Sobolev Spaces and Partial Differential Equations

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Lecture 28

Sobolev Spaces – Part 3

When we talk of a L^p function we really mean that we are talking of a representative from an equivalence class, throughout any computation we will of course work with the same representative and there will be no discrepancy. So, we generally talk of a function not of an equivalence class, though L^p elements are really equivalence classes.

So, in the spirit when you say an L^p function is continuous that means in that equivalence class there is a representative which is continuous. So, if you take an arbitrary element, it will be equal almost everywhere to a continuous function in that equivalence class. So, this is in that spirit we have the following theorem.

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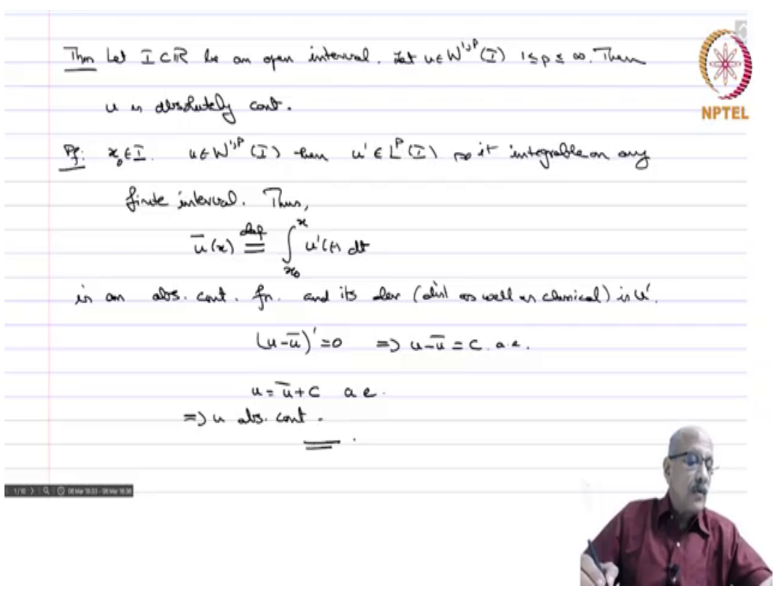

Theorem:

So, let $I \subset \mathbb{R}$ be an open interval. Let $u \in W^{1,p}(I)$, $1 \leq p \leq \infty$, then u is absolutely continuous.

Proof, so let $x_0 \in I$ so you choose, a point reference point so if $u \in W^{1,p}(I)$ then $u' \in L^p(I)$ so it is integrable on any finite interval, because L^p are locally integrable therefore its integral on any finite interval.

So, thus

$$\overline{u}(x) = \int_{x_0}^{x} u'(t)\, dt$$

which is defined in the following way so this is the definition is an absolutely continuous function, in fact this is one of the definitions of absolute continuous functions, you can have an epsilon delta version also but an absolutely continuous function is one which can be written as an indefinite integral of an integrable function, and in fact that integrand will be the derivative almost everywhere for the given function, absolutely continuous functions are differentiable almost everywhere in the classical sense, and we also saw before that for them the distribution and the classical derivatives coincide, so we have already seen that.

So, it is absolutely continuous function and its derivative distribution as well as classical is u dash. So,

$$(u - \overline{u})' = 0 \Rightarrow u - \overline{u} = C$$

almost everywhere, of course almost everywhere. And therefore, this implies that u is absolutely continuous.

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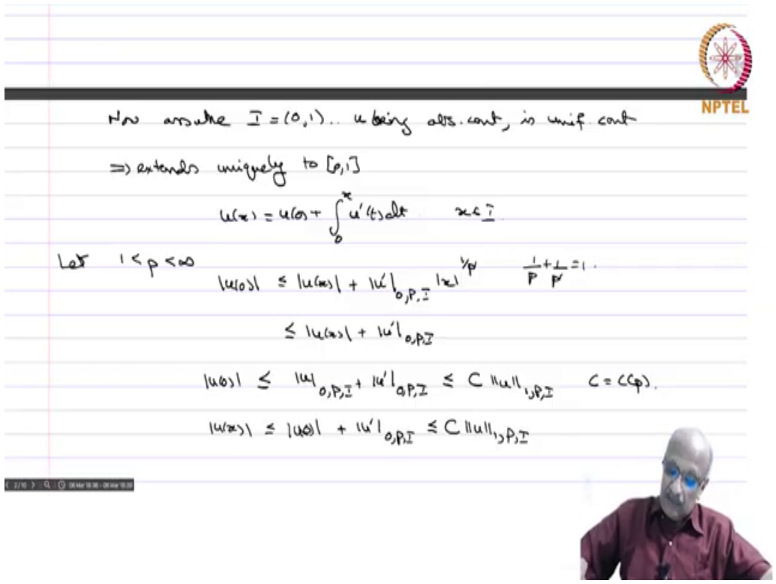


So, now assume that I is a bounded interval say I=(a, b) or rather let us say without loss of centrality, some bounded intervals so we will take the prototype of that I=(0, 1). So, u is, so u extent being absolutely continuous is uniformly continuous and this implies extends uniquely to the closed interval [0, 1].

