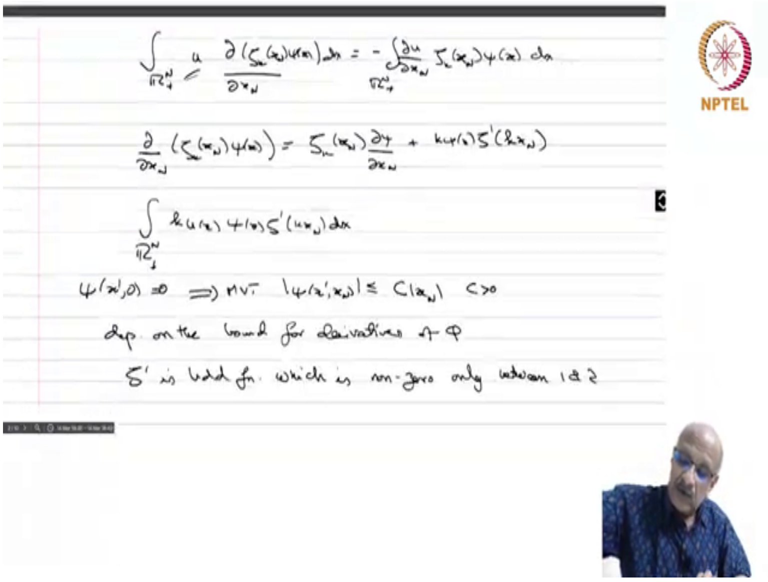

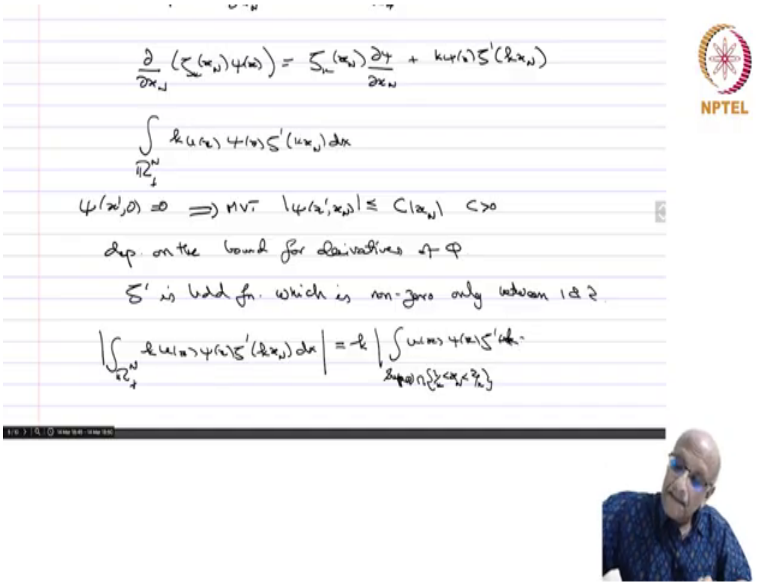

So, again this is not a, d Rn plus function and therefore, d Psi. Again we multiply it by Zeta k and therefore you have integral over Rn plus u of d Zeta k xn Psi x by dxn dx equal to minus integral du by dxn, over Rn plus of Zeta k xn, Psi of x dx.

So, now you look at the left hand side again so, d by dxn of Zeta k xn, Psi of x so, in the previous thing we were taking the derivatives up to n minus 1 so, this function was like a constant but, now you have to differentiate with respect to that also.

So, this will give you Zeta k xn, d Psi by dxn plus k times Psi of x into Zeta dash k xn. Because, we have defined Zeta k as Zeta kt Zeta of kt. So, when you differentiate that with respect to xn so, you get a k, a factor of k will come out in this expression. So, now you look at the we estimate so, this one is fine, we can pass to the limit, there is no problem, here you have k which tends to infinity and you have some Zeta k dash, etcetera.

So, let us look at that integral, integral over Rn plus of k ux which comes from here into Psi x Zeta dash k xn dx. So, this is one of the two terms on the left hand side and now Psi of x dash 0 is 0, because now if you put, what is the definition of Psi? It is Phi of x dash xn minus Phi of x dash minus xn.

