Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
Indian Institute of Mathematical Science
Chain Rule and Applications - Part 2

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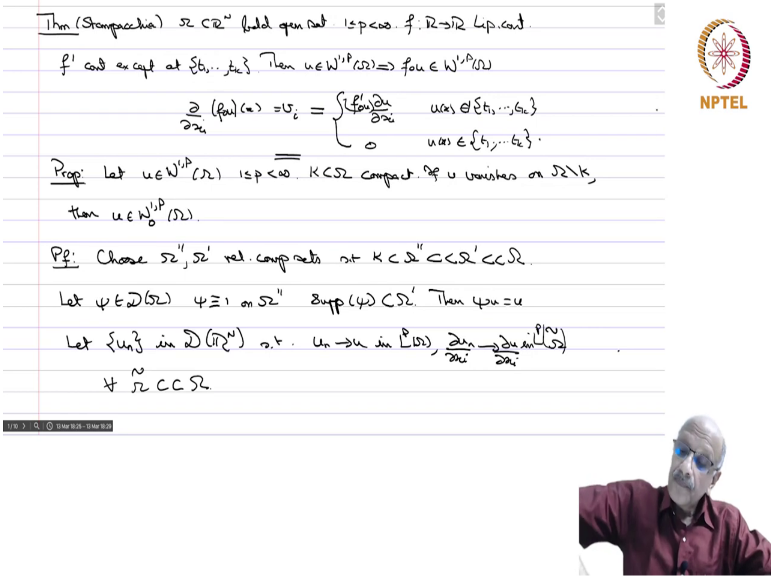

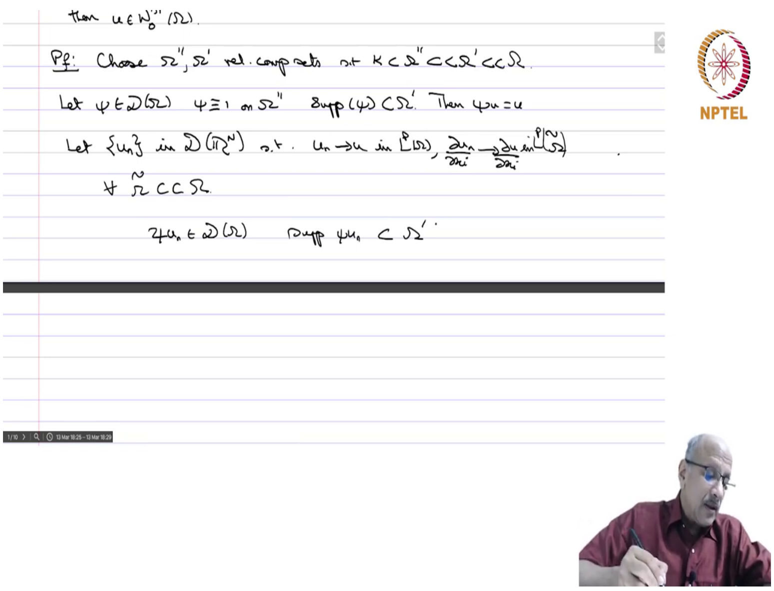

We stated Stampacchia’s theorem.

Theorem: $\Omega \subset \mathbb{R}^N$ bounded open set and $1 \leq p < \infty$, $f: \mathbb{R} \rightarrow \mathbb{R}$ Lipscitz continuous, f' continuous except at a finite number of points $\{t_1, t_2, ... t_k\}$. Then $u \in W^{1,p}(\Omega) \Rightarrow f \circ u \in W^{1,p}(\Omega)$.

$$\frac{\partial}{\partial x_i} f \circ u(x) = v_i = (f' \circ u)(x), \ \ if\ u(x) \notin \{t_1, t_2, ... t_k\},$$

$$= 0, \qquad if\ u(x) \notin \{t_1, t_2, ... t_k\}.$$

I did not prove this but the proof, complete proof is available in the book topics in functional analysis and applications which we are following for this course. So, proposition.