And you can write

$$u(x) = u(0) + \int_0^x u'(t)\, dt, \quad x \in I$$

So, let you, let us assume that $1 < p < \infty$. So,

$$|u(0)| \le |u(x)| + |u'|_{o,p,I} |x|^{1/p'}$$

and I want to estimate this integral I am going to use Holder's inequality. So,

$$\frac{1}{p} + \frac{1}{p'} = 1$$

So, this is the Holder's inequality.

And applying the triangle inequality to the two functions namely

u(x) is one function, and u, this is a constant function and so of course this can be

$$|u(0)| \leq |u(x)| + |u'|_{o,p,I}$$

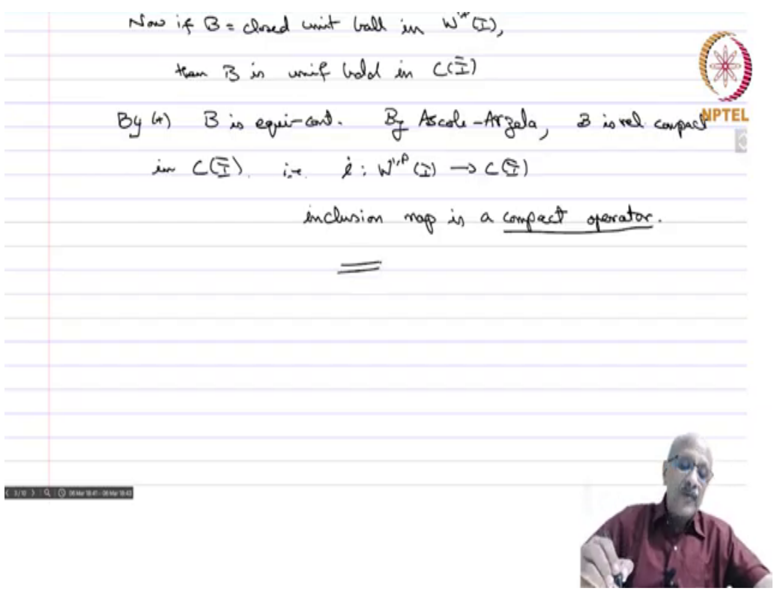
. So, now we apply it to a constant function and this and to this constant function, so if you get, you will get by the triangle inequality, the L^p norm of constant function is the constant itself less than or equal to you have

$$|u(0)| \leq |u|_{0,p,I} + |u'|_{o,p,I} \leq C||u||_{1,p,I}$$

I which is defined in terms of the Pth powers of this but as I told you the sum of the norms is an equivalent norm and therefore you can write it in terms of this. So, C depends only on p. Again, you have

$$|u(x)| \leq |u(0)| + |u'|_{o,p,I} \leq C||u||_{1,p,I}$$

this C may be whenever I write C in various inequalities they are not necessarily the same real



number but some generic constant that is all we mean. So, its independent of the variables.

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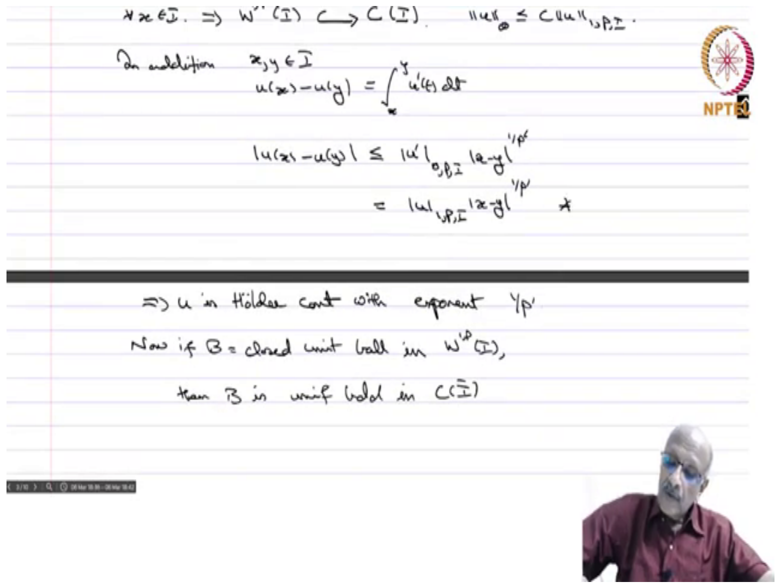