So, if you put 0 xn equal to 0 then these two will get cancelled so, we get Psi of x dash 0 is 0. So, by the mean value theorem you have that Mod of Psi of x dash xn will be lesser than equal to some constant times Mod xn, where C is a positive constant depending on the bound so, depending on the bound for derivatives of Phi.

Phi is the C infinity function with compact support so, it uniformly bounded in Rn so, I do not have to worry about any dependence on x dash and therefore this thing. And also, Zeta dash is a bounded function which is nonzero, only between 1 and 2 so, Zeta is 0 here up to 1 and then Zeta dash and Zeta dash will be 0 again so, it is only between t equals 1 and t equals 2 that you have that Zeta dash is nonzero so, everywhere else it is 0. Therefore, if you take the modulus integral over Rn plus of k ux Psi x Zeta dash kxn dx, I take the modulus this is equal to k times Mod integral.

Now, this integral on Rn plus I am going to restrict it, it is actually the integral in support of the Phi intersection of 1 by k less than xn, less than 2 by k. Because, only in that range you have that Zeta dash is nonzero so, Zeta dash kxn will be 0 only between 1 and 2 and therefore, xn must be between 1 by k and 2 by k so, if you that, and you have ux Psi x Zeta dash kxn dx.

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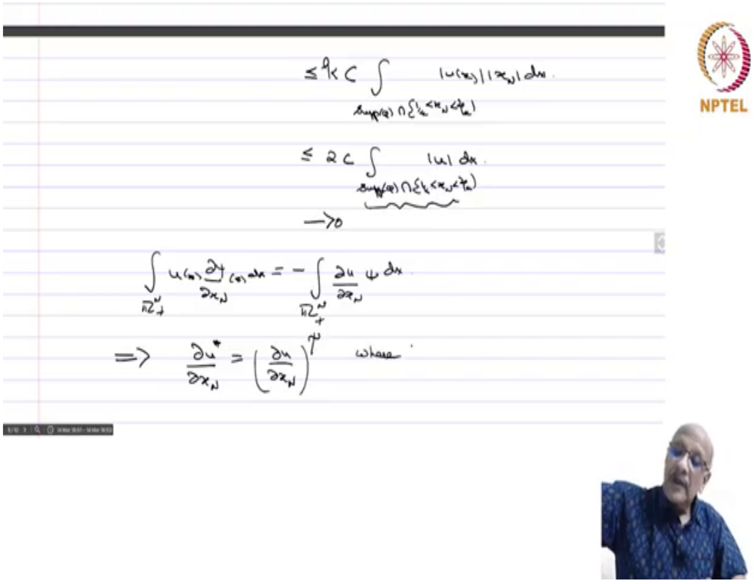

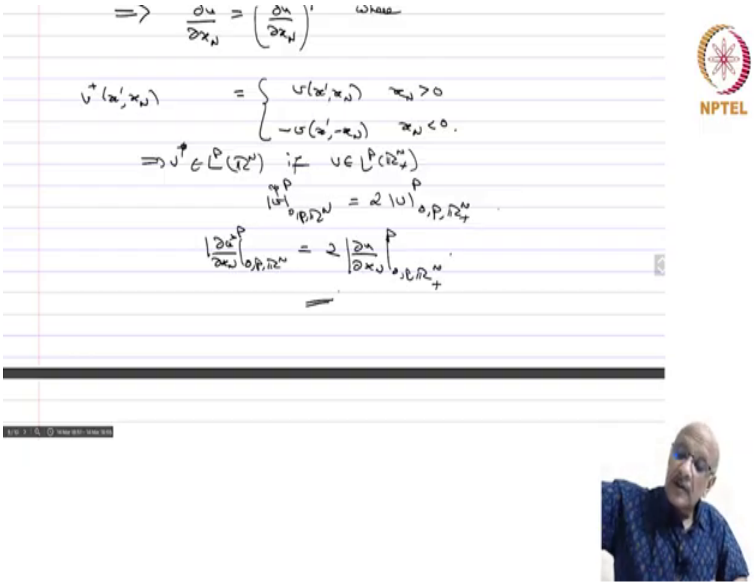

$$\leq kC \int_{supp(\phi)\cap\{\frac{1}{k}\leq x_N\}\leq\frac{2}{k}} |u(x)||x_N|dx \leq 2C \int_{supp(\phi)\cap\{\frac{1}{k}\leq x_N\}\leq\frac{2}{k}} |u|dx \to 0.$$

because; the major of this domain of integration goes to 0.

