Sobolev Spaces and Partial Differential Equations
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Lecture 27
Sobolev Spaces – Part 2

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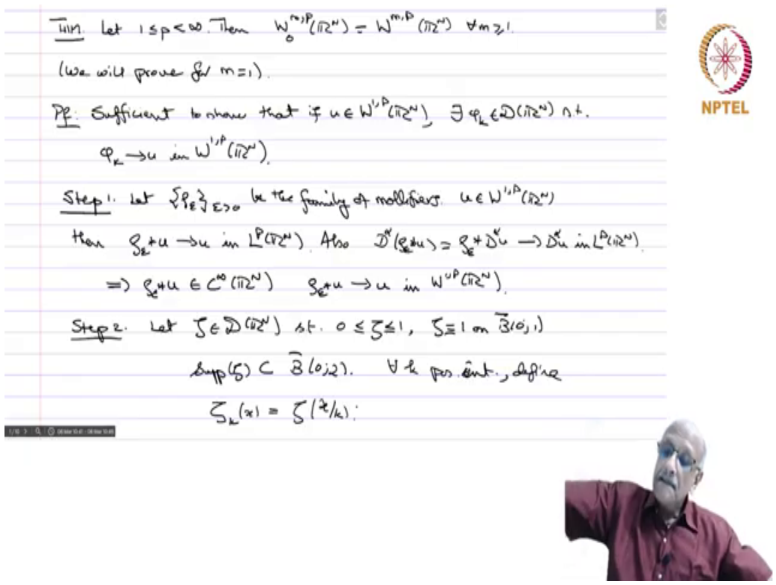

Theorem:

We now have the following theorem, let $1 \leq p < \infty$, then

$W_0^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$ for all $m \geq 1$. So, we will prove for $m = 1$, the rest will all follow by iteration and therefore you it is enough to prove for m=1, you can see from the proof. So, we will prove this somewhat carefully, so that this technique will be used frequently in the sequel and therefore it is good to know it…

proof, sufficient to show that if $u \in W^{1,p}(\mathbb{R}^N)$, $\exists \varphi_k \in D(\mathbb{R}^N)$ $such\ that$ $\varphi_k \to u\ in\ W^{1,p}(\mathbb{R}^N)$ this is the idea because we are going to show what is $W_0^{1,p}(\mathbb{R}^N)$ is closure of $D(\mathbb{R}^N)$ and therefore it is enough if you can show that every element in $W^{1,p}(\mathbb{R}^N)$ can be approximated by this, so that will show that $W^{1,p}(\mathbb{R}^N)$ is contained in $W_0^{1,p}$ $(\mathbb{R}^N)$ the reverse inclusion is always there and therefore we would have proved the equality. So, this proves that these two spaces are not distinct in $\mathbb{R}^N$, later on we will see about other spaces.

So, the **first step**, so let $\{\rho_\varepsilon\}_{\varepsilon>0}$, be the family of mollifiers so if $u \in W^{m,p}(\mathbb{R}^N)$ then we know

$$\rho_\varepsilon * u \to u \, in \, L^p(\mathbb{R}^N). \; Also \;\; D^\alpha(\rho_\varepsilon * u) = \rho_\varepsilon * D^\alpha u \to D^\alpha u \, in \, L^p(\mathbb{R}^N)$$

$$\Rightarrow \rho_\varepsilon * u \in C^\infty, \, \rho_\varepsilon * u \to u \;\; in \, W^{1,p}(\mathbb{R}^N).$$

namely the function and its derivatives they all converge to the correct one.

So, this is, but we have got a C^∞ function what we need is a C^∞ function with compact support, so we need to use the cut off technique. So,

Step 2:

Let $\varsigma \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \le \varsigma \le 1, \; \varsigma \equiv 1 \;\; on \;\; \overline{B(0;1)}$

$$supp\;(\varsigma) \subset B(0;1).$$

So, now for every k>0, positive integer define $\varsigma_k(x) = \varsigma(\frac{x}{k})$

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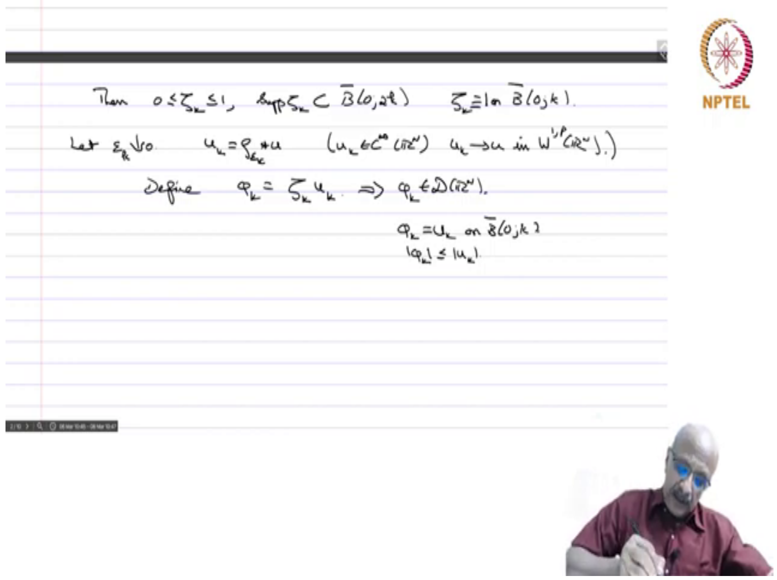

Then $0 \leq \varsigma_k \leq 1$, $supp(\varsigma_k) \subset \overline{B(0; 2k)}$, $\varsigma_k \equiv 1 \; in \; \overline{B(0; k)}$.