So, for all $x \in \bar{I}$. So, we have proved, we have that so this implies the

$W^{1,p}(I)$is continuously embedded in $C(\bar{I})$, namely every function is absolutely continuous, therefore continuous. And the norm in here namely the

$$||u||_{\infty} \leq C||u||_{1,p,I}$$

. This what we have shown because for u(0) and for every other u(x) we have shown this. So, you have this.

In addition, we have $x, y \in I$

$$u(x) - u(y) = \int_{x}^{y} u'(t)\,dt$$

and again by Halder inequality you have

$$|u(x) - u(y)| \leq |u'|_{0,p,I}|x - y|^{1/p'} = |u|_{1,p,I}|x - y|^{1/p'}$$

dash because the L^p norm of the first derivative is nothing but the mod 1 norm. So, this means that u is Halder continuous with exponent $\frac{1}{p'}$.

Now, if B= closed unit ball in $W^{1,p}(I)$ then B is uniformly bounded in $C(\bar{I})$ so call this star. By star B is equi-continuous. So, by Ascoli-Arzala, the image of B is relatively compact in $C(\bar{I})$ that is the inclusion map from $W^{1,p}(I) \to C(\bar{I})$ inclusion map is a compact operator. That means it takes a bounded set to a relatively compact set that is a definition of a compact operator and that is very important topic in functional analysis. So, these are the properties in case of $W^{1,p}(I)$ and which we will serve to us to as a guide to what to study in this chapter, so following is the road map of what we are going to do.

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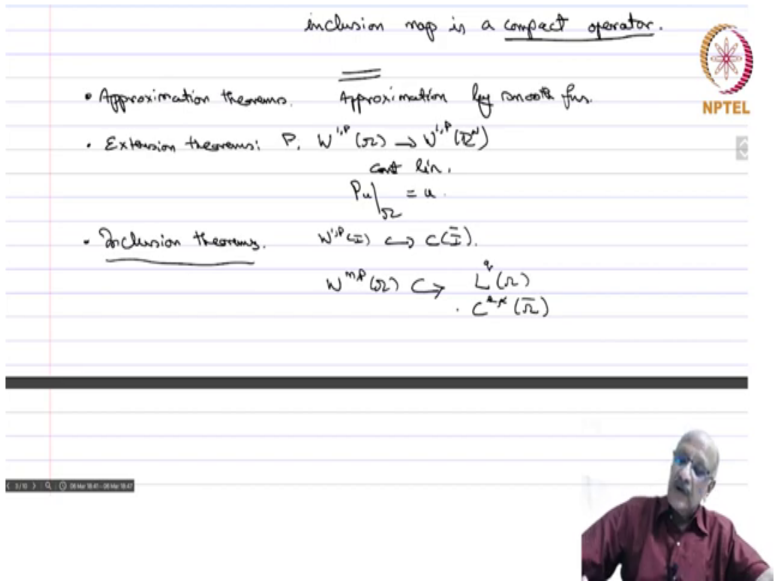

So, first we will study **approximation theorems.** Many results are easy to prove for smooth functions using calculus techniques and then complete it with a density argument for less smooth functions. Therefore, we would like to study when we can approximate $W^{m,p}(I)$, $W^{1,p}(I)$ $W^{m,p}(\Omega)$; $W^{1,p}(\Omega)$ omega by smooth functions so approximation by smooth functions.

Then two, extension theorems. Many results are easy to prove in $\mathbb{R}^N$ where you have no boundaries so you have plenty of elbow room, you can do what you like just as we saw $W_0^{1,p}(\mathbb{R}^N)$ is same as $W^{1,p}(\mathbb{R}^N)$, we just use convolution and we could use cut off functions and so on. So, convolution is a very important approximation technique and especially if you want approximation by smooth functions and for that you need to work in $\mathbb{R}^N$.

So, if you want to prove a result in omega we use one method is to extend the function to and then prove the result in $\mathbb{R}^N$ and try to restrict it to omega, so this is the standard method and so for this you need that the extension which you give is continuous in the sense that

$$P: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

we want extension operator P which is continuous linear and ,

$$Pu|_{\Omega} = u.$$

So, then this is called an extension operator and such operators are useful because as I said we can prove results in $\mathbb{R}^N$ and then we try to prove this.

