Sobolev Spaces and Partial Differential Equations

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Approximation by Smooth Functions

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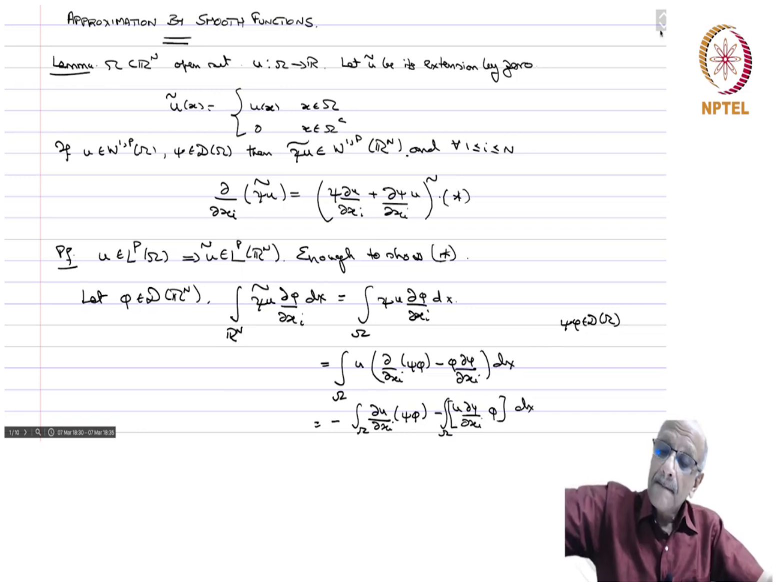

So, we will now study **Approximation by Smooth Functions**. So, we have already seen one result of this kind namely, if you are in $W^{1,p}(\mathbb{R}^N)$you can approximate it in that norm by being some functions which are in $\mathcal{D}(\mathbb{R}^N)$. So, we want to know to what extent this will carry out in other open sets. So, we first start with the technical

Lemma.

So, $\Omega \subset \mathbb{R}^N$ be an open set and ,$let\ u: \Omega \rightarrow \mathbb{R}^N$, then let $u^{\sim}$ be its extension by 0. So, what do you mean by that, so this is a notation we will use all the time. So,

$$u^{\sim} = u(x), \; x \in \Omega$$

$$= 0 \quad x \in \Omega^c$$

, so you just take the function blindly extended by 0. So, if

$$u \in W^{1,p}(\Omega), \; \psi \in D(\Omega) \text{then } (\psi u)^{\sim} \in W^{1,p}(\Omega)$$

So, you multiply ψu, so you multiplying by C infinity function with compact support, now you extend it by 0 this happens to be in $W^{1,p}(\Omega)$. And

$$\forall 1 \leq i \leq N \ \frac{\partial}{\partial x_i}(\psi u)^{\sim} = \left(\psi \frac{\partial u}{\partial x_i} + \frac{\partial \psi}{\partial x_i} u\right)^{\sim}$$

the usual product formula applied tilde. So, this is the formula. So

proof, so if $u \in L^{,p}(\Omega)$ this always implies that $u^{\sim} \in L^{,p}(\mathbb{R}^N)$.

In fact, the norm will be the same, in fact. So, the norm even the norm will not change. So, you have that this is, so in view of the statement, so if $(\psi u)^{\sim}$, $(\psi u)^{\sim} \in D(\Omega)andu \in W^{1,,p}(\Omega)$, so the product will be in the $L^{,p}(\Omega)$. So, its extension by 0 is an $L^{,p}(\mathbb{R}^N)$. So, if I can prove this formula for derivatives, since all this inside the bracket is still in $L^{,p}(\Omega)$, so the tilde will be in $L^{,p}(\mathbb{R}^N)$ and therefore, the derivatives are in $L^{,p}(\mathbb{R}^N)$ and the function is in $L^{,p}(\mathbb{R}^N)$ and that will prove that psi u tilde is in $W^{1,,p}(\mathbb{R}^N)$.

