Sobolev Spaces and Partial Differential Equations
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Lecture 3
Distributions

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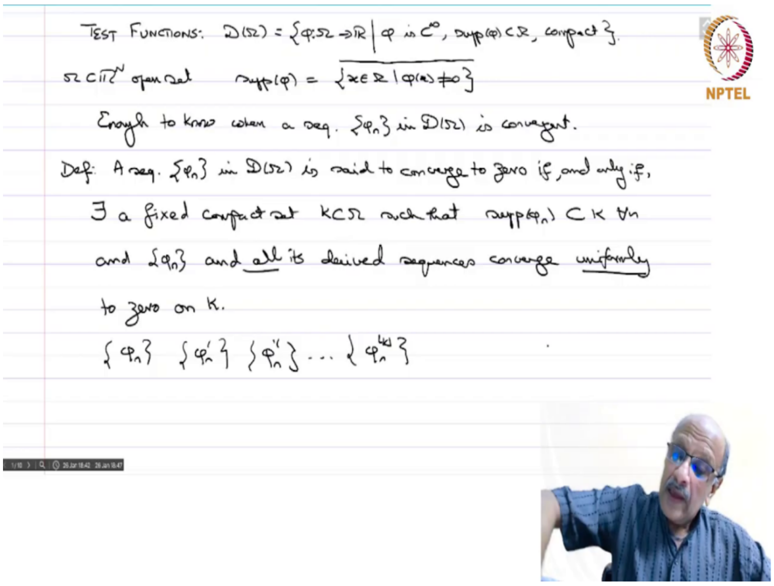

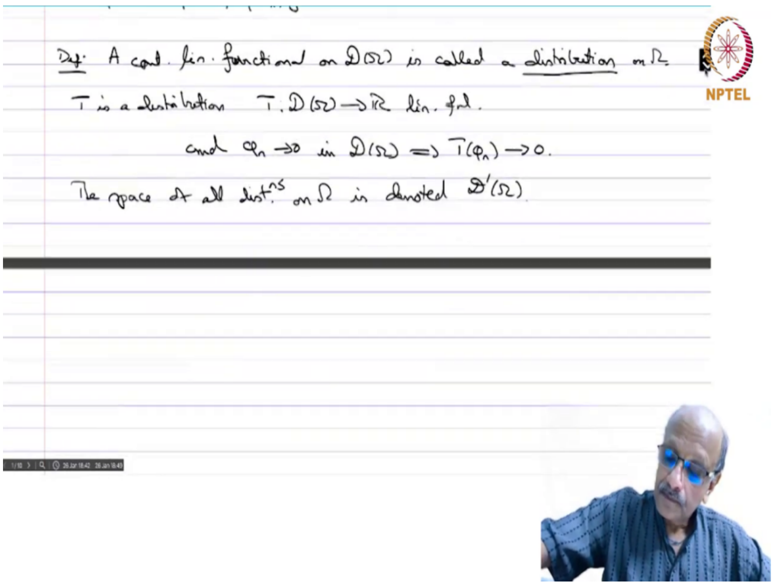

Let me recall the space of test functions.

$$D(\Omega) = \left\{ \phi : \Omega \rightarrow \mathbb{R} \mid \phi \; is \; C^{\infty} \; in \; \Omega \; and \; supp(\phi) \subset \Omega \; is \; compact \right\}.$$

This is a vector space with the usual pointwise addition and the scalar multiplication and we have seen that it is very well endowed; there are lots of functions which we can construct with specified properties. Let us also recall that

$$\text{supp}(\phi) = \overline{\{x \in \Omega:\ \phi(x) \neq 0\}}.$$

So, it is always a closed set and if it is compact then you say it is a function with compact support. Now, we want to give a topology on this space. So, the topology which we are going to give is called the inductive limit topology.

I do not want to give the details of this topology. Again, you can read it from the book which I cited before, Topics in Functional Analysis and Applications and it does not really matter for us. What we want to know is that this topology, which makes $D(\Omega)$ into a topological vector space, is in fact for the purpose of continuity, it is enough to deal with sequences.