So, now let $\varepsilon_k \downarrow 0$, some sequence of positive numbers so let us if, so let us take $u_k = \rho_{\varepsilon_k} * u$, we wanted a sequence we add ρ_ε.

So, recall that $u_k \in C^\infty(\mathbb{R}^N)$ and $u_k \to u \; in \; W^{1,p}(\mathbb{R}^N)$. So, we have, remember that from step 1.

Now, define $\varphi_k = \varsigma_k u_k \Rightarrow \varphi_k \in \mathcal{D}(\mathbb{R}^N)$. Since ς_k is C^∞function with compact support.

$\varphi_k = u_k$ on $B(0; k)$ and so $|\varphi_k| \leq |u_k|$.

So, now our aim of course is to show that φ_k is also going to converge to u in $W^{1,p}(\mathbb{R}^N)$ and that will complete the proof. So, let us first look at the L^p convergence.

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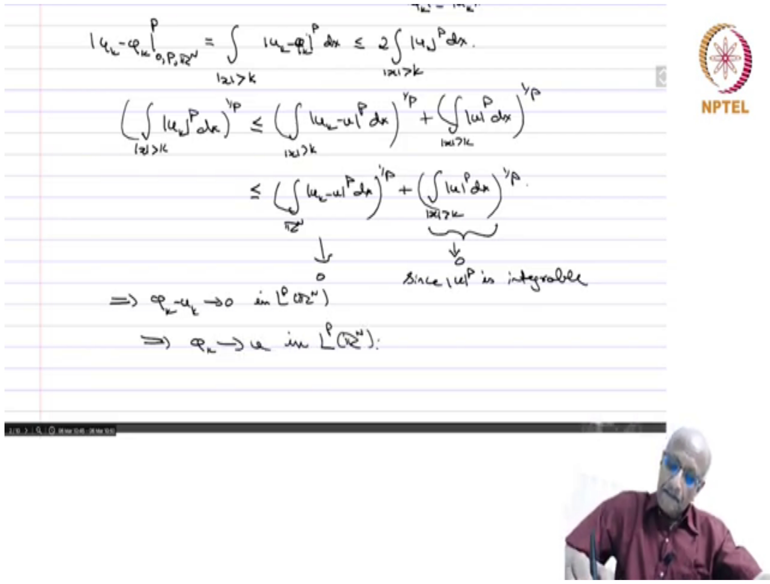

So,

$$|u_k - \varphi_k|^p_{0,p,\mathbb{R}^N} = \int_{|x|>k} |u_k - \varphi_k|^p \, dx \leq 2^p \int_{|x|>k} |u_k|^p \, dx.$$

so now I will use my notation 0, p, $\mathbb{R}^N$ that means the L^p norm in $\mathbb{R}^N$ whole power p

unfortunately both this and the ball both are dependent on k and I want to get rid of that. So, we now have apply the Makowski inequality on $|x| > k$.

$$\left(\int_{|x|>k} |u_k|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{|x|>k} |u_k - u|^p dx \right)^{\frac{1}{p}} + \left(\int_{|x|>k} |u|^p dx \right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbb{R}^N} |u_k - u|^p dx \right)^{\frac{1}{p}} + \left(\int_{|x|>k} |u|^p dx \right)^{\frac{1}{p}}$$

Now, we saw that u_k converges to u in $W^{1,p}(\mathbb{R}^N)$ in particular $L^p(\mathbb{R}^N)$, so this term goes to 0 and this term also goes to 0 since u^p is integrable and this is the tail of a convergence integral and therefore it has to go to 0 as $k \to \infty$. So, therefore these things go to 0 and so you have that

$$\varphi_k - u_k \to 0 \text{ in } L^p(\mathbb{R}^N)$$

$$\varphi_k \to u \text{ in } L^p(\mathbb{R}^N)$$

So, we have got at least a C infinity function with compact support which goes in $L^p(\mathbb{R}^N)$.

