Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
Indian Institute of Mathematical Science
Chain Rule and Applications - Part 1

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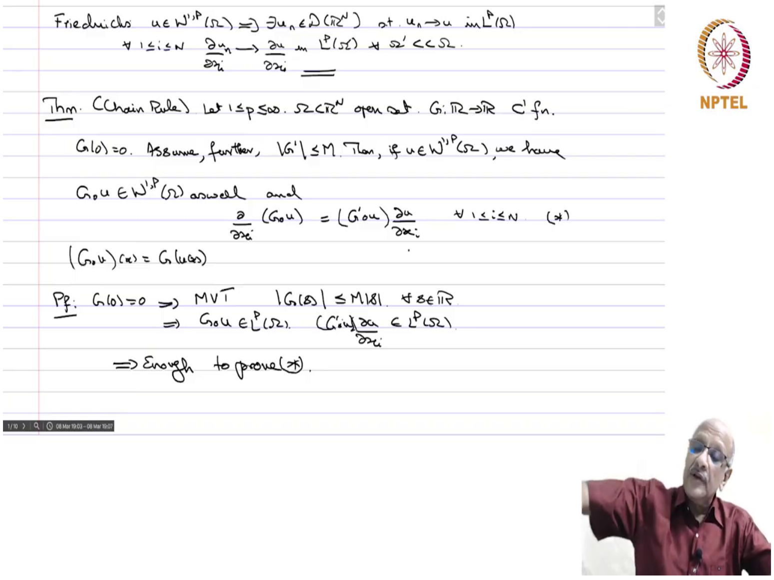

So, we were proved the following theorem of Friedrichs. So, $u \in W^{1,p}(\Omega)$ then there exists $u_n \in D(\mathbb{R}^N)$ such that $u_n \to u \; in \; L^P(\Omega)$ and for all $1 \leq i \leq N$, $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^P(\Omega^{'})$, for all $\Omega^{'} \subset \Omega$. relatively compact subsets of Ω . So, this was the theorem which we proved. So, let us now look at some applications of this.

So, the first

Theorem: so this is called the chain rule. The function of a function rule in calculus which we have here, so let $1 \leq p \leq \infty, \Omega \subset \mathbb{R}^N$ open set and

$G: \mathbb{R} \to \mathbb{R}, \; C^1$

function, that means continuously differentiable and such that $G(0) = 0$. Assume further, $|G^{'}| \leq M$. So, it has a bounded derivative then if$u \in W^{1,p}(\Omega)$ we have $G \circ u \in W^{1,p}(\Omega)$ as well.

And So, you first differentiate with respect to whatever you have and then differentiate then. So,

$$\frac{\partial}{\partial x_i}(G \circ u) = (G^{'} \circ u)\frac{\partial u}{\partial x_i}, 1 \leq i \leq N.$$

So, what is $(G \circ u)(x) = G(u(x))$. So, you see this the function of the function rule or the chain rule you have if you want to differentiate this in normal calculus, you would have done $(G^{'} \circ u)\frac{\partial u}{\partial x_i}$ and that is exactly we are having the formula which we have in mind here.

So, **proof,** so let, so G(0)=0 implies by the mean value theorem you have that

$$|G(s)| \leq M|s|.$$

So, that is the for all $s \in \mathbb{R}^N$. So, this is just application of the mean value theorem which says that $G(s) - G(0) \leq M|s - 0|$, where M is the maximum of the derivatives. So, this implies that $(G \circ u) \in L^p(\Omega)$, automatically because $|G(u(x))| \leq M|u(x)|$.

and therefore, if $u \in L^p(\Omega)$, $G \circ u \in L^p(\Omega)$.

Also, $(G^{'} \circ u)\frac{\partial u}{\partial x_i} \in L^p(\Omega)$. So, this also belongs to $\in L^p(\Omega)$. So it is enough to prove star. Once we have this formula for the derivative, it automatically implies that $G \circ u$ is an L^p and first derivatives are all in L^p and therefore, it is in $W^{1,p}$ and that is exactly what we want in addition to the formula which we have.