So that is the nice property of this topology and therefore it is enough to know when the sequence in $D(\Omega)$ is convergent. So it is enough to know when a sequence $\{\phi_n\}$in $D(\Omega)$ is convergent. So, this is all that we know and therefore we want to know, since it is a vector space it is enough to know when it converges to 0, because if ϕ_n converges to ϕ, then $\phi_n - \phi$ converges to 0 and therefore it is enough to know when the sequence is convergent.

Definition: A sequence $\{\phi_n\}$ in $D(\Omega)$ is said to converge to 0 if and only if there exists a fixed compact set $K \subset \Omega$ such that $\text{supp}(\phi_n) \subset K$for all n and $\{\phi_n\}$ and all it's derived sequences converge uniformly to zero on K.

So, what do you mean by derived sequences? So let us take for instance in one dimension if$n = 1$, so if you have $\Omega \subset \mathbb{R}$ that means let us say an interval, so you have $\{\phi_n\}$ is a sequence and then, $\{\phi'_n\}$ is a sequence and then $\{\phi''_n\}$ is a sequence and so on……..$\{\phi^{(k)}_n\}$ gives another sequence. So, these are all the derived sequences. All of them must converge uniformly to 0 on this compact set. So, if you have higher dimensions then you think of sequences got by all kinds of partial derivatives. Every one of them has to converge to 0. So, this is a very costly condition. So, lots of things have to happen in order that sequence $\{\phi_n\}$ converges to 0 in $D(\Omega)$.

So, with this we can, so this is the topology and now we have a topological vector space and as I said continuity on this topological vector space for linear functionals is dependent only on behaviour of sequences. Therefore, well, generally in a topological space except in metric spaces you cannot say continuity characterizes the entire topology . But, in this case, it happens that you can do it here also.

So, the dual space of this is called the space of distribution. So, we have the next definition.

Definition: A continuous linear functional T on $D(\Omega)$, i.e., T: $D(\Omega) \to \mathbb{R}$ is a linear functional, is called a distribution on Ω if

$$\{\phi_n\} \to 0 \text{ in } D(\Omega) \Rightarrow T(\phi_n) \to 0.$$

So, this is the condition for a distribution. So, this is what we need.

So, the space of all distributions on Ω is denoted as $D(\Omega)'$, so it is a dual space.

So now we want to give lots of examples of distributions and of course the first is to redeem our promise that we wanted to generalize the notion of a function to a distribution. So, we want to know what functions can be considered as distributions and then we show that it is a more general thing.

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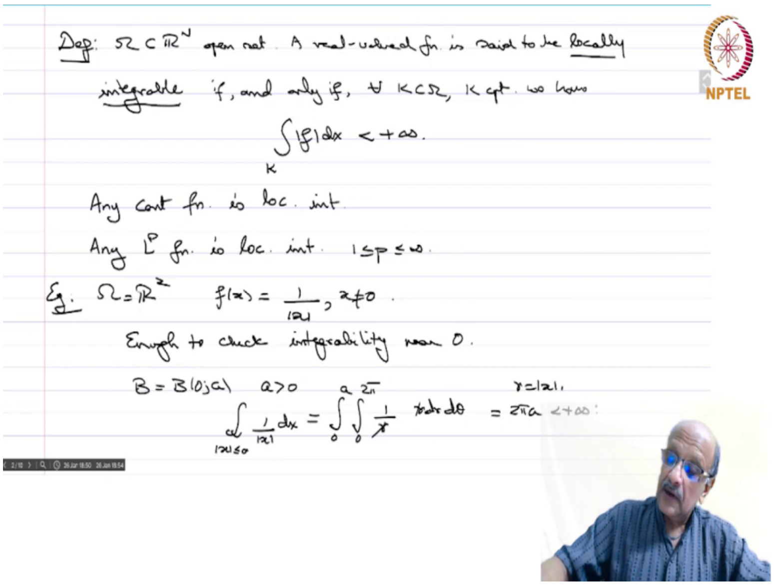

So, we have one more definition now.

