Sobolev Spaces and Partial Differential Equations
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Lecture 26
Sobolev Spaces – Part 1

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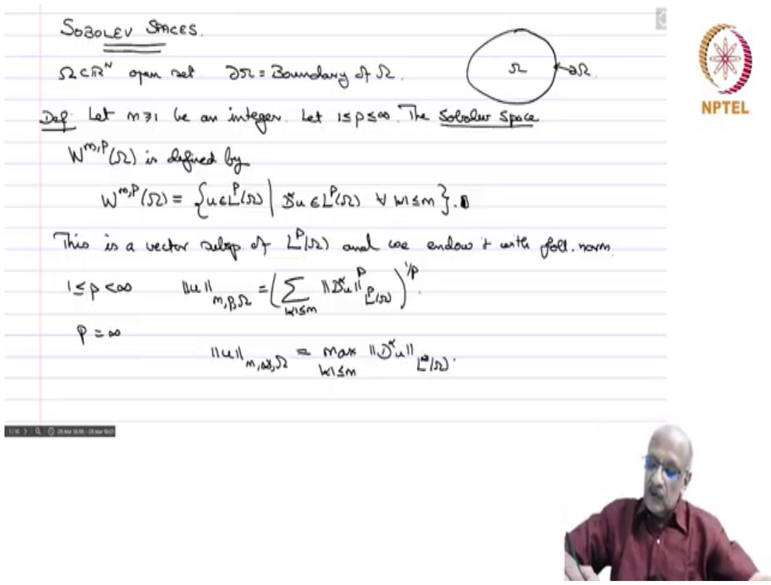

Sobolev Space:

We will now move to the core topic of this course, so we will discuss Sobolev Spaces, these are subspaces of the L^p spaces and they form a natural setting, a functional analytic setting for the study of partial differential equations of many kinds. So, in all that follows $\Omega \subset \mathbb{R}^N$ will be an open set, and then $\partial\Omega =$ boundary of omega.

So, if you have for instance a bounded open set, so this will be Ω and this will be $\partial\Omega$. So, definition let $m \geq 1$ be an integer.

Let $1 \leq p \leq \infty$ The sobolev space

$$W^{m,p}(\Omega) = \left\{u \in L^p \mid \ D^\alpha u \in L^p(\Omega),\ \forall|\alpha| \leq m\right\}$$

so the two indices here m is for the order, p is for the exponent of the Lebasque as we will see in a moment is defined by $\forall|\alpha| \leq m$, so these are the multi indices as you know.

So, M is the order of up to the which you are considering the derivatives. So, u is an $L^p(\Omega)$ of omega function so it is a distribution and it has distributions of all derivatives of all orders. So, we want that the distribution derivative may or may not be $L^p(\Omega)$ function so you want to show that they are all $L^p(\Omega)$ functions or if, I mean if they are all $L^p(\Omega)$ functions up to order M, then you say the space is $W^{m,p}(\Omega)$.

So, this is a vector space, so this is the definition of the space. So, this is a vector space, vector subspace of $L^p(\Omega)$ of omega and we endo it with the following norm.

So, if $1 \leq p < \infty$, then you say norm

$$||u||_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} ||D^\alpha u||_{L^p(\Omega)}{}^{p} \right)^{\frac{1}{p}}$$

Now, you could define it in other ways also like for instance you could just define it as some of the $L^P(\Omega)$ norms but they are all equivalent and we will see that this is a more convenient way to have it in. So, this is especially in a when P=2 this is a good way to write it rather than just a sum.

But otherwise you could write since given is a certain number of norms, norm linear spaces when you want to associate with it jointly a norm then this is you could do it in myriad base, but this is the one which you are going to choose. And if

$$p = \infty \quad ||u||_{m,p,\Omega} = max_{|\alpha| \leq m} ||D^\alpha u||_{L^P(\Omega)}$$

Now, this max makes sense because we are only taking the maximum over finite number of components. So, this is the space and this is the norm.