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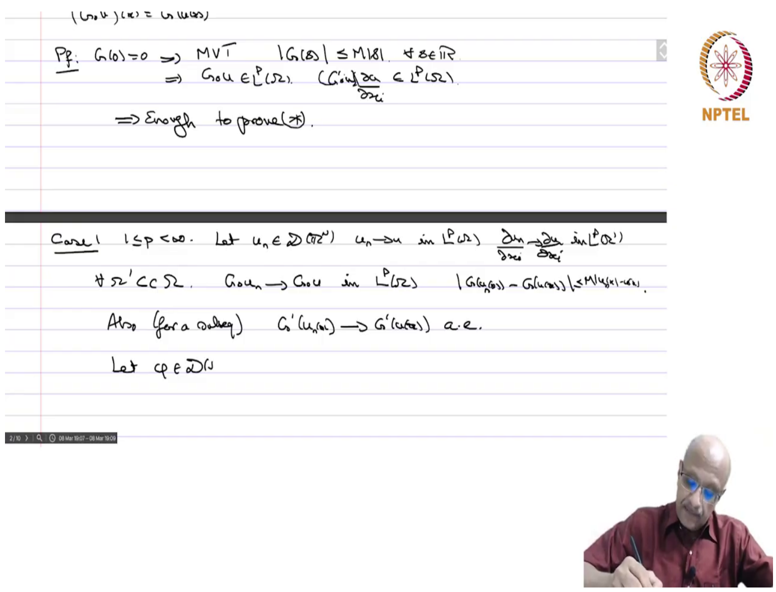

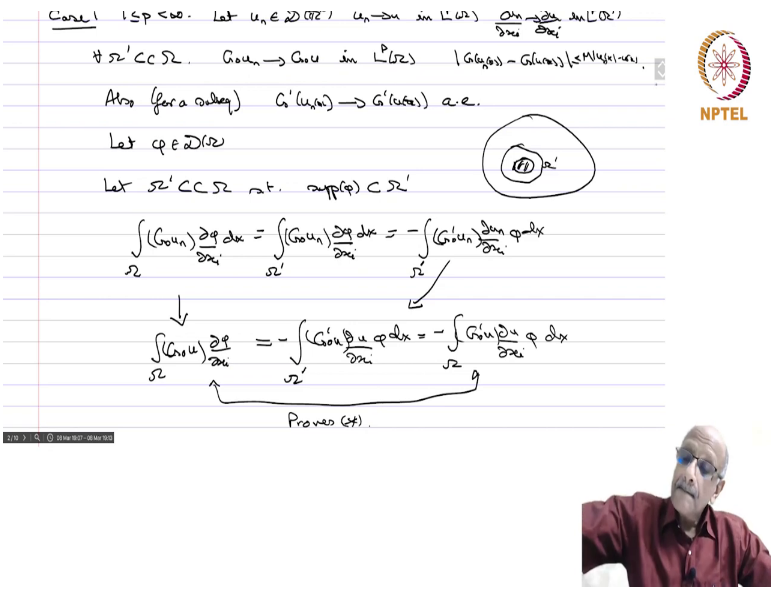

So, let us do it in two cases. So, case 1 where we have $1 \leq p < \infty$. So, let $u_n \in D(\mathbb{R}^N)$, $u_n \to u \; in \; \in L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^P(\Omega')$ for all $\Omega' \subset \Omega$. So, we are applying Friedrichs theorem. So, then $G \circ u_n \to G \circ u \; in \; \in L^p(\Omega)$. Why, because $|Gu_n(x) - Gu(x)| \leq M|(u_n - u)(x)|$. And so, if you integrate both sides the power p you will get that this goes to 0. So, this also has to go to 0.

Also, for a subsequence you have that $G' \circ u_n \to G' \circ u \; in \; \in L^p(\Omega)$because we have for a subsequence $u_n(x) \to u(x)$ point wise almost everywhere and therefore, $G' \; is \; a \; C^1$ function and therefore, the G is C^1 function, so G' is continuous and therefore, this also goes to 0. Now, we will henceforth work with the sub sequence, so I am not putting the sub sequence notation here $\{u_{n_k}\}$, I am going to work with that sub sequence, so I will call it un itself.