Definition: Let $\Omega \subset \mathbb{R}^N$ be an open set. A real valued function f is said to be locally integrable if and only if for every $K \subset \Omega$, K-compact, we have

$$\int_K |f|\, dx < \infty.$$

As I said, complex valued is also fine if you are dealing with complex things, but we are going to deal with reals most of the time and it is all applicable to complex valued functions.

So, if you have any continuous function, it is locally integrable. Because any continuous function on any compact set is bounded and compact sets a finite Lebesgue measure and therefore this integral is always finite. Any L^p $(1 \le p \le \infty)$ function is locally integrable. Because if you have a function which is in L^p then it is in L^p(any compact) also and if your compact sets have finite measure, so if it is L^p(any compact), then it is also L^1(any compact). Therefore again these are all locally integrable. So let us give another example of function which is not covered by these two examples.

Example: Let $\Omega = \mathbb{R}^2$ and $f(x) = \frac{1}{|x|}$ for $x \neq 0$. So, it is only defined away from the origin and we want to show that this is not a continuous function and it is not in L^p function because at the origin you have a problem. And therefore, you want to show that this is locally integrable. Now, why is this locally, this blows up a torch, so why is it locally integrable? Well, if you take any compact set K which does not contain the origin then of course this is a nice continuous function and therefore the integral will be automatically finite. So enough to check integrability near the origin.

Let $B = B(a; 0)$, which is a ball centered at origin and radius a and we want to look at

$$\int_{|x|\leq a} \frac{1}{|x|}\, dx = \int_0^a \int_0^{2\pi} \frac{1}{r} r\, dr\, d\theta, \quad [\text{ polar coordinate; } r = |x|]$$

$$= 2\pi < \infty$$

And therefore, you have that this function is a locally integrable function. Similarly, you can construct locally integrable functions in higher dimensions also.

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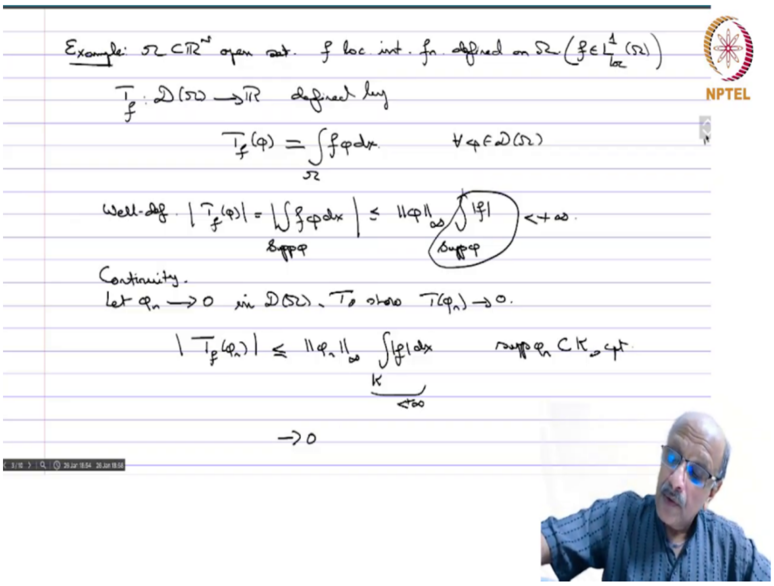

So now we are going to give the first example of a distribution.

Example: Let $\Omega \subset \mathbb{R}^N$ be an open set and f be a locally integrable function defined on Ω ($f \in L^1_{loc}(\Omega)$). Define T_f: $D(\Omega) \to \mathbb{R}$ by

$$T_f(\phi) = \int_\Omega f\phi\, dx, \quad \text{for all } \phi \in D(\Omega).$$

So, we have to check that this is a distribution. So first of all, we need to check well-definedness.

well defined: $|T_f(\phi)| = |\int_{supp(\phi)} f\phi\, dx| \leq ||\phi||_\infty \int_{supp(\phi)} |f| < \infty.$

So, it is well defined, it is also linear. So that is why we need locally integrable functions, so that you have, that this is well defined. You cannot define it otherwise unless the function is integrable over any compact set.