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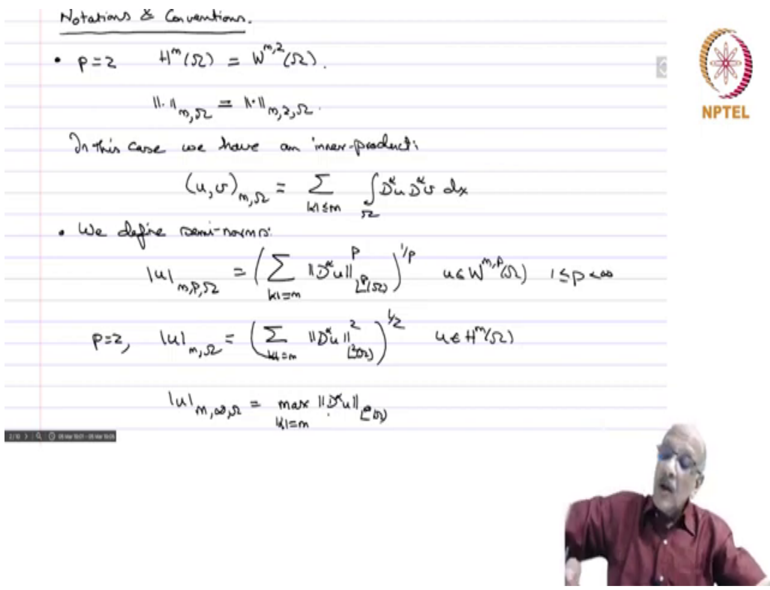

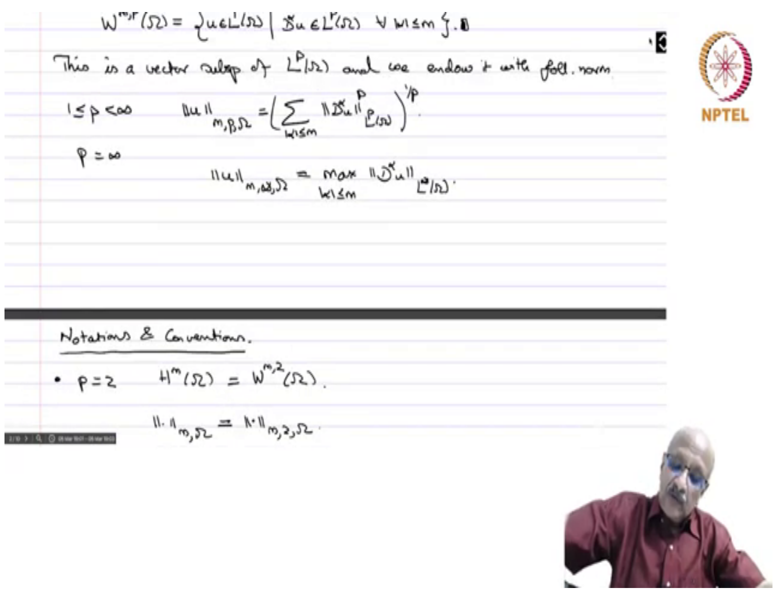

So, now we have some

notations and conventions. So, the first one if $p = 2$, $H^m(\Omega) = W^{2,m}(\Omega)$

p=2, so then it is somewhat special as we will see.

$$|| \cdot ||_{m,\Omega} = || \cdot ||_{m,2,\Omega}.$$

And the corresponding norm we will say norm m, Ω will be stand for norm $m, 2, \Omega$ so the norm $\Omega, m, 2$ so this is just a slightly shorter notation so we drop the index p, the parameter p=2 so if this only two indices appear here then it is just the order and the domain because we know we are in L^2, so this is the first one.

$$< u,\ v>_{m,\Omega} = \sum_{|\alpha|\le m} \int_{\Omega} D^{\alpha}u D^{\alpha}v\ dx$$

$$|u|_{m,p,\Omega} = \left(\sum_{|\alpha|=m} \int_{\Omega} ||D^{\alpha}u||^{p}_{L^{P}(\Omega)} \right)^{1/p},\ u \in W^{m,p}(\Omega),\ 1 \le p < \infty.$$

$$p = 2\ |u|_{m,\Omega} = \left(\sum_{|\alpha|=m} \int_{\Omega} ||D^{\alpha}u||^{2}_{L^{2}(\Omega)} \right)^{1/2}\quad u \in H^{m}(\Omega).$$

H is for Hilbert and therefore we will see that in a moment.

So, both are L^2 functions so you can integrate and then that gives you an inner product and we are assuming without loss of generality that we are all real valued, so the we are working in over ℝ for this. So, this gives you the norm, norm u square will be precisely the norm which we defined for P=2,

$$|u|_{m,\infty,\Omega} = max_{|\alpha|=m}\ ||D^{\alpha}u||_{L^{\infty}(\Omega)}\delta$$

So, why are these semi norms they have otherwise all the properties of the norm except when this quantity is 0, it does not mean that the distribution is 0 because if the first derivative is 0 for instance then you know it is only a constant and therefore you do not get it that it is 0, so these are only semi norms.