So, enough to show star. So, it is enough to prove the formula for the derivatives. So, that we straightaway calculate. So let $\varphi \in D(\mathbb{R}^N)$. So, what is integral over

$$\int_{\mathbb{R}^N} (\psi u)^{\sim} \frac{\partial \varphi}{\partial x_i} dx = \int_{\Omega} (\psi u) \ \frac{\partial \varphi}{\partial x_i} dx = \int_{\Omega} u\left(\frac{\partial(\psi\varphi)}{\partial x_i} - \varphi \frac{\partial \psi}{\partial x_i}\right) dx$$

$$= \int_{\Omega} \frac{\partial u}{\partial x_i}(\psi\varphi) - \int_{\Omega} u \frac{\partial \psi}{\partial x_i} \varphi dx$$

$$= \int_{\mathbb{R}^N} \left(\psi \frac{\partial u}{\partial x_i} + \frac{\partial \psi}{\partial x_i} u\right)^{\sim} \varphi dx$$

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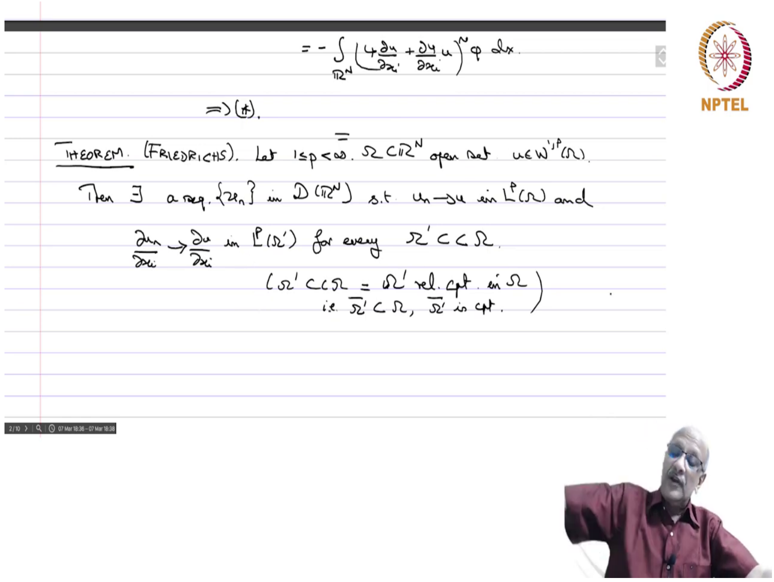

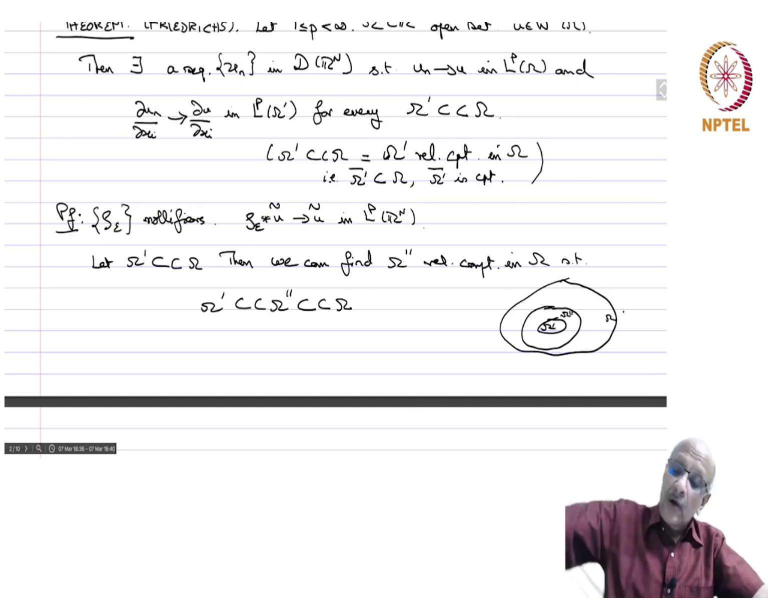

and that proves the fact therefore, the derivative So, this implies this implies (*), so that immediately you have, so this is just a very simple lemma. So, now we have the