Now, let $\varphi \in D(\Omega)$. So, let Ω' be relatively compact in Ω such that $supp(\varphi) \subset \Omega'$. So, you have Ω and you have a compact set K here which is a support of φ I can always find by the properties of the topology in $\mathbb{R}^N$ and another which contains, Ω' this support. So, now you have

$$\int_{\Omega} (G \circ u_n) \frac{\partial \varphi}{\partial x_i} dx = \int_{\Omega'} (G \circ u_n) \frac{\partial \varphi}{\partial x_i} dx,$$

And now everything by the standard un is a smooth function G is a smooth function, so this is smooth function, this also smooth function. So, by Greens theorem which is integration by parts you get that this

$$= -\int_{\Omega'} (G' \circ u_n) \frac{\partial u_n}{\partial x_i} \varphi dx.$$ There is no boundary term because phi vanishes on the boundary since it is C infinity function with compact support therefore, it vanishes on the boundary.

So, there is no boundary term on Ω'. So, now, we are going to pass to the limit $G \circ u_n \to G \circ u$ in L^p, φ is a fixed C infinity function with compact support. So, we have seen these many times already. So, this will converge to $G \circ \frac{\partial u_n}{\partial x_i}$. And this one will converge,

$$\int_{\Omega} (G \circ u) \frac{\partial \varphi}{\partial x_i} dx = -\int_{\Omega'} (G' \circ u) \frac{\partial u}{\partial x_i} \varphi dx,$$

goes in L^p and phi is a fixed function C infinity function with compact support and therefore, by again you can apply whatever favourite theorem you like the dominated convergence theorem or any other theorem.

So, you have if you have for instance a sequence which is uniformly bounded in L infinity converging point wise and function which is converging in L^p then the product will converge in L^p. So, this and consequently you are multiplying it by φ and therefore, this will go to $= -\int_{\Omega'} (G' \circ u) \frac{\partial u_n}{\partial x_i} \varphi dx$ and then we can replace now Ω' by Ω. So this is equal to

$$= -\int_{\Omega} (G' \circ u) \frac{\partial u_n}{\partial x_i} \varphi dx.$$ and that is exactly what we wanted to prove and this will prove star.

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Proves (*).

Case 2 $p = \infty$ $\varphi \in \mathcal{D}(\Omega)$ $\text{supp}\,\varphi \subset \Omega' \subset\subset \Omega$.

$u \in W^{1,\infty}(\Omega) \Rightarrow u \in W^{1,q}(\Omega')\ \forall\ 1 \le q < \infty$

and case (i) applies.

Rem. $G(0)=0$ used to show that $|G(s)| \le M|s| \Rightarrow G\circ u \in L^p$.

Suppose Ω bounded. MVT

$$|G(s) - G(0)| \le M|s|$$

$$\Rightarrow |(G\circ u)(x)| \le |G(0)| + M|u(x)|$$

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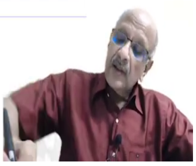
Thm $\Omega \subset \mathbb{R}^N$ bdd. open set. $1 \le p < \infty$. $u \in W^{1,p}(\Omega)$

$\Rightarrow |u| \in W^{1,p}(\Omega)$

$$\frac{\partial |u|}{\partial x_i} = \text{sgn}(u)\frac{\partial u}{\partial x_i} \qquad 1 \le i \le N.$$



So, **case 2**, this is easier, $p = \infty$. So, if again $\varphi \in D(\Omega)$ and $supp(\varphi) \subset \Omega'$ which is relatively compact in Ω as before and then if $u \in W^{1,\infty}(\Omega)$ then what do you have, in fact, in fact this implies $u \in W^{1,\infty}(\Omega')$, for all $1 \leq p \leq \infty$ also, but I do not want to want that now. And now, we are in the previous case and case 1 applies and the result for us so, we have the chain rule.

So now, remark important remark $G(0) = 0$used to show that

$$|G(s)| \leq M|s|, \Rightarrow G \circ u \in L^p$$

this was the only place where we needed the fact that $G(0) = 0$.

Suppose Ω is bounded then by the mean value theorem, you have

$$|G(s) - G(0)| \leq M|s| \Rightarrow |G \circ u(x)| \leq |G(0)| + M|u(x)|$$

and Ω bounded means finite measure implies $|G(0)|$ constant function is in L^p and therefore, you do not need the and therefore, this implies that $G \circ u$ is also in L^p .