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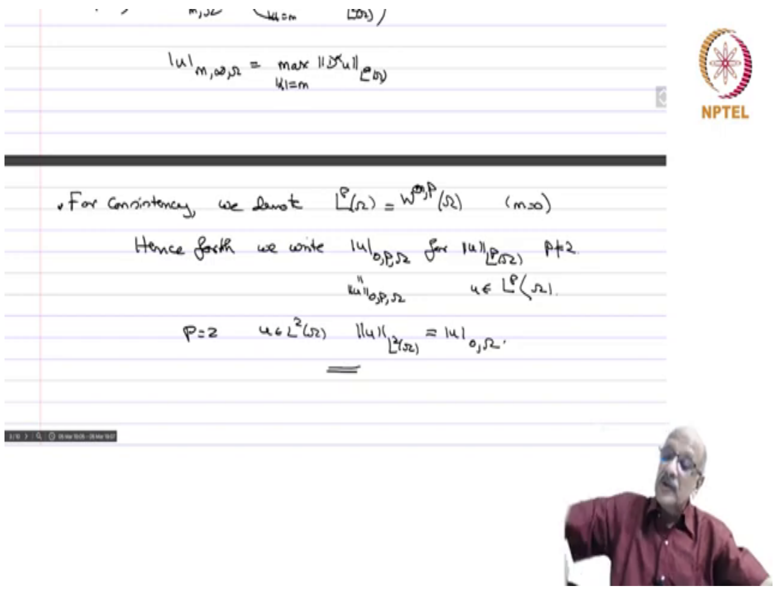

Then for consistency we denote $L^p(\Omega) = W^{0,p}(\Omega)$, so if m, that is m=0, we have defined $m \geq 1$ in the Sobolev space, now we want, we will see later why this is consistent so the $L^p(\Omega)$ is just the case when m=0 that means no derivative is used, so that means just a function.

And if you look at the definition of the norms and so on that is precisely corresponding to

$\alpha = 0$ and so $|\alpha| = 0$and therefore $L^p(\Omega)$. So, hence fourth, we write

$$|u|_{o,p,\Omega} \quad \text{for } ||u||_{L^p(\Omega)}, \quad p \neq 2$$

$$||u||_{L^p(\Omega)} = |u|_{o,p,\Omega}$$

$$p = 2, \ u \in L^2(\Omega), \ ||u||_{L^2(\Omega)} = |u|_{o,,\Omega}$$

So, this is the, these are the definitions which we want to keep, notations and notational conventions which we are doing.

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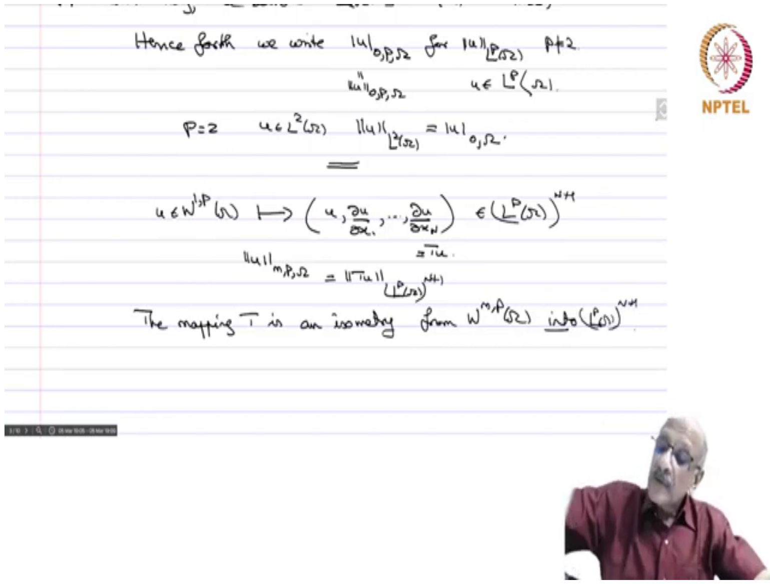

So, now if you take $u \in W^{m,p}(\Omega)$ sorry that means so what is this, this means u and all its first derivatives are in $L^p(\Omega)$. So, now I am going to map it to the following thing so that

$$u \in W^{m,p}(\Omega) \, maps \, to \left(u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right) = Tu \in (L^p(\Omega))^{N+1}$$

So, this means that norm.

$$||u||_{m,p,\Omega} = ||Tu||_{(L^p(\Omega))^{N+1}}$$

So, this mapping the mapping T is an isometry from $W^{m,p}(\Omega)$ into $(L^p(\Omega))^{N+1}$. So, this tells us the first theorem which we want to do.