Theorem: of Friedrichs. Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^N$ open set and $u \in W^{1,p}(\Omega)$ then there exists a sequence un in $D(\mathbb{R}^N)$ such that $u_n \to u$ in $L^p(\Omega)$and $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^p(\Omega')$ for every Ω' relatively compact in Ω'. So that means, so Ω' what does this mean, Ω' relatively compact in Ω that is the notation which means that is $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact. So, in the case $\Omega = \mathbb{R}^N$ we had $u_n \in D(\mathbb{R}^N)$ such that $u_n \to u$ in $D(\mathbb{R}^N)$ and $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^p(\mathbb{R}^N)$. Now, here we because we are in our open set in the domain and not in the entire set we have to pay a price for it and that is we lose the convergence of the derivatives in the entire domain, but we can do it in any relatively compact subset of it. So, that is the what this theorem is telling. So, let us try to prove this. So, as before we will always denote by $\sim$ the extension by the, so $\{\rho_\epsilon\}$ mollifiers and we know that $\rho_\epsilon * u^\sim \to u^\sim$ in $L^p(\mathbb{R}^N)$. So, we all know this. So, now let $\Omega^{'}$ be relatively compact in Ω remember this is the definition of relatively, relative compactness. So, then we can find $\Omega^{''}$ open relatively compact in Ω such that $\Omega^{'}$ is relatively compact in $\Omega^{''}$ which is relatively compact in Ω. So, let us assume, so you have Ω is here and this is $\Omega^{'}$. So, I can put one more relatively compact set $\Omega^{''}$ inside. So, that is what this question of $\mathbb{R}^N$ and its topology, you know we have T3, T4 spaces and all that this comes from those properties.

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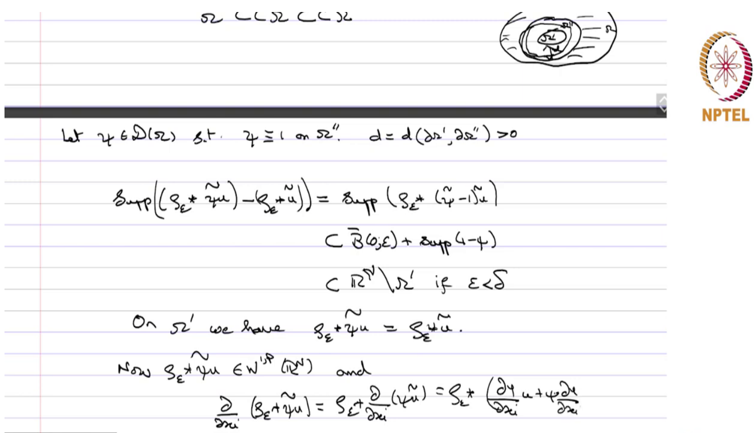