So, if you are in a bounded domain then you can still use this the chain rule without the hypothesis $G(0) = 0$ which we are now going to do. So, we are going to prove another interesting

theorem, very nice theorem. So, let $\Omega \subset \mathbb{R}^N$ open set $1 \leq p < \infty$, $u \in W^{1,p}(\Omega) \Rightarrow |u| \in W^{1,p}(\Omega$

and $\frac{\partial |u|}{\partial x_i} = sgn(u)\frac{\partial u}{\partial x_i}$, $1 \leq i \leq N$.

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$$\frac{\partial |u|}{\partial x_i} = \text{sgn}(u)\frac{\partial u}{\partial x_i} \qquad 1\le i\le N$$

where $\text{sgn}\,u(x) = \begin{cases} +1 & u(x)>0 \\ 0 & u(x)=0 \\ -1 & u(x)<0. \end{cases}$

Pf. Let $\varepsilon>0$. $f_\varepsilon(t) = \sqrt{t^2+\varepsilon}$ $\quad f_\varepsilon \in C^1(\mathbb{R})$

$$f_\varepsilon'(t) = \frac{t}{\sqrt{t^2+\varepsilon}} \Rightarrow |f_\varepsilon'| \le 1.$$

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$f_\varepsilon(0)\neq 0$ However Ω bdd $\Rightarrow$ chain rule holds.

$$u\in W^{1,p}(\Omega) \Rightarrow f_\varepsilon(u) = f_\varepsilon\circ u \in W^{1,p}(\Omega)$$

$$\frac{\partial}{\partial x_i}(f_\varepsilon\circ u) = \frac{u}{\sqrt{u^2+\varepsilon}}\frac{\partial u}{\partial x_i} \qquad 1\le i\le N.$$

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$$\frac{\partial}{\partial x_i}(f_\varepsilon\circ u) = \frac{u}{\sqrt{u^2+\varepsilon}}\frac{\partial u}{\partial x_i} \qquad 1\le i\le N.$$

$|f_\varepsilon(t) - |t|| \le \sqrt{\varepsilon}$ (check!) $\quad (t^2+\varepsilon \le (|t|+\sqrt{\varepsilon})^2)$.

$$f_\varepsilon\circ u \to |u| \text{ in } L^p(\Omega).$$

$$\frac{u}{\sqrt{u^2+\varepsilon}}\frac{\partial u}{\partial x_i} \to \text{sgn}\,u\,\frac{\partial u}{\partial x_i} \quad \text{pt wise.}$$

Further

where

$$sgn(u)(x) = \begin{cases} +1, & if\ u(x) > 0, \\ 0, & if\ u(x) = 0, \\ -1, & if\ u(x) < 0. \end{cases}$$

Proof: so, we will prove this result now. So, let $\varepsilon > 0$,, small positive number and define

$f_\varepsilon(t) = \sqrt{t^2 + \varepsilon}.$

So, then $f_\varepsilon \in C^1(\mathbb{R})$ and $f_\varepsilon' = \frac{t}{\sqrt{t^2+\varepsilon.}}$,

and this shows that $|f_\varepsilon(t)'| \le 1$, $f_\varepsilon(0) \neq 0$, however Ω bonded implies that chain rule applies.

And therefore, you have that $u \in W^{1,p}(\Omega) \Rightarrow f_\varepsilon \circ u \in W^{1,p}(\Omega)$ and

$$\frac{\partial(f_\varepsilon \circ u)}{\partial x_i} = \frac{u}{\sqrt{u^2+\varepsilon.}}\frac{\partial u}{\partial x_i},\ 1 \le i \le N.$$

So now, $|f_\varepsilon(t) - |t|| \le \sqrt{\varepsilon}$, that is a very trivial inequality for you to check.

So, check just comes from the fact that $\sqrt{t^2 + \varepsilon \le}(t + \sqrt{\varepsilon})$, that is all, it just comes from that. So, $(t^2 + \varepsilon) \le (t + \sqrt{\varepsilon})^2$.