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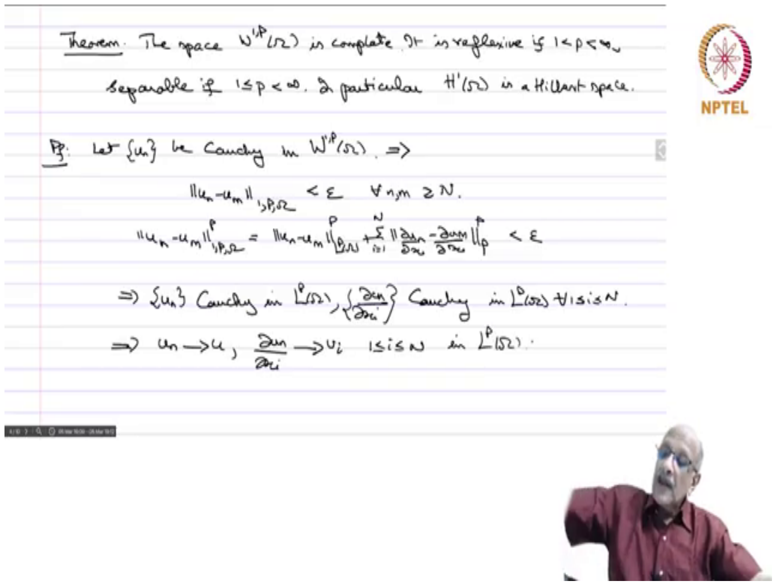

Theorem:

The space $W^{1,p}(\Omega)$ is complete so it is a Banach space. It is reflexive if $1 < p < \infty$. Separable if $1 \leq p < \infty$. In particular $H^{1}(\Omega)$ is a Hilbert space because it is a complete inner product space that is okay. So, proof, so let u n, so what we have to, we have just show that it every Cauchy sequence converges.

Proof:

So, let $\{u_n\}$ be Cauchy in $W^{1,p}(\Omega)$. So, then this what does this imply? From the norm so that means

$$||u_n - u_m||_{1,p,\Omega} < \varepsilon \ , \forall n, m \geq N$$

$$\Rightarrow ||u_n - u_m||^p_{1,p,\Omega} < ||u_n - u_m||^p_{L^p(\Omega)} + \sum_{i=1}^{N} ||\frac{\partial u_n}{\partial x_i} - \frac{\partial u_m}{\partial x_i}||^p_p < \varepsilon$$

$\Rightarrow \{u_n\}$ is a Cauchy sequence in $L^p(\Omega)$ and $\left\{\frac{\partial u_n}{\partial x_i}\right\}$ is a Cauchy sequence in $L^p(\Omega)$, $\forall 1 \leq i \leq N$

$$\Rightarrow u_n \to u,\ \frac{\partial u_n}{\partial x_i} \to v_i,\ in\ L^p(\Omega),\ \forall 1 \le i \le N$$

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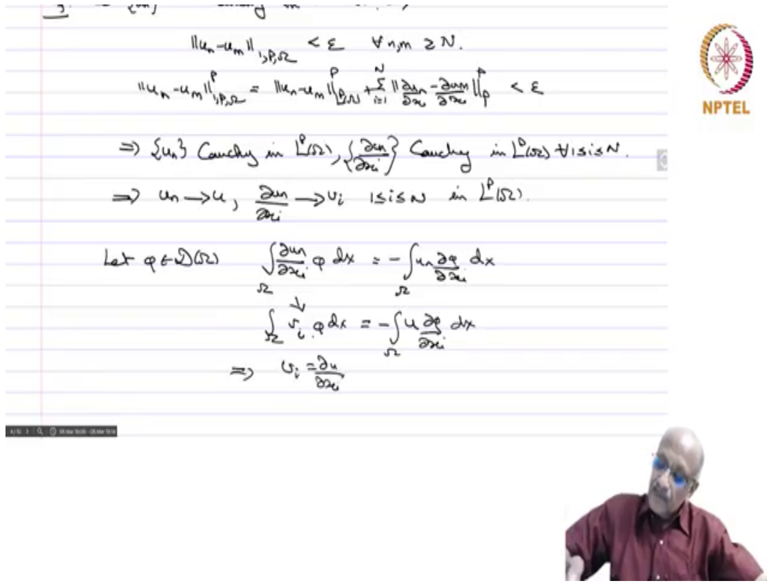

Now, what is the meaning of the distribution derivative so then so let

Let $\varphi \in D(\Omega)$ then what do you have, that $\int_\Omega \frac{\partial u_n}{\partial x_i} \varphi dx = - \int_\Omega u_n \frac{\partial \varphi}{\partial x_i} dx$

Now, I want to pass to the limit $\frac{\partial u_n}{\partial x_i} \to v_i$ in L^p.