Now we have to check continuity.

continuity: Let $\{\phi_n\} \to 0$ *in* $D(\Omega)$.

to show: $T(\phi_n) \to 0$.

Now $|T_f(\phi_n)| = |\int_{supp(\phi_n)} f\phi_n\, dx| \leq ||\phi_n||_\infty \int_K |f| < \infty$, where supp$(\phi_n) \subset K$ -compact $\to 0$

Therefore T_f defines a distribution.

Now, we are going to state a proposition.

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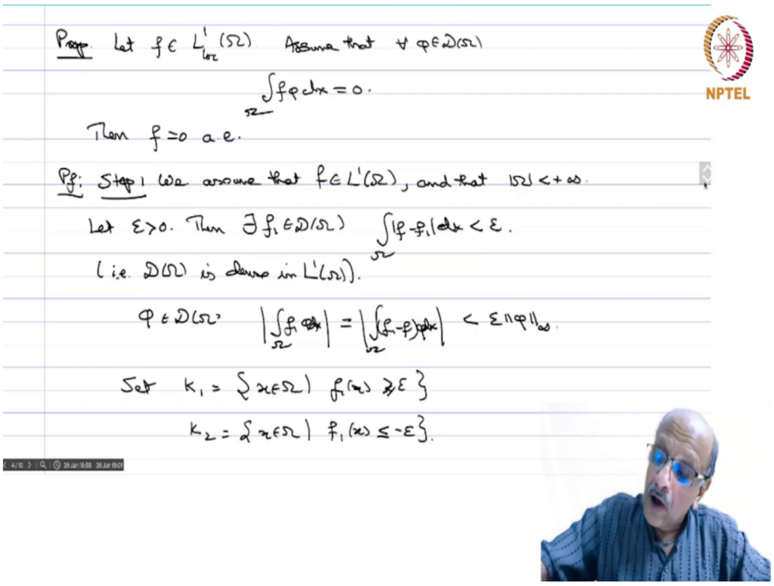

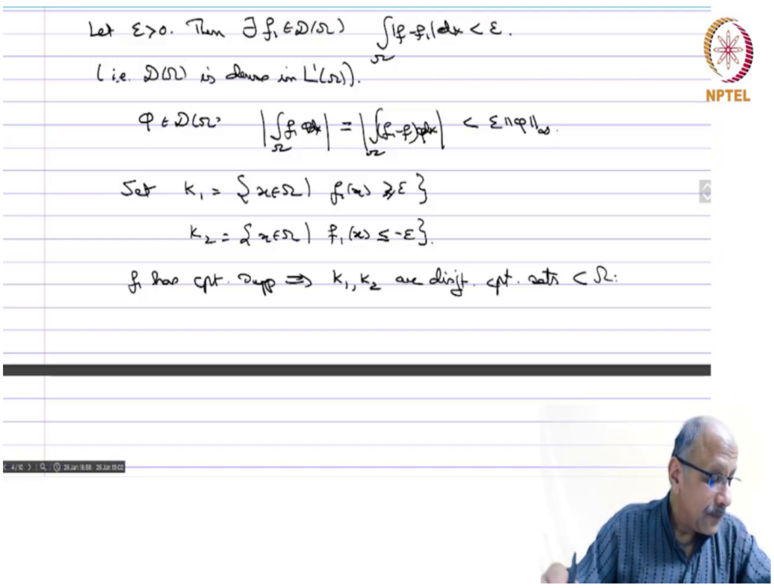

Proposition: Let $f \in L^1_{loc}(\Omega)$. Assume that $\forall \phi \in D(\Omega)$,

$$\int_\Omega f\phi \, dx = 0.$$

Then $f \equiv 0$a.e.

proof: *step I:* We assume that $f \in L^1(\Omega)$ and $|\Omega| < \infty$.