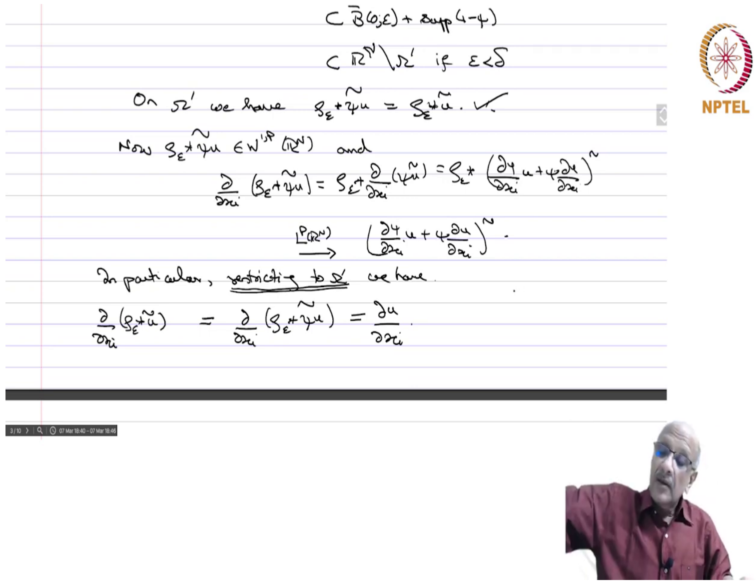

So, now let $\varphi \in D(\Omega)$ such that $\varphi \equiv 1$ on Ω'' and you take $d = d(\partial\Omega', \partial\Omega'') > 0$. So, it is the distance between these 2 boundaries, the smallest distance which u have between the sets there. So now, what about, so this will be strictly positive because you have 2 disjoint compact sets and consequently the distance is positive.

Now, support of

$$supp(\rho_\epsilon * (\psi u)^{\sim} - \rho_\epsilon * u^{\sim}) = supp(\rho_\epsilon * (\psi^{\sim} - 1)u^{\sim}) \quad \subset B(0;\epsilon) + supp(1 - \psi)$$

just I have taken out the common factor and now support of a convolution and that is the same as a support of $1 - \psi$ because $\psi^{\sim}$, $\psi$is a function which is C infinity with compact support in Ω and outside Ω, It only have extended by 0, so I have really not done anything to it.

So, the $1 - \psi$, now support a $1 - \psi$ is a, is somewhere here because on Ω'', $\psi = 1$ and therefore, the support of Ω''will be contained, support of $1 - \psi$ will be outside $1 - \Omega''$ and to that you are adding ϵ distance ball of radiusϵ. So, you will have to include a small portion here. So, this will be

$$\subset \mathbb{R}^N \setminus \Omega' \text{ if } \epsilon < d.$$

So, as long as they do not add something more than the distance between these 2 boundaries, so the small thin layer which I am adding is going to be well near the other boundary outer boundary and therefore, this will be contained in$\Omega^{'}$. So, support of these 2 functions lies in$\mathbb{R}^N \backslash \Omega^{'}$. So, on $\Omega^{'}$ we have $\rho_{\epsilon} * (\psi u)^{\sim} = \rho_{\epsilon} * u^{\sim}$ in $\Omega^{'}$. Now, $\rho_{\epsilon} * (\psi u)^{\sim} \in W^{1,p}(\mathbb{R}^N)$.

And what about its derivatives,

$$\frac{\partial(\rho_{\varepsilon} * (\psi u)^{\sim})}{\partial x_i} = \rho_{\epsilon} * \frac{\partial(\psi u)^{\sim}}{\partial x_i} = \rho_{\epsilon} * \left(\frac{\partial\psi}{\partial x_i} u + \psi \frac{\partial u}{\partial x_i}\right)^{\sim}$$

and this converges in $L^p(\mathbb{R}^N)$.

So, what will this converge to, this will converge to

$$\left(\frac{\partial\psi}{\partial x_i} u + \psi \frac{\partial u}{\partial x_i}\right)^{\sim}$$

So, in particular restricting to $\Omega^{'}$ we have $\frac{\partial(\rho_{\epsilon} * u^{\sim})}{\partial x_i} = \frac{\partial(\rho_{\epsilon} * (\psi u)^{\sim})}{\partial x_i} \rightarrow \frac{\partial u}{\partial x_i}$

. And if you use the properties of ψ what is psi is identically 1 in $\Omega^{''}$, so in particular$\Omega^{'}$ so $\frac{\partial\psi}{\partial x_i} \equiv 0$ and $\psi \equiv 1$and that is just equal to $\frac{\partial u}{\partial x_i}$.