Now, phi being C^∞ function with compact support will be in all L^p spaces in particular in the dual space of L^p and therefore and that is a fixed function therefore I can pass to the limit so this will converge to

$$\int_\Omega v_i \varphi dx = - \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx$$

Now, $u_n \to$u and $\frac{\partial \varphi}{\partial x_i}$ is in C^∞ function, fixed C^∞ function with compact support which is in all the in the dual space and therefore this goes to $u \frac{\partial \varphi}{\partial x_i}$. And this implies that

$$\Rightarrow v_i = \frac{\partial u}{\partial x_i}$$

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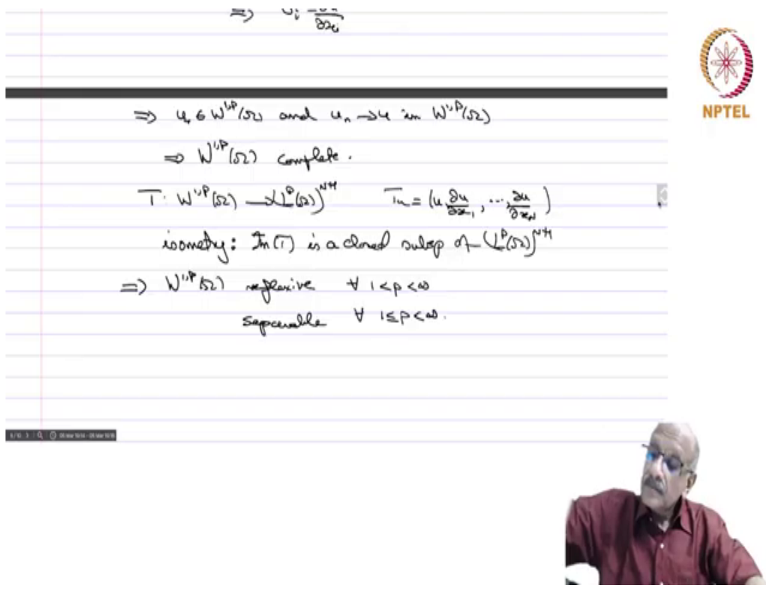

So, this means that $u_{n,}$ $u \in W^{1,p}(\Omega)$ because all its derivatives are in L^p and consequent and of $u_{n,}$ $u \in W^{1,p}(\Omega)$ because $u_n \to u$, $\frac{\partial u_n}{\partial x_i} \to v_i$, $in\ L^p(\Omega)$ and that means u n goes to u in W 1, P of omega and this implies that $W^{1,p}(\Omega)$ is complete.

Now, you take the mapping

$$T: W^{1,p}(\Omega) \to (L^p)^{N+1}(\Omega),\ T(u) = \left(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) \text{ is isometry.}$$

$Im(T)\ is\ a\ closed\ subspace\ of\ (L^p)^{N+1}(\Omega).$

So, a close subspace will inherit all the reflexivity and separability properties of the original space and an isometric isomorphic image of reflexive space is reflexive, separable space is separable. And therefore, this implies that $W^{1,p}(\Omega)$ is reflexive for all $\forall\ 1 < p < \infty$ and separable $\forall\ 1 \le p < \infty$ so this comes from the inheritance properties of that.

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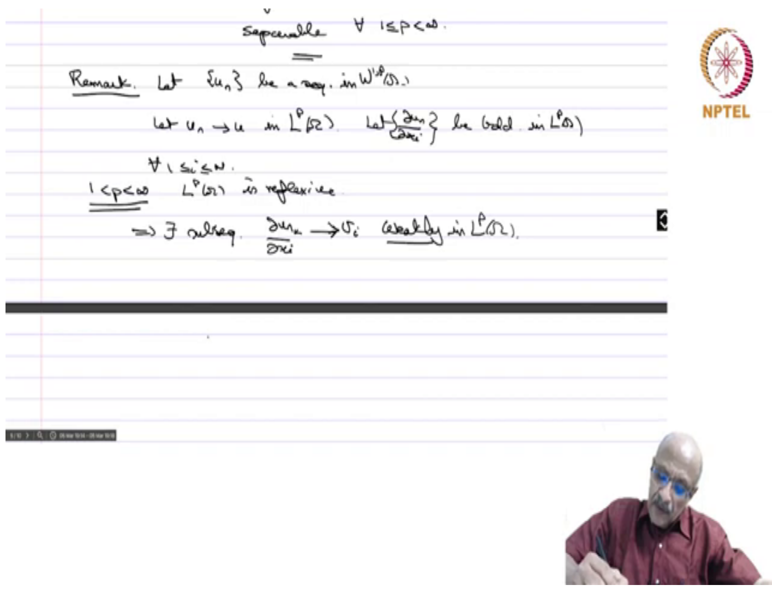

So**, remark** so we have something useful from the proof which we have seen the above so let us assume that this is a very useful technique which we remember.