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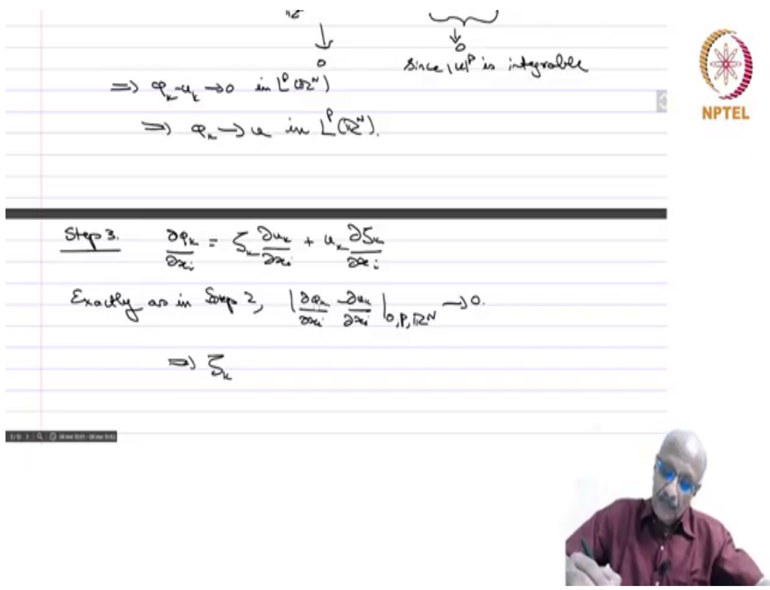

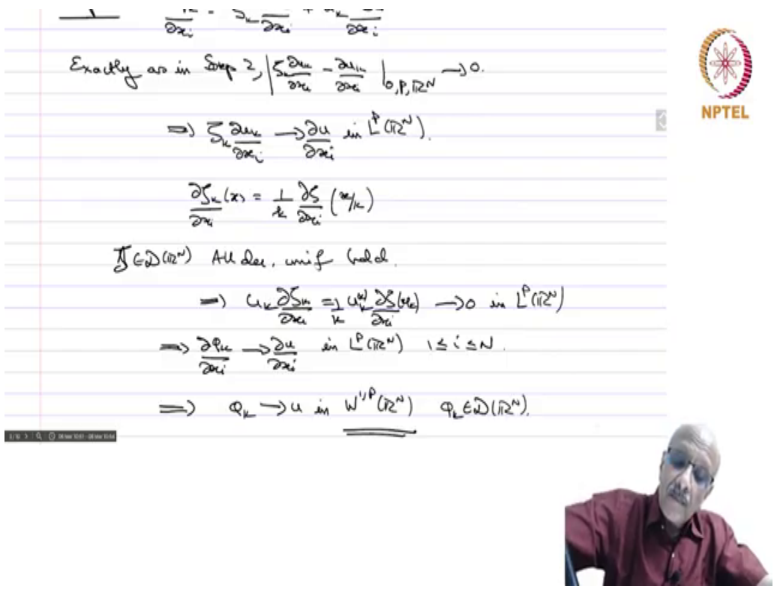

So, now for the derivatives so

step 3, so now let us look at

$$\frac{\partial \varphi_k}{\partial x_i} = \varsigma_k \frac{\partial u_k}{\partial x_i} + u_k \frac{\partial \varsigma_k}{\partial x_i}, 1 \leq i \leq N$$

this is the usual product rule. Now, exactly as in step 2, we have that

$$\left| \varsigma_k \frac{\partial u_k}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \right|_{0,p,\mathbb{R}^N} \to 0$$

$$\Rightarrow \varsigma_k \frac{\partial u_k}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ in } L^p(\mathbb{R}^N).$$

this is exactly as in step 2 and therefore there is no need to elaborate it further. So, we only have to show the remaining term goes to 0.

$$\frac{\partial \varsigma_k}{\partial x_i} = \frac{1}{k} \frac{\partial \varsigma}{\partial x_i}\left(\frac{x}{k}\right).$$

Now, all so zeta is in $\mathcal{D}(\mathbb{R}^N)$, so all derivatives uniformly bounded and therefore this implies

$$\Rightarrow u_k \frac{\partial \varsigma_k}{\partial x_i} = \frac{1}{k} u_k \frac{\partial \varsigma}{\partial x_i}\left(\frac{x}{k}\right) \to 0 \text{ in } L^p(\mathbb{R}^N).$$

because this is a bounded function u_k goes, u_k is a convergent function and $\frac{1}{k}$ is there for you to send everything to 0.

$$\Rightarrow \frac{\partial \varphi_k}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ in } L^p(\mathbb{R}^N) \quad 1 \le i \le N.$$

Therefore, you have this and therefore consequently you have the $1 \le i \le N$.

$$\Rightarrow \varphi_k \to u \text{ in } W^{1,p}(\mathbb{R}^N) \; and \; \varphi_k \in \mathcal{D}(\mathbb{R}^N)$$

and that completes the proof of this theorem. So, this is a very useful technique which we will use several times convolution with mollifiers to produce a C^∞ function which approximates a given function and then cutting, multiplying with the cutoff function to produce something with C^∞ with compact support, so this is a very useful technique and we have to estimate, so we have carefully estimated the integrals this time. So, we will use this later.