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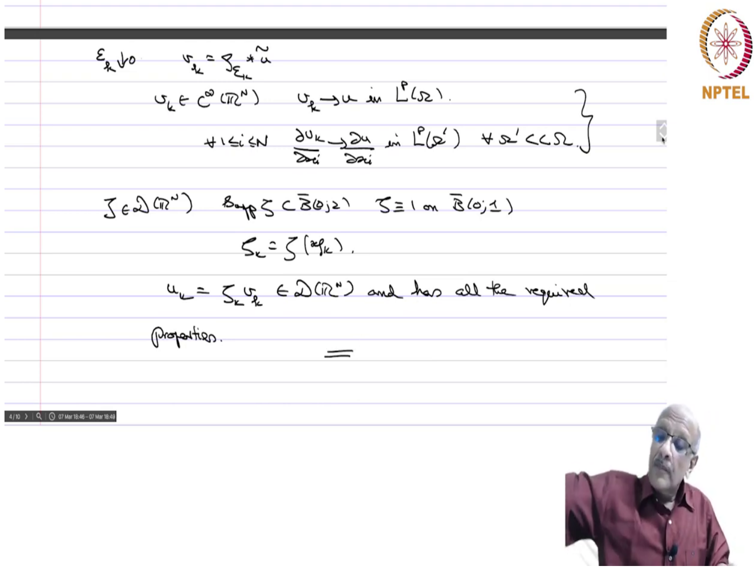

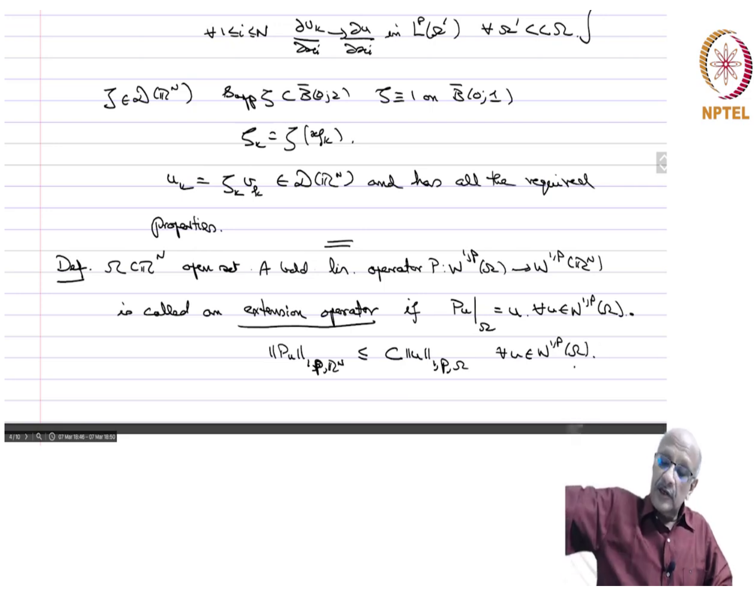

So, now you choose ϵ_k to be a sequence of real numbers going to 0 and then you take $v_k = \rho_{\epsilon} * \tilde{u}$, then what, what do you know, we know that v_k belongs to C infinity of $\mathbb{R}^N$, $v_k \to u$ in $L^p(\Omega)$ and for all $1 \leq i \leq N$, $\frac{\partial v_k}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^p(\Omega')$ for all Ω' relatively compact in Ω. So now, we have got C infinity functions, from C infinity functions to go to C functions with compact support that is exactly the theorem which we have done already once, so we do not have to repeat the technique.

So, all we have to do is let $\varsigma \in D(\mathbb{R}^N)$, $supp(\varsigma) \subset B(0;2)$, $\varsigma \equiv 1 \; on \; B(0;1)$ and you take $\varsigma_k = \varsigma(\frac{x}{k})$ exactly as before and then you take $u_k = v_k \varsigma_k \in D(\mathbb{R}^N)$, and has all the required properties this just a repetition of the previous theorem's proof because once you have proved something which, which is this then multiplication by ς_k will produce exactly the same properties and therefore, you have nothing else to do.