Proposition: Let $u \in W^{1,p}(\Omega)$ and $1 \leq p < \infty$, $K \subset \Omega\ compact$. If u vanishes on $\Omega \backslash K$, then $u \in W_0^{1,p}(\Omega)$.

proof: So let, so choose Ω'', Ω' - relatively compact sets such that $K \subset \Omega'' \subset\subset \Omega' \subset\subset \Omega$.

Let $\psi \in D(\Omega)\ s.t.\ \psi \equiv 1\ on\ \Omega''$, $supp(\psi) \subset \Omega'$. Then $\psi u = u$. Let $\{u_n\} \in D(\mathbb{R}^N)\ s.t.$

$$u_n \to u\ in\ L^p(\Omega)\ and\ \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}\ in\ L^p(\widetilde{\Omega}),\ \forall\ \widetilde{\Omega} \subset \Omega.$$

So, then $\psi u_n \in D(\Omega)$, $supp(\psi u_n) \subset \widetilde{\Omega}$. Also $\psi u_n \to \psi u\ in\ L^p(\Omega)\ and\ \frac{\partial \psi u_n}{\partial x_i} \to \frac{\partial \psi u}{\partial x_i}\ in\ L^p(\Omega') \Rightarrow\ also\ in\ L^p(\Omega)$.

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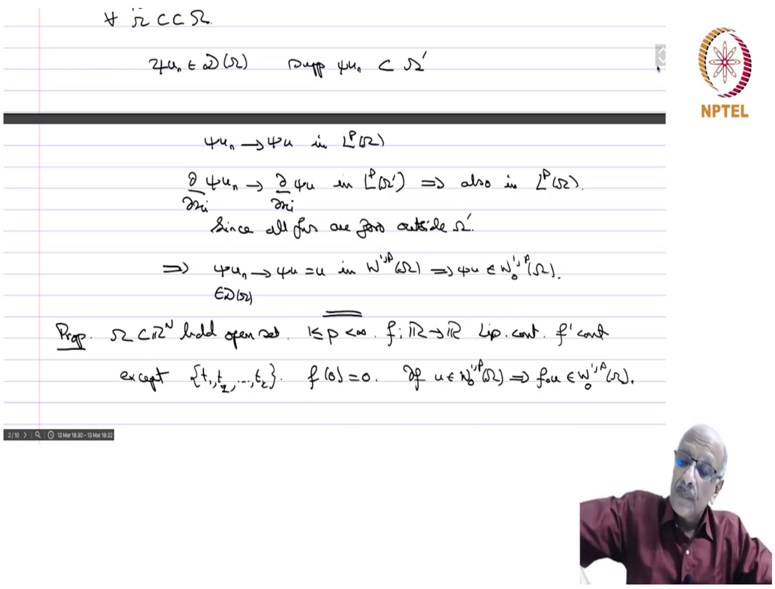

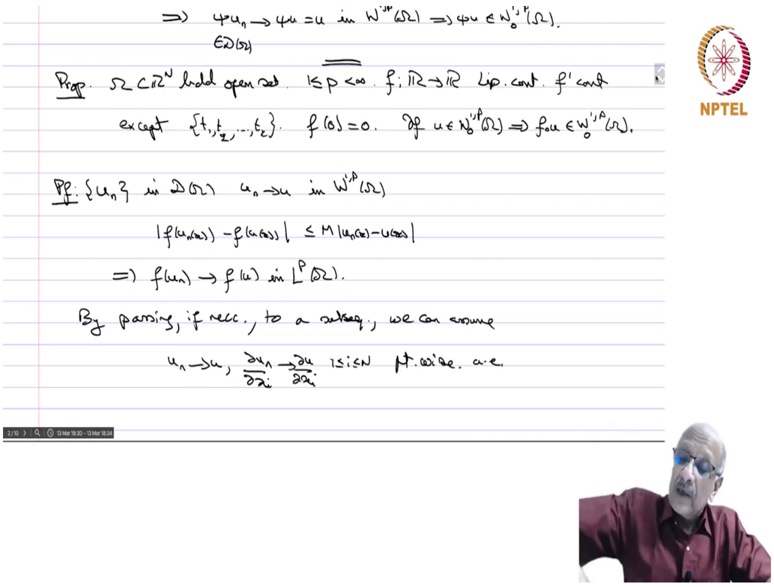

$$\Rightarrow \psi u_n \to \psi u \ = u \; in \; W^{1,p}(\Omega) \ \Rightarrow \psi u \in W_0^{\ 1,p}(\Omega).$$