Then inclusion theorems, so we saw that $W^{1,p}(I)$ is continuously embedded in $C(\bar{I})$. So, we would like to know in general if we have some $W^{1,p}(\Omega)$ is it included in some other well-known space, it may not be always spaces of continuous functions for that you may have to go to very high order Sobolev spaces $W^{m,p}(\Omega)$where $m$is very large but you may be able to do so in other Lebasque spaces better integrability properties and so on, so that will be the idea.

So, $W^{m,p}(\Omega)$ will become included in some either $L^q(\Omega)$ for some q or it will be in C^{α}, $C^{\alpha,k}(\overline{\Omega})$, which means differentiable k times and Halder continuous exponent is alpha and so on, so such kind of results we will try to prove in the inclusion theorem.

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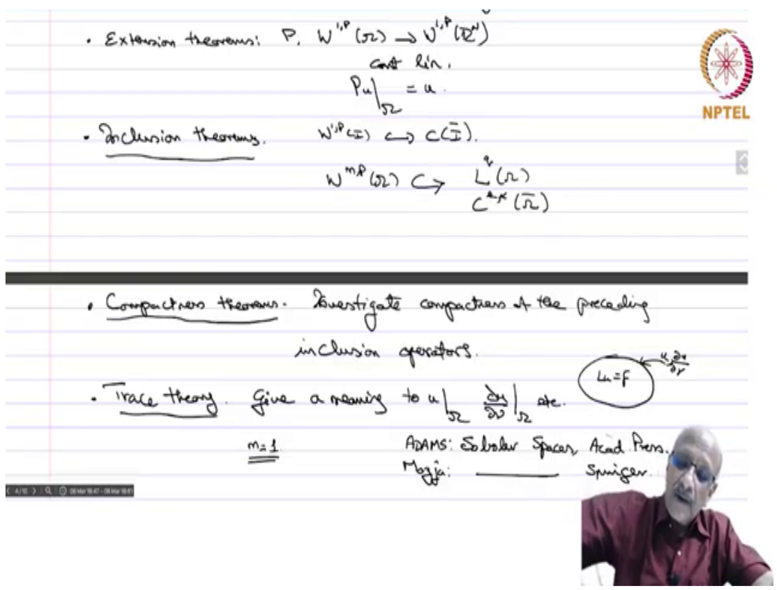

Then compactness theorems. So, we saw that this inclusion $W^{1,p}(I) \to C(\bar{I})$ was compact. So, we have investigate compactness of the preceding inclusion operators. And compactness is very important because once you have compactness then you have sequences with convergent subsequences and so on and therefore especially in the study of PDEs non-linear PDEs and eigen value problems this will be a very useful idea to have.

So, find the trace theory. So, I said that the Sobolev spaces form a natural functional analytic framework to look at solutions of partial differential equations. Now, most PDEs come as boundary value problems. So, if you have Ω bounded domain you will have certain $Lu = F$ and u or $\frac{du}{dv}$ such thing will be prescribed on the boundary. So, we want to know.

Now, if in the case of $C(\bar{I})$ then the function extends naturally to the endpoints and so the value of u at the end point is well defined. But if you have an L^p function in general because the measure of the boundary is boundary is 0, the boundaries of measure 0 and therefore and L^p functions are only defined almost everywhere, so it is not realistic, it is observed to talk of the value of u on the boundary for a L^p function, so it is not possible.

But on the other hand, we are making, we have to make use of the fact that we just do not have any arbitrary L^p function, we have L^p functions whose distribution derivatives up to some order are alsoL^p functions, so we have to exploit that extra knowledge and then somehow give a meaning to $u|_{\Omega}$ or$\frac{du}{dv}|_{\Omega}$, the external normal derivative restricted to Ω, etc and that is called trace theory.

So, we will mostly deal with m=1 because the exposition is simple and we will present the results in the simplest of cases but a very comprehensive reference for this is ADAMS Sobolev Spaces Academic Press, but of course I must warn you or Mozja also Sobolev Spaces, I think this is Springer, these are very difficult books to read but anyway these are where you will find some like an encyclopedia all kinds of results connected to all these spaces.

But we will give a fairly self-contained and simple treatment which will suffice for most of the applications. So, that is what we plan to do. So, this is the roadmap we are going to follow in this chapter and we will execute them one by one.