Let $\epsilon > 0$. Then there exists $f_1 \in D(\Omega)$ such that

$$\int_{\Omega} |f - f_1|\, dx < \epsilon$$

(as $D(\Omega)$ is dense in $L^1(\Omega)$). So, we are using this fact. We will prove this later.

Let us take $\phi \in D(\Omega)$. Then

$$|\int_{\Omega} f_1 \phi dx| = \int_{\Omega} (f_1 - f)\phi dx < \epsilon ||\phi||_{\infty}.$$

Set
$$K_1 = \{x \in \Omega : f_1(x) \geq \epsilon\}$$

$$K_2 = \{x \in \Omega : f_1(x) \leq - \epsilon\}$$

Now recall that f_1 has compact support. So, this implies that K_1 and K_2 are disjoint compact sets contained in Ω.

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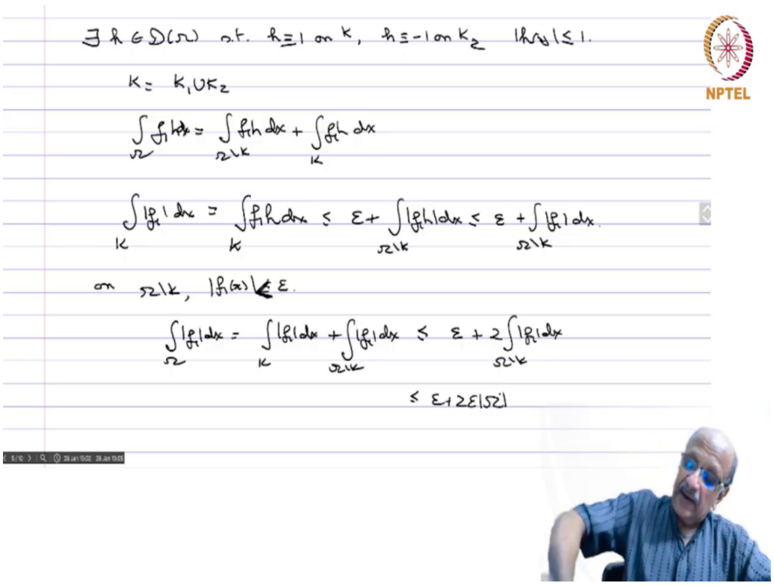

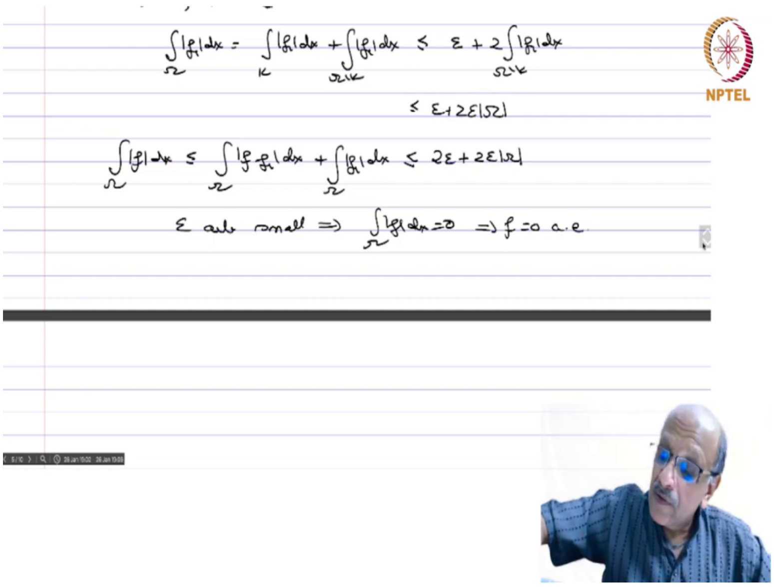

gave this exercise last time: there exists a $h \in D(\Omega)$ such that $h \equiv 1$ on K_1, $h \equiv -1$ on K_2 and $|h(x)| \leq 1$.