Proposition. $\Omega \subset \mathbb{R}^N$ bounded open set and $1 \le p < \infty$, $f: \mathbb{R} \ \to \mathbb{R}$ Lipscitz continuous, f' continuous except at a finite number of points $\{t_1, t_2, ... t_k\}$, $f(0) = 0$. If $u \in W_0^{\ 1,p}(\Omega)$, then

$$f \circ u \in W_0^{\ 1,p}(\Omega).$$

proof. Let $\{u_n\} \in D(\mathbb{R}^N) \; s.t. \; u_n \to u \; in \; W^{1,p}(\Omega)$.

$$|f(u_n(x)) - f(u(x))| \le M|u_n(x) - u(x)|$$

$$\Rightarrow f(u_n) \to f(u) \; in \; L^p(\Omega).$$

By passing, if necessary, to a subsequence we can assume that

$$u_n \to u, \ \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}, \ 1 \le i \le N \; pointwise \; a.e.$$

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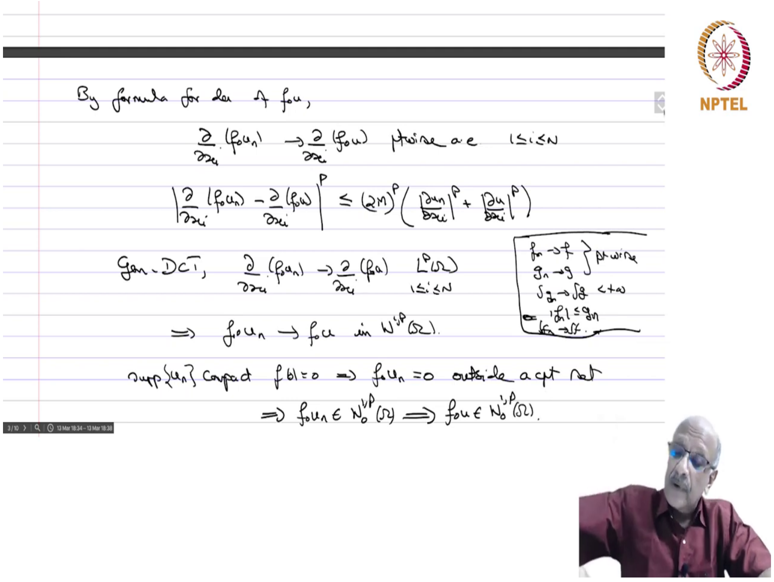

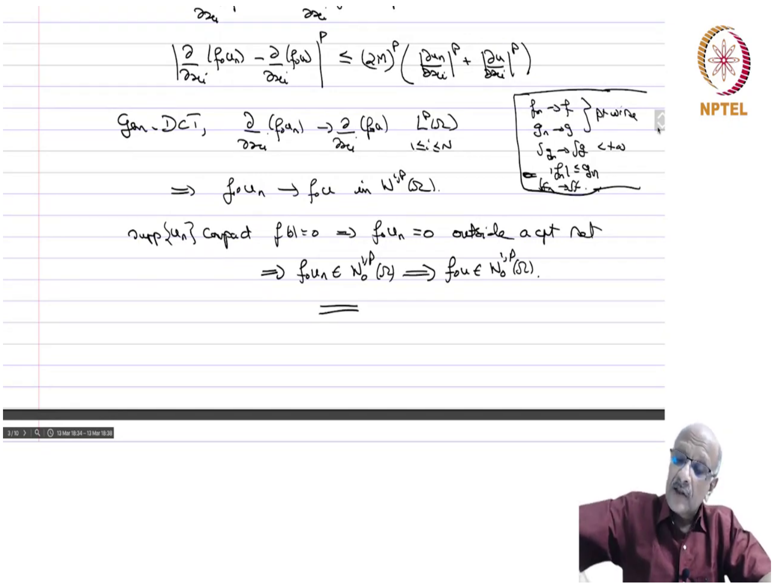

So, then f is continuous and by formula for derivative of $f \circ u$,

$$\frac{\partial(f\circ u_n)}{\partial x_i} \to \frac{\partial(f\circ u)}{\partial x_i},\ 1 \le i \le N \ pointwise\ a.e.$$

$$|\frac{\partial(f\circ u_n)}{\partial x_i} - \frac{\partial(f\circ u)}{\partial x_i}|^p \le (2M)^p||\frac{\partial u_n}{\partial x_i}|^p - |\frac{\partial u}{\partial x_i}|^p$$