So, let $\{u_n\}$ be a sequence in $W^{1,p}(\Omega)$.

$$u_n \to u, \ \ in\ L^p(\Omega). \text{ Let } \left\{\frac{\partial u_n}{\partial x_i}\right\} \text{be bounded } in\ L^p(\Omega).$$

$\forall 1 \leq i \leq N$. So, now if you take $1 < p < \infty$ then $L^p(\Omega)$ is reflexive implies there exists a subsequence you can choose a common subsequence for all of them, they are only finite number such that $\left\{\frac{\partial u_{n_k}}{\partial x_i}\right\} \to v_i$ weakly in $L^p(\Omega)$.

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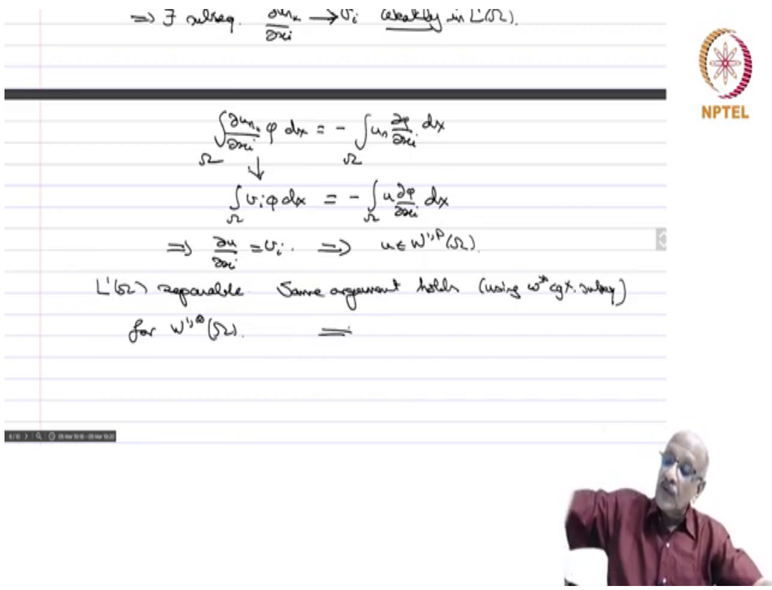

So, now let us again do what we did in the theorem so you have

$$\int_\Omega \frac{\partial u_{n_k}}{\partial x_i} \varphi dx = -\int_\Omega u_n \frac{\partial \varphi}{\partial x_i}\, dx$$

So, now you have a weak convergence and phi is in the dual space fixed function by definition of the weak convergence this goes to $\int_\Omega v_i \varphi dx$ and therefore that is $\int_\Omega v_i \varphi dx = -\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx$.

And consequently, once more we show, we see again that

$$\Rightarrow v_i = \frac{\partial u}{\partial x_i} \Rightarrow u \in W^{1,p}(\Omega)$$

So, $\{u_n\}$ converges u in $L^p(\Omega)$ and all the derivatives are bounded in $L^p(\Omega)$ itself says that the limit is in $W^{1,p}$ (Ω) so you are able to get a lot of extra information from this, this is a very useful consultation.

So, if p=1, $L^1(\Omega)$ is separable so if something is bounded in the dual space it will have a weak star convergence subsequence and therefore same argument holds using weak star convergent subsequence for $W^{1,\infty}(\Omega)$. So, if you have $\{u_n\} \to u$ in L^∞ and all the derivatives first derivatives are bounded in L^{∞} then you will get that $u \in W^{1,\infty}(\Omega)$ and this is again with useful. So, we are able to predict when a function will belong to this thing.