Let $K = K_1 \cup K_2$, which is again a compact set. Then we have

$$\int_{\Omega} f_1 h\, dx = \int_{\Omega\setminus K} f_1 h\, dx + \int_{K} f_1 h\, dx.$$

Now $$\int_{K} |f_1|\, dx = \int_{K} f_1 h\, dx \leq \epsilon + \int_{\Omega\setminus K} f_1 h\, dx \leq \epsilon + \int_{\Omega\setminus K} |f_1|\, dx$$

On $\Omega\setminus K$, we have $|f_1(x)| < \epsilon$. Therefore,

$$\int_{\Omega} |f_1|\, dx = \int_{\Omega\setminus K} |f_1|\, dx + \int_{K} |f_1| dx \leq \epsilon + 2$$

$$\int_{\Omega\setminus K} |f_1|\, dx \leq \epsilon + 2\epsilon|\Omega|.$$

Therefore, it follows that

$$\int_{\Omega} |f| \, dx = \int_{\Omega} |f - f_1| \, dx + \int_{\Omega} |f_1| \, dx \le 2\epsilon + 2\epsilon|\Omega| .$$

So, if ϵ is very small, we have $\int_{\Omega} |f| \, dx = 0 \Rightarrow f \equiv 0$ a.e.

So, we have proved this in the case when f is an integrable function on Ω and Ω has finite measure.

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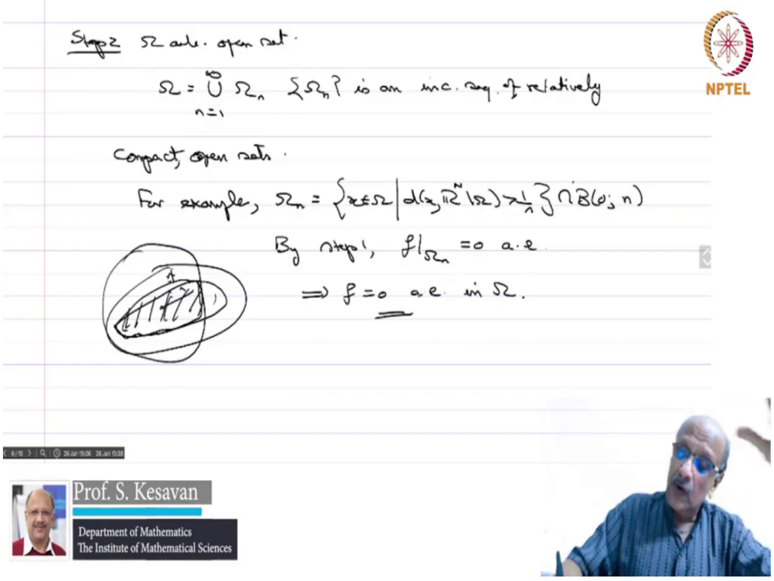