So, by generalized dominated convergence theorem,

$$\frac{\partial(f\circ u_n)}{\partial x_i} \to \frac{\partial(f\circ u)}{\partial x_i},\ in\ L^p(\Omega),\ \ 1 \le i \le N.$$

$$\Rightarrow f \circ u_n \to f \circ u\ \ in\ W^{1,p}(\Omega).$$

Now, support of u_n is compact f(0)=0. So, this implies that $f \circ u_n = 0$ outside a compact set $\Rightarrow f \circ u_n \in W_0^{1,p}(\Omega) \Rightarrow f \circ u \in W_0^{1,p}(\Omega)$.

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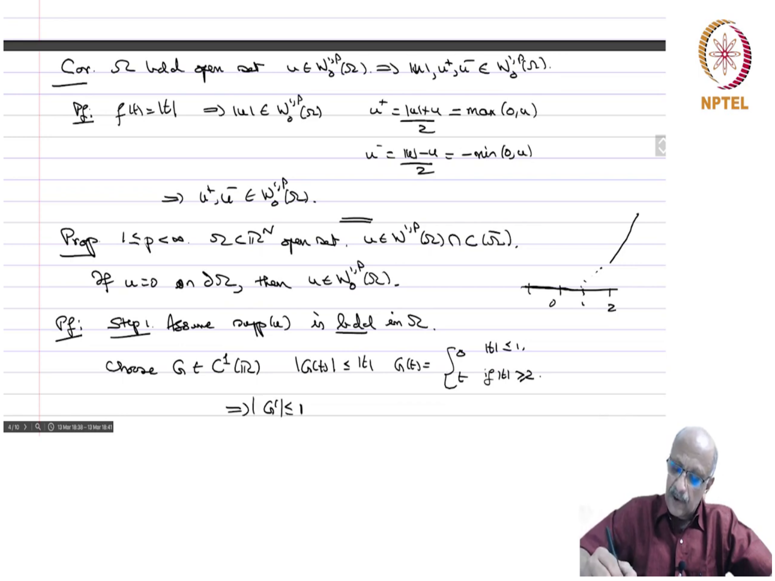