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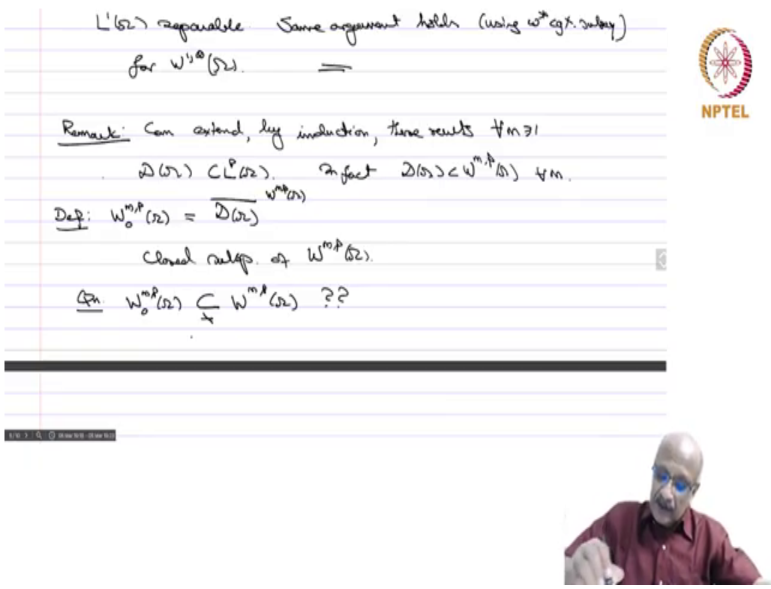

Remark:

So, then another remark can extend by induction these results $\forall m \geq 1$. So, in most of this course I will only prove results for $m = 1$, sometimes I will prove for other m also but most of the time it will be clear by iterating the arguments you can go on to higher and higher orders over the spaces it is just technically more horrendous but otherwise no new ideas are involved and therefore we will do that.

Definition: So, now important definition so we know that d omega is contained in $L^p(\Omega)$ and for $D(\Omega)$ the distribution derivatives are just the classical derivatives and therefore they are also C^{∞} functions with compact support and in fact $D(\Omega) \subset W^{1,p}(\Omega),\ \forall m.$

So, now comes the definition, we denote,

$$W_0^{m,p}(\Omega) = \overline{D(\Omega)}^{W^{m,p}(\Omega)}$$

so if you take that closure so this is a closed subspace of $W^{1,p}(\Omega)$. So, important question which we will not be able to fully answer, we will answer partially. So,

question

$$W_0^{m,p}(\Omega) \subset \overline{D(\Omega)}??$$

so we will take some time to answer this question, we will soon answer it in the case of $\mathbb{R}^N$ itself but for open sets we will answer it after sometime.

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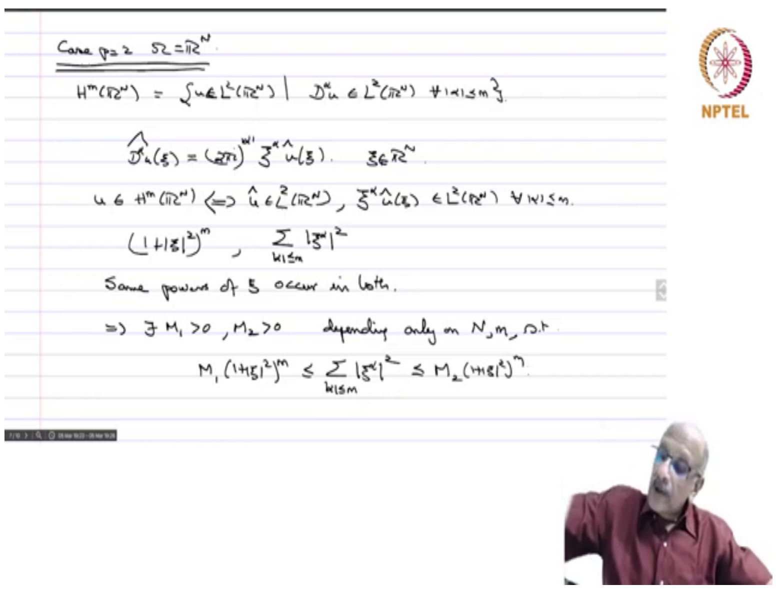

Case $p = 2,\ \Omega = \mathbb{R}^N$

So, now the some notations for so Case $p = 2,\ \Omega = \mathbb{R}^N$ So, now if Case Case $p = 2,$

then if you look at

$$H^m(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N)\ |\ D^\alpha u \in L^2(\mathbb{R}^N),\ \forall\, |\alpha| \le m\right\}$$

Now, on $L^2(\mathbb{R}^N)$ we have the Fourier transform, so and the Fourier transform is, so square integrable and for partial theorem the L 2 norm of a function and its Fourier transform will be the same.

Therefore, and you also know that

$$\widehat{D^\alpha u}(\xi) = (2\pi i)^{|\alpha|}\xi^\alpha \hat{u}(\xi), \quad \xi \in \mathbb{R}^N$$

$$u \in H^m(\mathbb{R}^N) \Leftrightarrow \hat{u} \in L^2(\mathbb{R}^N),\ \xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^N),\ \forall |\alpha| \le m.$$

So, and so this is and the converse also true and consequently you have another way of looking at the space.