So, it just a consequence of this fact and therefore, you have that $f_\varepsilon \circ u \to |u|\ in\ L^p(\Omega)$. Now, $\frac{u}{\sqrt{u^2+\varepsilon.}}\frac{\partial u}{\partial x_i} \to sgn(u)\frac{\partial u}{\partial x_i}$, because if $u > 0$ this will go to you +1, if $u < 0$ this will go to -1, if $u = 0$ it is 0 anyway. So, so this goes to sig(u)$\frac{\partial u}{\partial x_i}$ a pointwise.

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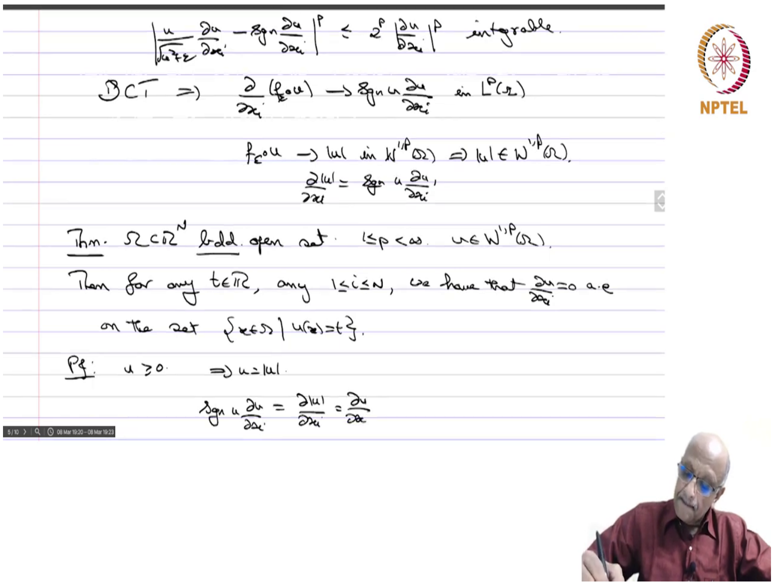

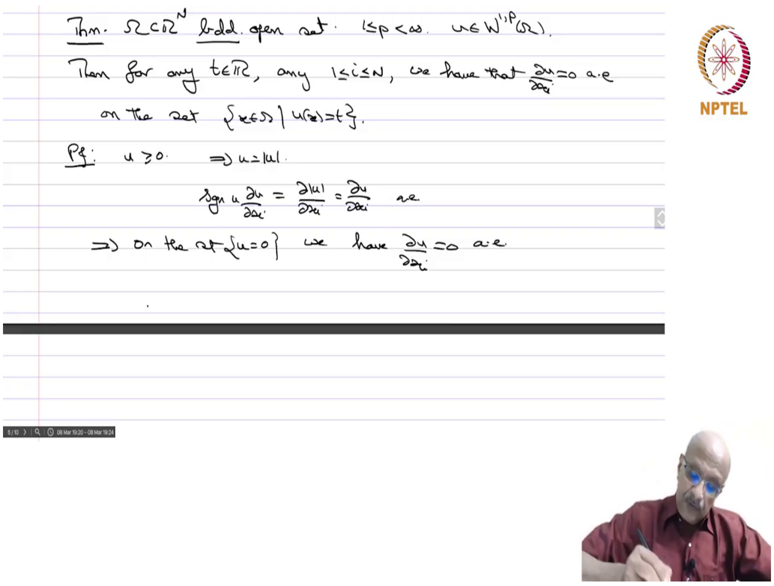

Further you have that

$$\left| \frac{u}{\sqrt{u^2+\varepsilon.}} \frac{\partial u}{\partial x_i} - sign(u) \frac{\partial u}{\partial x_i} \right|^p \leq 2^p \left| \frac{\partial u}{\partial x_i} \right|^p$$

Therefore, by the dominated convergence theorem, dominated convergence theorem implies that $\frac{\partial}{\partial x_i}(f_\varepsilon \circ u) \rightarrow sgn(u) \frac{\partial u}{\partial x_i} \ L^p(\Omega)$.