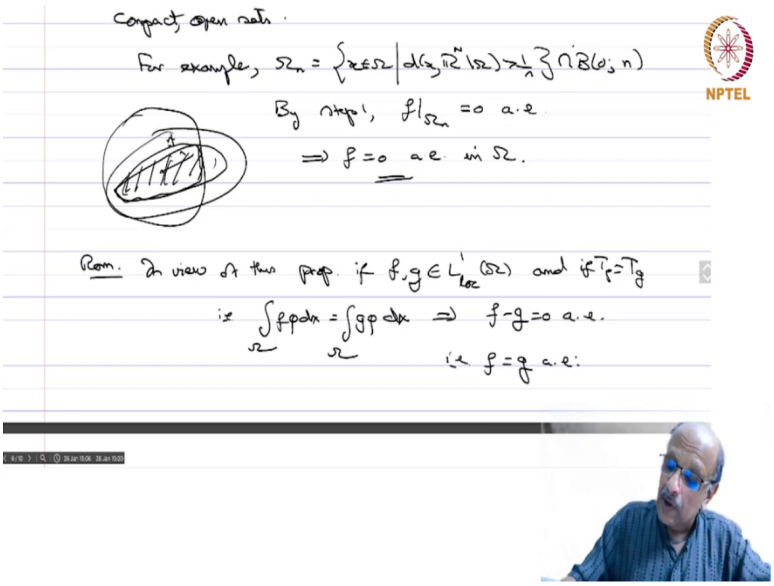

step II: Let Ωbe an arbitrary open set. Then you can write

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n$$, where $\{\Omega_n\}$ is a sequence of relatively compact open sets.

For example, $\Omega_n = \{x \in \Omega : d(x, \mathbb{R}^N \setminus \Omega) > \frac{1}{n}\} \cap B(0; n)$.

So, we are really looking at these sets and as n increase the ball becomes bigger and bigger. It will ultimately cover the whole of $\mathbb{R}^N$ and therefore you will be able to write x in this fashion.

So now by step I, $f|_{\Omega_n} \equiv 0$ a.e. $\Rightarrow f \equiv 0$ a.e. in Ω.

So, what does this proposition tell us?

Remark: So, in view of this proposition, if $f, g \in L^1_{loc}(\Omega)$ and if $T_f = T_g$, i.e.,

$$\int_\Omega f \phi \, dx = \int_\Omega g\phi \, dx \Rightarrow f - g = 0 \text{ a.e.}$$

i.e., $f = g$ a.e.

And when you are looking at measurable functions, equality almost everywhere is enough and therefore they are essentially the same function.

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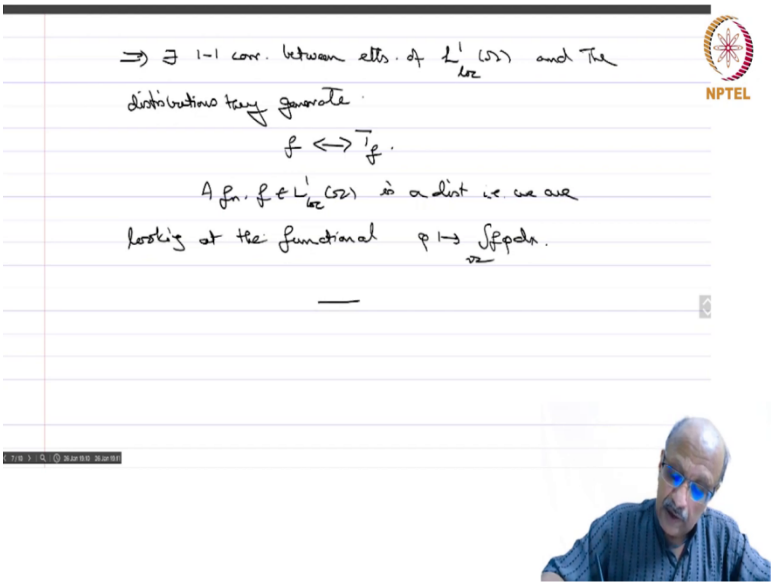

So, this implies that there exists 1-1 correspondence between elements of $L^1_{loc}(\Omega)$and the distributions they generate. So, from $f \rightarrow T_f$ is a bijection.

So, in future, I may, if necessary, I will say T_f to be absolutely clear if there is a reason for confusion. Otherwise, we will just say the distribution itself is f.

So, when I say a function is a distribution, I mean a function $f \in L^1_{loc}(\Omega)$ is distribution, means we are looking, that is we are looking at the functional

$$\phi \rightarrow \int_\Omega f \ \phi \, dx.$$

And this gives you a one to one correspondence between the functions. So, the function is well defined. So therefore, this is how we generalize the notion of a function, a locally integrable function can be thought of as a distribution. So, we will now continue other examples of distributions.