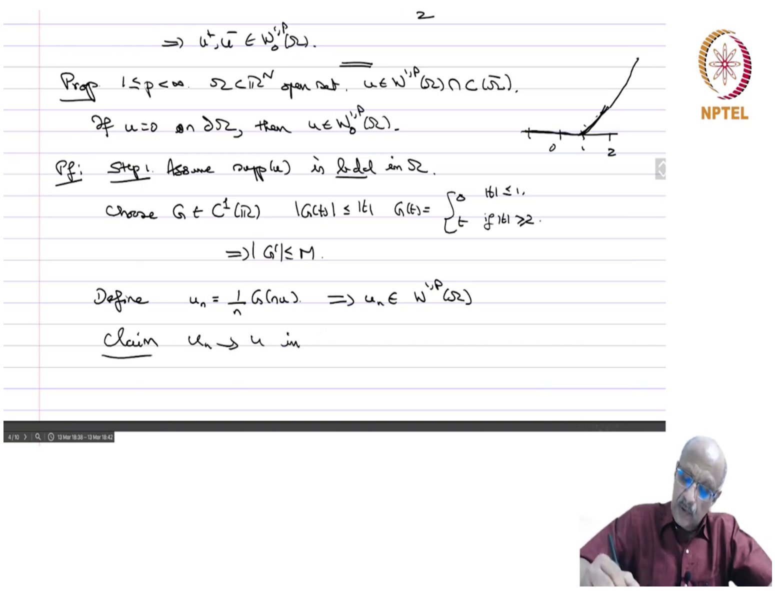

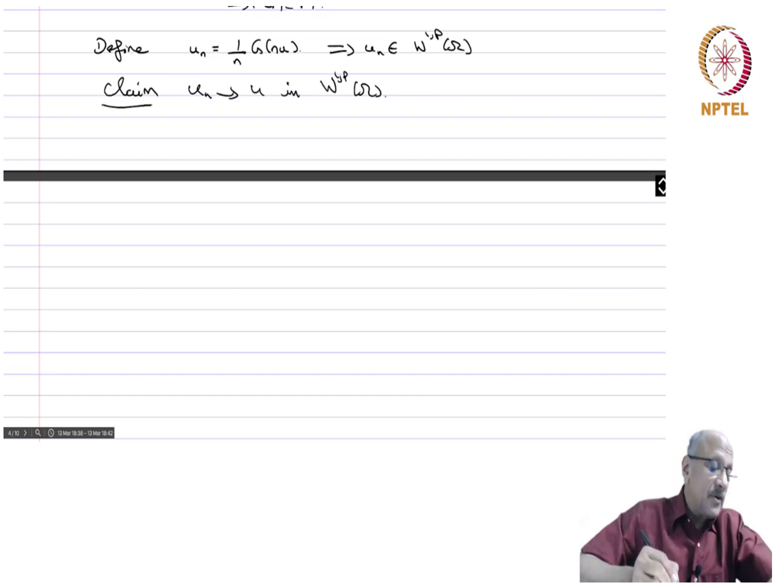

So, nice corollary of this theorem.

Corollary. $\Omega \subset \mathbb{R}^N$ bounded open set and $u \in W_0^{1,p}(\Omega) \Rightarrow |u|, u^+, u^- \in W_0^{1,p}(\Omega)$.

proof. $f(t) = |t|$, it satisfies all the conditions of the previous theorem.

$$\Rightarrow |u| \in W_0^{1,p}(\Omega).$$

Also, $u^+ = \frac{u+|u|}{2} = \max\{u, 0\}$ *and* $u^+ = \frac{|u|-u}{2} = -\min\{u, 0\}$.

Therefore $u^+, u^- \in W_0^{1,p}(\Omega)$.

This very important observation though it is very simple to prove that, in fact, when you are studying second order partial differential equations, there are some very important results called maximum principles. And the maximum principles come from the observation we have made in this corollary.

So, therefore, it is a very useful corollary which we have. So now, one more proposition,

Proposition. $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^N$ bounded open set and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. If $u = 0$ on $\partial\Omega$, *then* $u \in W_0^{1,p}(\Omega)$.

Proof. So, we will do it in 2 steps.

So, the first step: we assume $supp(u)$ is bounded in Ω. Now choose

$$G \in C^1(\mathbb{R}), \quad s.t.\ |G(t)| \le |t| \ and$$

$$G(t) = 0,\ if\ |t| \le 1,$$

$$= t,\ if\ |t| \ge 2.$$

$$\Rightarrow\ |G'(t)| \le M.$$

So now, you define $u_n = \frac{1}{n} G(nu) \ \Rightarrow u_n \in W^{1,p}(\Omega)$.

$$claim:\ u_n \to u\ in\ W^{1,p}(\Omega).$$

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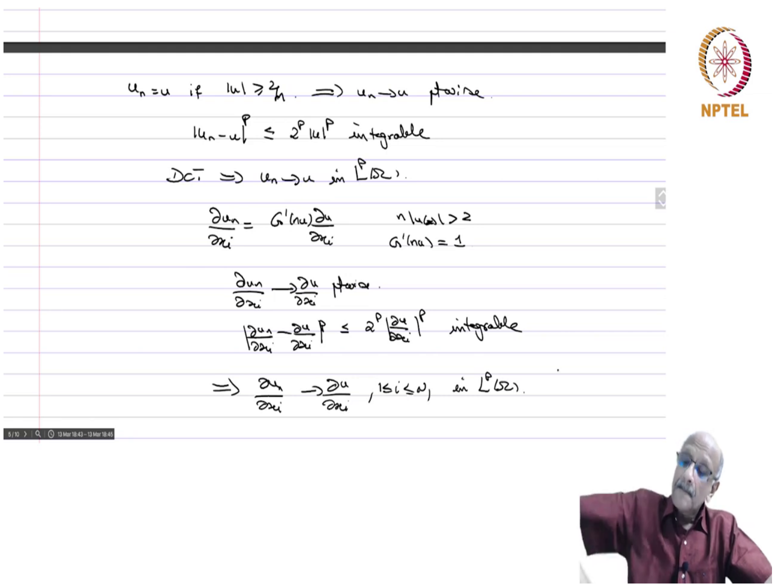