So, this means that

$$(f_\varepsilon \circ u) \rightarrow |u| \ in \ W^{1,p}(\Omega) \Rightarrow |u| \in W^{1,p}(\Omega).$$

And of course, the derivative is what we have because we converges, so therefore, the derivative and $\frac{\partial |u|}{\partial x_i} = sgn(u)\frac{\partial u}{\partial x_i}$, $1 \leq i \leq N$. So, that proves this, it has very nice applications which we will see later on. So, theorem again, consequences of this theorem.

Theorem: So, $\Omega \subset \mathbb{R}^N$ bounded open set $1 \leq p < \infty$. Let $u \in W^{1,p}(\Omega)$ then for any $t \in \mathbb{R}$ and any $1 \leq i \leq N$, we have that $\frac{\partial u}{\partial x_i} = 0$ almost everywhere on the set of all $\{x \in \Omega \mid u(x) = t\}$. So

proof, so let us take $u \geq 0$, so that this implies that $u = |u|$. So,

$$sgn(u)\frac{\partial u}{\partial x_i} = \frac{\partial |u|}{\partial x_i} = \frac{\partial u}{\partial x_i}$$

So, this implies that on the set $\{x \in \Omega \mid u(x) = 0\}$ we must have, we have $\frac{\partial u}{\partial x_i} = 0$ when I say this is equal to this these are of course, L^p functions, so they are all equal almost everywhere. Because $sgn(u)\frac{\partial u}{\partial x_i} = 0$ on the set $\{x \in \Omega \mid u(x) = 0\}$. So, this is $\frac{\partial u}{\partial x_i} = 0$ almost everywhere.

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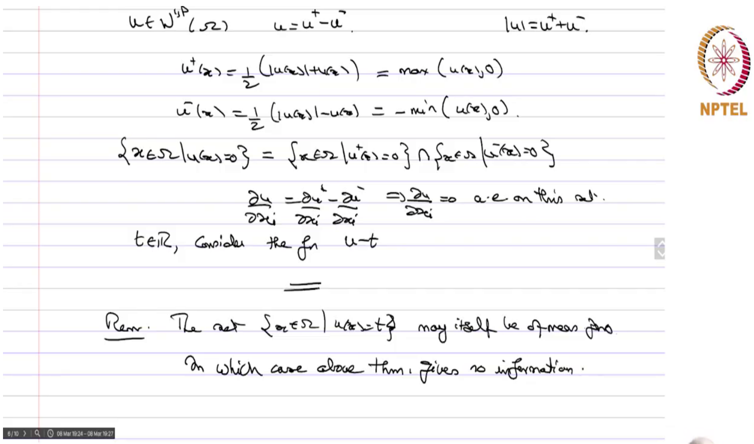

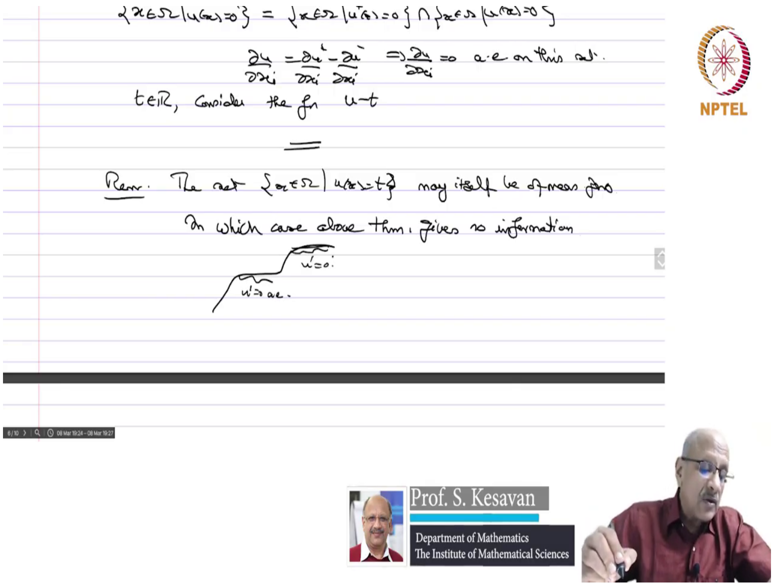

Now, if $u \in W^{1,p}(\Omega)$, we write $u = u^{+} - u^{-}$. So,

$$u^{+} = \frac{1}{2}(|u(x)| + u(x)) = max\{u(x), 0\},$$

$$u^{-} = \frac{1}{2}(|u(x)| - u(x)) =- min\{u(x), 0\}.$$

So, both are non-negative functions and you can write it and then of course, you have $u = u^{+} - u^{-}$ this is something which you would have seen many times when doing real analysis and therefore, if you have