So, this, so we can approximate u by $D(\mathbb{R}^N)$ function in L^p, but we want to approximate the derivatives then you have to pay a little price you can only do in relatively compact subsets. So, the question you ask is can I ever do in the whole space, can I, without losing on Ω and that you can do provided you have what is called an extension operator. So, we have

Definition, so $\Omega \subset \mathbb{R}^N$, open set a bounded linear operator

$$P: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

is called an extension operator. I already mentioned this yesterday, so

$$Pu|_{\Omega} = u, \;\; \forall u \in W^{1,p}(\mathbb{R}^N).$$

So, this is the notion of extension operator. So, you note that you have

$$||Pu||_{1,p,\mathbb{R}^N} \leq C||u||_{1,p,\Omega}, \quad \forall u \in W^{1,p}(\Omega),$$

this is the condition that it is a bounded linear operator.

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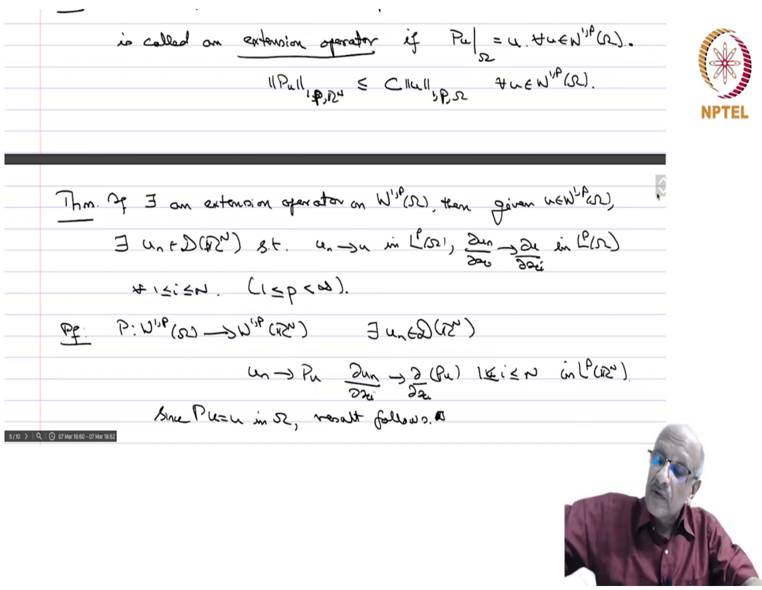

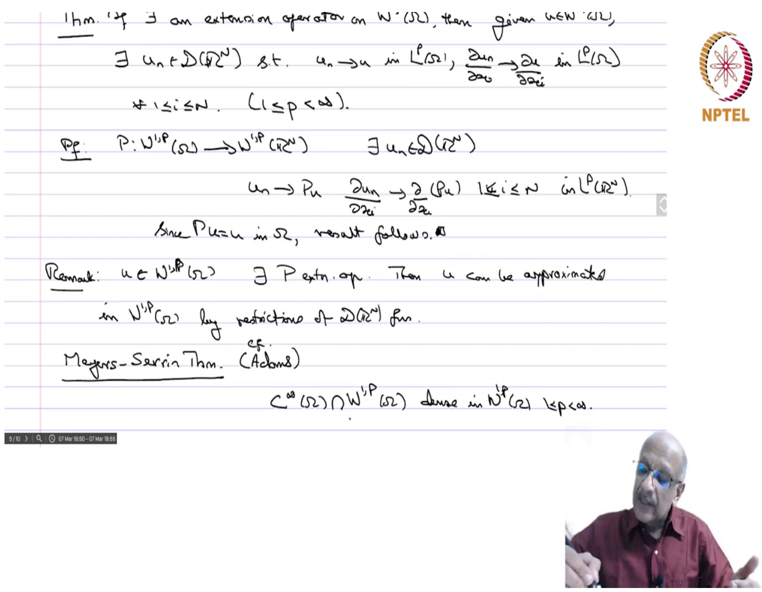

So now, the

Theorem: says if there exists an extension operator on $W^{1,p}(\Omega)$ then given $u \in W^{1,p}(\Omega)$ there exists $u_n \in D(\mathbb{R}^N)$, such that $u_n \to u,\ in\ L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x}\ \ in\ L^p(\Omega)$. Now, you do not have to worry about relatively compact sets for all $1 \leq i \leq N$.