∂x_i $\Rightarrow u_n \to u$ in $L^p(\Omega)$.

$$\frac{\partial u_n}{\partial x_i} = G'(nu)\frac{\partial u}{\partial x_i} \qquad n|u(x)|>2, \quad G'(nu)=1$$

$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ pointwise.

$\left|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right|^p \le 2^p \left|\frac{\partial u}{\partial x_i}\right|^p$ integrable

$\Rightarrow \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$, $1 \le i \le N$, in $L^p(\Omega)$.

$\Rightarrow u_n \to u$ in $W^{1,p}(\Omega)$.

$Supp(u_n) \subset \{x \in \Omega \mid |u(x)| > 1/n\}$

But $u(x) = 0$ on ∂

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$Supp(u_n) \subset \{x \in \Omega \mid |u(x)| > 1/n\}$

But $u(x) = 0$ on $\partial\Omega$.

$\Rightarrow$ $Supp(u_n)$ is a bdd, closed & strictly contained in Ω.

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But $u(x) = 0$ on $\partial\Omega$.

$\Rightarrow$ $Supp(u_n)$ is a bdd, closed & strictly contained in Ω.

$\Rightarrow u_n$ vanishes outside a compact set $\subset \Omega$,

$\Rightarrow u_n \in W_0^{1,p}(\Omega)$

So, let us try to establish this claim. So, you have that $u_n = u,\ if\ |u| \geq \frac{2}{n}$.

$$\Rightarrow u_n \rightarrow u\ pointwise\ and\ |u_n - u|^p \leq 2^p u^p -\ integrable.$$

So, by the dominated convergence theorem, this implies that

$$\Rightarrow u_n \rightarrow u\ in\ L^p(\Omega).$$

$$\frac{\partial u_n}{\partial x_i} = G'(nu)\frac{\partial u}{\partial x_i}.$$

So, $\frac{\partial u_n}{\partial x_i} \rightarrow \frac{\partial u}{\partial x_i}\ pointwise$ and $|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}|^p \leq 2^p|\frac{\partial u}{\partial x_i}|^p -\ integrable.$

By DCT, $\frac{\partial u_n}{\partial x_i} \rightarrow \frac{\partial u}{\partial x_i}\ in\ L^p(\Omega),\ 1 \leq i \leq N.$

Consequently, you have, $u_n \rightarrow u\ in\ W^{1,p}(\Omega)$.

Now, $supp(u_n) \subset \{x \in \Omega: |u(x)| \geq \frac{1}{n}\}$. But $u(x) = 0\ on\ \partial\Omega$. Therefore, this implies that $supp(u_n)$ is bounded , closed, it is closed we know, it is bounded because support of u is bounded that is where we are using this hypothesis and strictly contained in omega because of the distance which it has. And consequently, you have that u_n vanishes outside a compact set $\subset \Omega$. Therefore, by the earlier proposition you have that $u_n \in W_0^{1,p}(\Omega)$.