Because, it support of Phi is a compact set and you are intersecting it with a small strip whose width is only 1 by k and therefore you have compact set with strip of 1 by k so, the measure will go to 0 as k goes to 0 and since, Lebesgue integral is absolutely continuous with respect to a absolutely continuous and therefore this has to go to 0.

$$\int_{\mathbb{R}^N_+} u(x)\frac{\partial\psi}{\partial x_N}dx = -\int_{\mathbb{R}^N_+} \psi\frac{\partial u}{\partial x_N}dx.$$

$$\Rightarrow \frac{\partial u^*}{\partial x_N} = \left(\frac{\partial u}{\partial x_N}\right) .$$

$$v(x', x_N) = v(x', x_N),\ if\ x_N > 0,$$

$$= \ - v(x', - x_N),\ if\ x_N < 0.$$

$$\Rightarrow v^* \in L^p(\mathbb{R}^N)\ if\ v \in L^p(\mathbb{R}^N_+),\quad |v^*|^p_{0,p,\mathbb{R}^N} = 2\,|v|^p_{0,p,\mathbb{R}^N_+} .$$

$$\left|\frac{\partial u^*}{\partial x_N}\right|^p_{0,p,\mathbb{R}^N} = 2\left|\frac{\partial u}{\partial x_N}\right|^p_{0,p,\mathbb{R}^N_+} .$$

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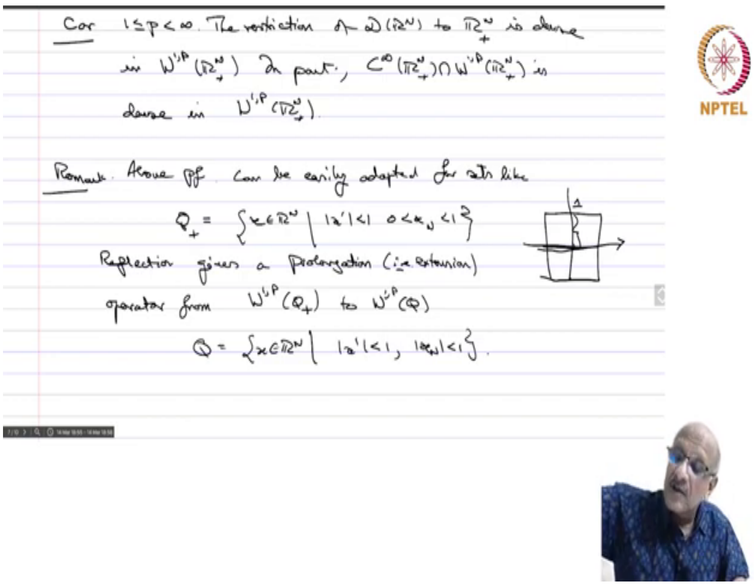

Corollary: $1 \leq p \leq \infty$. The restriction of $D(\mathbb{R}^N)$ to $\mathbb{R}^N_+$ is dense $W^{1,p}(\mathbb{R}^N_+)$. In particular, $C^\infty(\mathbb{R}^N_+) \cap W^{1,p}(\mathbb{R}^N_+)$ is dense in $W^{1,p}(\mathbb{R}^N_+)$.

Remark: so, this proves so, above can be easily adapted for sets like

$$Q_+ = \{x \in \mathbb{R}^N:\ |x'| < 1,\ 0 < x_N < 1\}.$$

so, reflection gives a prolongation operator or extension operator, a prolongation operator that is extension operator from $W^{1,p}(Q_+)$ to $W^{1,p}(Q)$, where Q is the set

$$Q = \{x \in \mathbb{R}^N : |x'| < 1, |x_N| < 1\}.$$

Now, we can use this trick to define prolongation operators for some sets like.