So again. all these results is for $1 \leq i < \infty$. So, proof is just 1 line. So, if P is the extension operator, then there exists $u_n \in D(\mathbb{R}^N)$ such that $u_n \to Pu$, and $1 \leq i \leq N\ in\ L^p(\mathbb{R}^N)$. So, now you just take un so that, so this un itself will work, therefore, so since $Pu = u,\ \ in\, \Omega,$ result follows. So, the question is when does it, when do we have extension operators we will see that that it depends on the nature of the set Ω in particular on the nature of its boundary and therefore, how we can extend depends on the domain and we will see results which talk about that.

Now, what is, what does this theorem mean, it means that if $u \in W^{1,p}(\Omega)$ So, remark, this is small p and there exists capital P extension operator then u can be approximated in $W^{1,p}(\Omega)$by restrictions of $D(\mathbb{R}^N)$ functions which are C infinity functions in particular in omega. So, we have a better a more difficult theorem called the Meyers Serrin theorem find a proof in Adams the book which I cited, CF Adams, Mayers Serrin theorem says $C^{\infty}(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ 1 less than equal to P less than infinity.

So, if you have a C infinity function which is dense in which is also a $W^{1,p}(\Omega)$, see because if Ω is unbounded you cannot really say that C infinity function is in L^p, so u have to put this intersection, so it should be a C infinity function and it should also be such it is in $W^{1,p}(\Omega)$such functions may or may not be restrictions of $D(\mathbb{R}^N)$ functions, so that it says that if you have such C infinity functions which are in L^p they are also dense, so you can approximate it by those functions.

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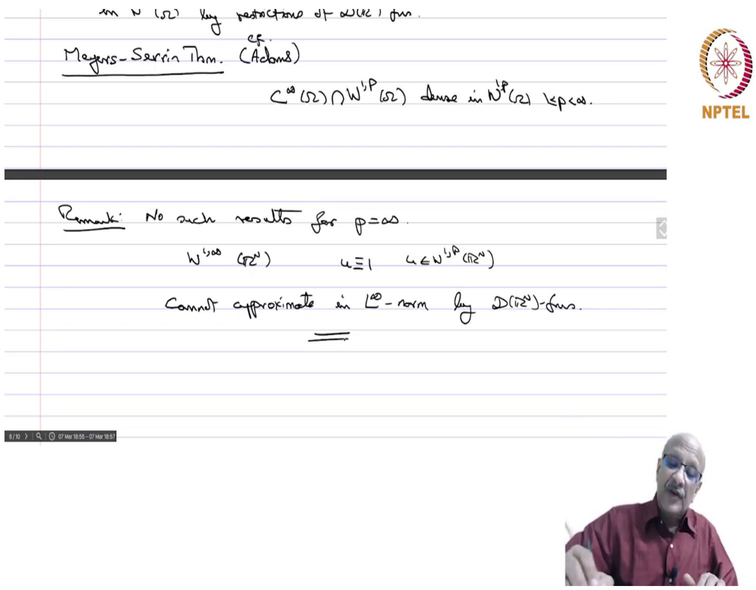

Now, final remark. No such results for $p = \infty$. You do not have any approximations. For instance, if you take $W^{1,\infty}(\mathbb{R}^N)$ you have $u \equiv 1,\ in\, W^{1,p}(\mathbb{R}^N)$. Now, you cannot, cannot approximate in L infinity norm by $D(\mathbb{R}^N)$. Because if you have functions with compact support C infinity functions with compact support are the in particular continuous functions with compact support and the limit in L infinity norm would give you a function which vanishes at infinity.

Therefore, u identically 1 does not vanish at infinity it is 1 everywhere and therefore, you can never approximate this function by means of $D(\mathbb{R}^N)$. So, you cannot expect any of the results which we have proved for infinity, so we have to exclude it. So, now we will look at some applications of Friedrichs theorem, which are very interesting.