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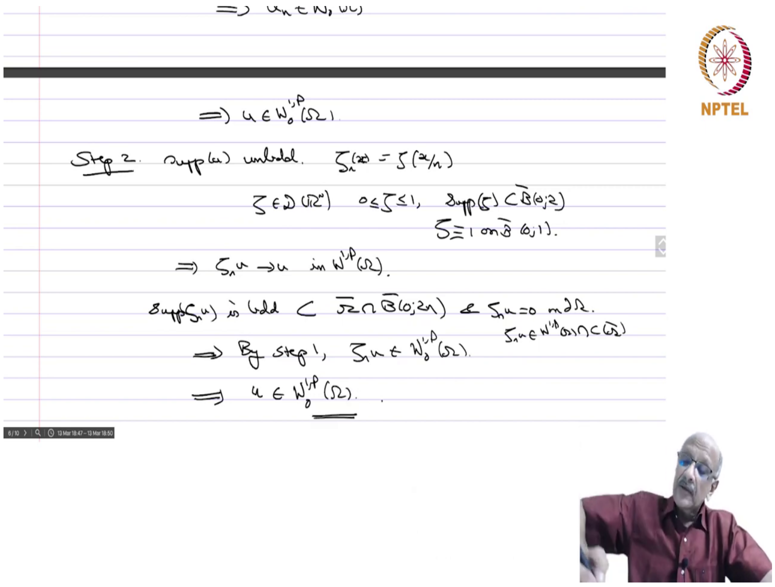

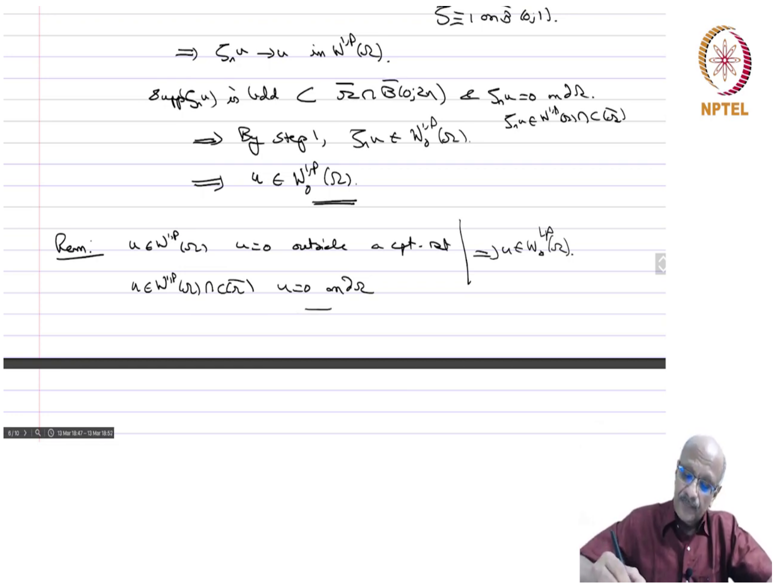

Now, step 2: $supp(u)$ is unbounded. So, then you have $\zeta_n(x) = \zeta(\frac{x}{n}),\ \zeta \in D(\mathbb{R}^N),$

$0 \leq \zeta \leq 1,\ supp(\zeta) \subset B(0;\ 2),\ \zeta \equiv 1\ on\ \overline{B(0;1)}.$

$$\Rightarrow \zeta u_n \to u\ in\ W^{1,p}(\Omega)$$

$supp(\zeta_n)$ is bounded $\subset \overline{\Omega} \cap \overline{B(0;2n)},\ \zeta_n u = 0\ on\ \partial\Omega.$

$$\Rightarrow By\ step\ 1,\ \zeta_n u \in W_0^{1,p}(\Omega) \Rightarrow u \in W_0^{1,p}(\Omega).$$

Remark. So, we have been proving theorems like if $u \in W^{1,p}(\Omega)$, u=0 outside a compact set or $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, $u = 0$ on $\partial\Omega$, then $u \in W_0^{1,p}(\Omega)$.

Later on we will show that $W_0^{1,p}(\Omega)$ is precisely the function set of all functions in $W^{1,p}(\Omega)$ which vanish on the boundary in the sense of trace. So, we will trace this what the generalizes the notion of value on the boundary because as I remarked long ago that if since the boundary is a measure 0 it does not make sense for an L^p function for you to talk about the value on the boundary, but because we know that the function $W_0^{1,p}(\Omega)$ We use supplementary information to show that you do have something like the boundary value can which can be defined in a nice way.

And precisely the boundary value vanishing is $W_0^{1,p}(\Omega)$ and that will coincide with the usual boundary value when you have continuous functions and so on. And therefore, that is our aim. So, we are making little progress towards that question and we have proved some particular results.