So, now if you look at

$$(1 + |\xi|^2)^m, \quad \sum_{|\alpha| \le m} |\xi^\alpha|^2$$

the same powers of ξ occur in both. So, this implies $\exists M_1 > 0, M_2 > 0$, which depending only on the dimension N and small m such that

$$M_1 (1 + |\xi|^2)^m \le \sum_{|\alpha| \le m} |\xi^\alpha|^2 \le M_2 (1 + |\xi|^2)^m,$$

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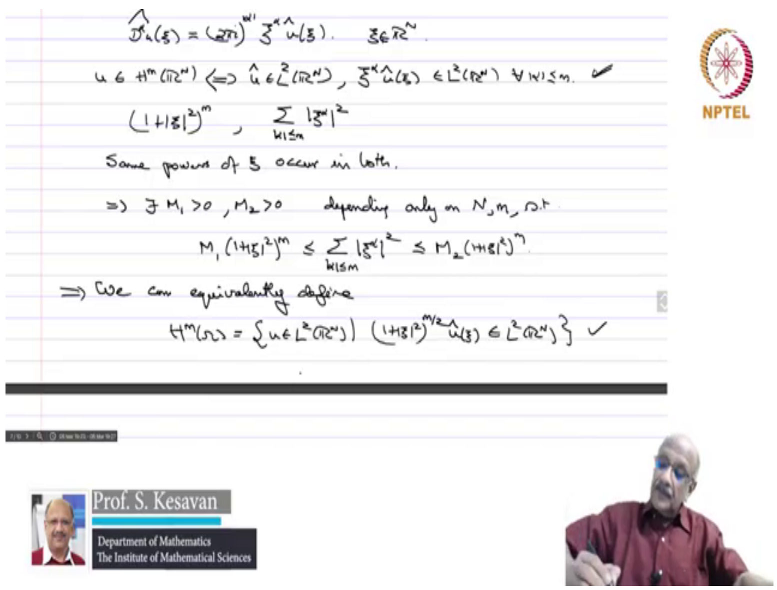

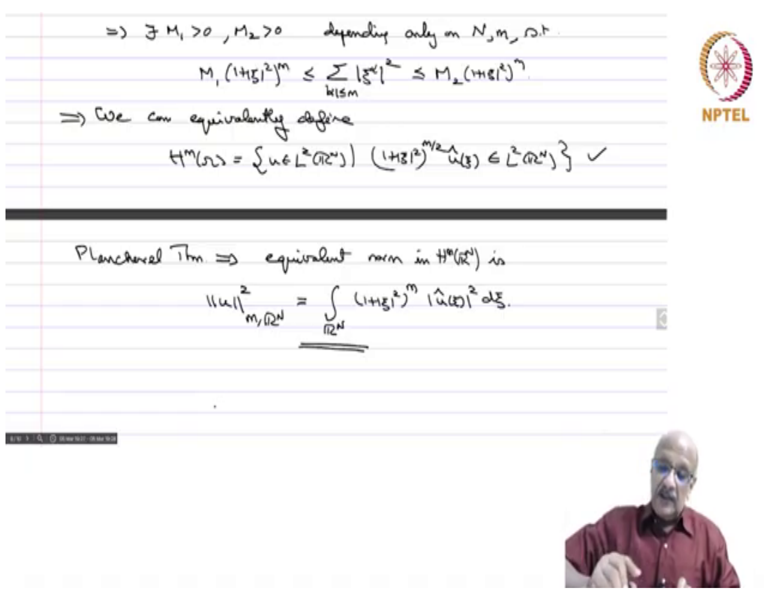

So, consequently we can equally, equal, so therefore we can equivalently define

$$H^m(\Omega) = \left\{ u \in L^2(\mathbb{R}^N) \mid (1 + |\xi|^2)^{m/2} \hat{u}(\xi) \in L^2(\mathbb{R}^N) \right\}$$

So, this is just the same as this statement, they are equivalent statements and therefore you can write it this way.

And using the Plancherel theorem this implies that an equivalent norm in $H^m(\mathbb{R}^N)$ is and we will denote it by the same symbol we will not give it another thing, so you will have

$$||u||^2_{m,\mathbb{R}^N} = \int_{\mathbb{R}^N} (1 + |\xi|^2)^m |\hat{u}(\xi)|^2 \, d\xi$$

So, this another way of writing the $H^m(\mathbb{R}^N)$ when $p = 2$ and its norm. So, this is another equivalence and sometimes we will find that this is useful to know. So, we ask this question is $W_0^{m,p}(\Omega) = W^{m,p}(\Omega)$. So, we will answer that question in the affirmative for $\Omega = \mathbb{R}^N$, which we will do next.