set of all $\{x \in \Omega \ | u(x) = 0\} = \{x \in \Omega | \ u^{+}(x) = 0\} \cap \{x \in \Omega | \ u^{-}(x) = 0\}$. And

$$\frac{\partial u}{\partial x_i} = \frac{\partial u^{+}}{\partial x_i} - \frac{\partial u^{-}}{\partial x_i} = 0$$

almost everywhere on the set. So, for any $t \in \mathbb{R}$ consider the function $u - t$. So, then apply the theorem for 0 and then you get this. So, this is thing which says that if you have a level set namely $u(x) = 0$, then the derivative of u distribution derivative will vanish almost everywhere on that set.

Of course, **remark** the set $\{x \in \Omega \ | \ u(x) = t\}$ itself be of measure 0 because very often you have smooth functions they will unless the function is flat, given any value the set where it takes this value will be a set of measures 0. In which case above theorem gives no new information, no information because you are saying something is 0 almost everywhere. Well, it may not even be 0 at all because the set itself is of measure 0.

But however, if you have a function which is like this, so in portions like this where the constant value is attained, so there you will have that u dash will be 0 almost everywhere just like in the calculus case. So, this is what we have.

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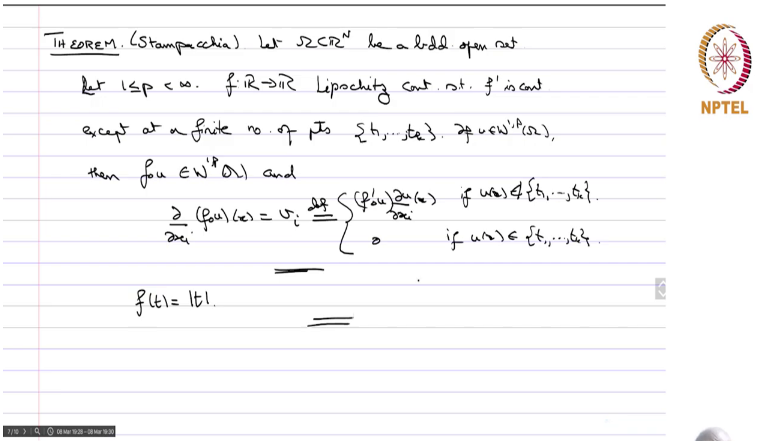

Now, we will state the theorem. I would not prove it the proof you can read in the book which I suggested to you topics and functional analysis and applications, but the proof is almost similar to the chain rule proof we have a little more, you have to be a little careful here and there, but otherwise there is no new ideas. So, we will save some time. So, we will just take it. So, this is called

Stampacchia's theorem.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 \le p < \infty$ and $f: \mathbb{R} \to \mathbb{R}$ Lipschitz continuous such that, so Lipschitz continuous function is of bounded variation absolutely continuous etc. and therefore, you have that it is differentiable almost everywhere and the derivative is bounded by the Lipschitz constant.

So, f' is continuous except at a finite number of points $\{t_1,..., t_k\}$ if $u \in W^{1,p}(\Omega)$, then $f \circ u \in W^{1,p}(\Omega)$ and

$$\frac{\partial (f \circ u)}{\partial x_i}(x) = v_i \overset{def}{=} (f' \circ u) \frac{\partial u}{\partial x_i}(x) \text{ , } if \ u(x) \notin \{t_1,..., t_k\},$$

$$\overset{def}{=} 0 \qquad \text{ , } if \ u(x) \in \{t_1,..., t_k\}.$$

this we have already seen this in the, suppose a typical example for us is a function $f(t) = |t|$ and then there we saw in fact this is precisely sgn(u) is precisely given by this expression.

So, this is a generalization of the previous theorem. And as I said it does not produce any new ideas in the proof. It is only technically a little more complicated and therefore, we will omit the proof and then we will continue with some other properties based on the chain rule later.