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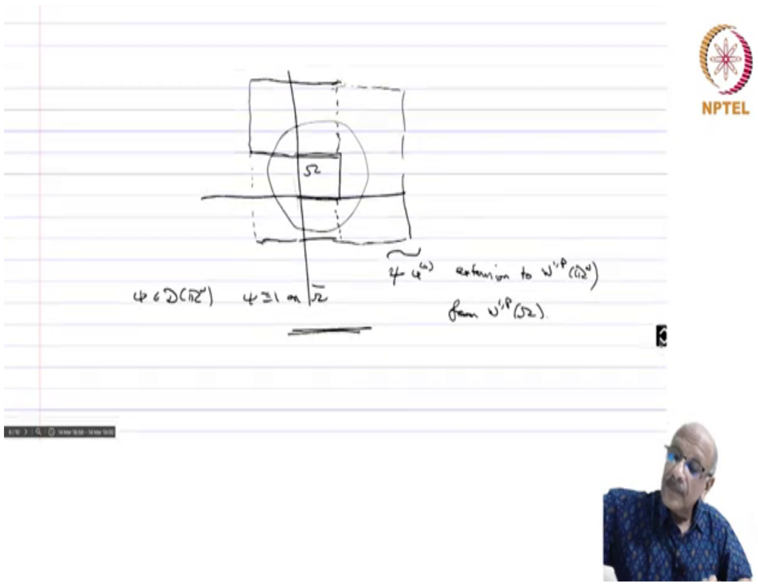

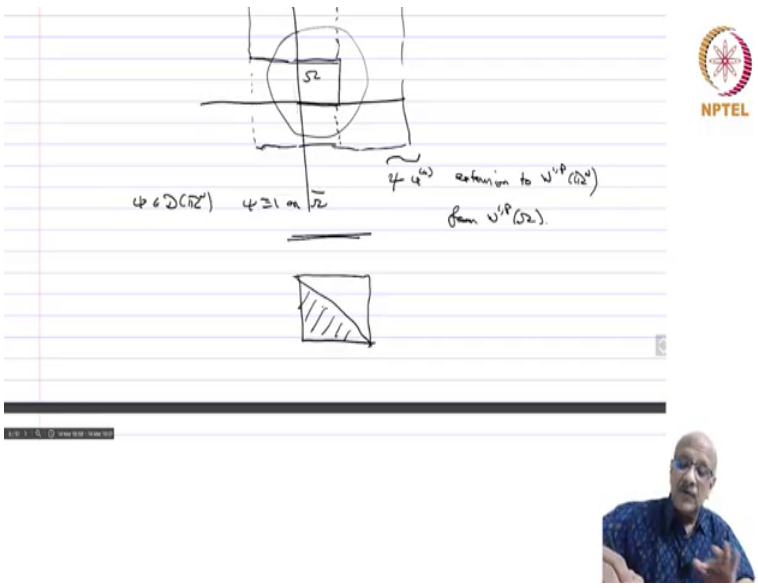

For instance, take the square, suppose I have a square like this now, so this is Ω now by reflection on this line, I can extend it to this rectangle. Now, by reflection on this line, I can further extend it to this rectangle. Now, by reflection on this line, I can extend it further. Finally, by reflection on this line, I can extend the function 4. So, each time I would have taken the factor of 2 in the powers of, in the integrals of the derivatives and their functions. So, this 4 times I have done it 1, 2, 3, 4 so, that will be 16 and therefore, that is a constant which is well under control.

Now, what you do is, you take a $\psi \in D(\mathbb{R}^N)$ with $\psi = 1$ on $\overline{\Omega}$ and the neighborhood of it and then you take. So, you have this u star so, I let you call it u4 the 4 fold extension of u and then the multiply it by ψ that will give you an extension. And then extension, outside the square by 0 will give you an extension to $W^{1,p}(\mathbb{R}^N_+)$ from $W^{1,p}(\Omega)$. So, you have u, you have u1 which is the extension here, then u2 which is the extension to the square, u3 which is the extension to the third one and u4 is the fourth extension. So, you have 4 and then you multiply by Psi and therefore, you will get the function which is vanishing outside the compact set and therefore, you will then extend it by 0 nothing no harm is done and it will continue to be equal to Omega, u inside Omega.

So, this way we can define, a use this for instance so, now we can for instance suppose I have a triangle, a right-angled triangle like this, then by reflection on the high part new, so I can extend it to the square and then from the square I know how to extend it to the whole plain. So, I also have an extension operator for the triangle. So, the method of reflection is very useful when you have such flat portions on the boundary to do it.

Next, we will consider a general method for domains so, for what kind of domains we can define extension operators, reflection was very special and now we will see for other kinds of domains where, we will use this reflection idea and the partition of unity and then see how that is to be done.