Sobolev Spaces and Partial Differential Equations

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Lecture 25
Exercises – Part 3

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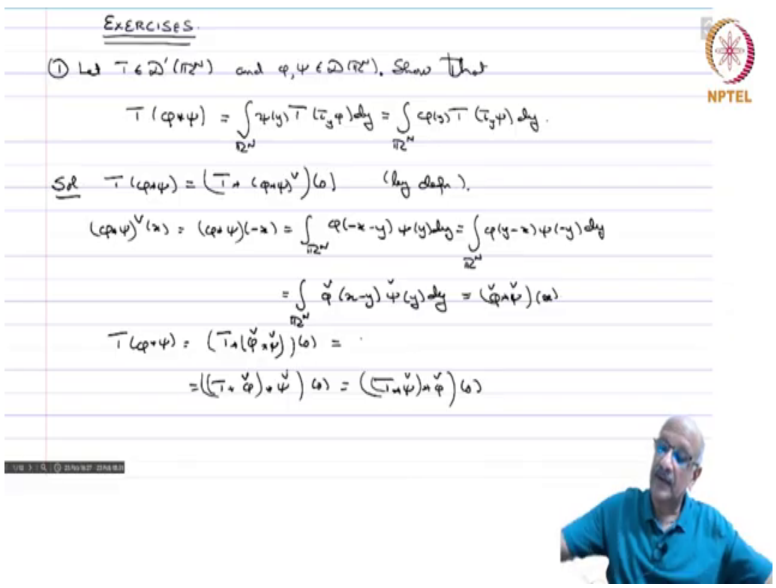

EXERCISES:

(1)

We will now do some exercises. So, the first one,

let $T \in D'(\mathbb{R}^N)$ and $\varphi, \psi \in D(\mathbb{R}^N)$. Show that

$$T(\varphi * \psi) = \int_{\mathbb{R}^N} \psi(y)\, T(\tau_y \varphi)\, dy$$

Well, $T(\varphi * \psi) \in D(\mathbb{R}^N)$, and therefore this makes sense that is

$$= \int_{\mathbb{R}^N} \varphi(y)\, T(\tau_y \psi)\, dy$$

So, what is $T(\varphi * \psi)$?

Solution:

$$T(\varphi * \psi) = \left(T * (\varphi * \psi)^{\vee\vee}\right)(0)$$

double chesh is back to the original function and therefore this is precisely, so by definition, by definition of T star something.

$$(\varphi * \psi)^{\vee}(x) = (\varphi * \psi)(-x) = \int_{\mathbb{R}^N} \varphi(-x-y)\,\psi(y)\,dy = \int_{\mathbb{R}^N} \varphi(y-x)\,\psi(-y)\,dy$$

$$= \int_{\mathbb{R}^N} \varphi^{\vee}(x-y)\,\psi^{\vee}(y)\,dy = (\varphi^{\vee} * \psi^{\vee})(x)$$

$$T(\varphi * \psi) = \left(T * (\varphi^{\vee} * \psi^{\vee})\right)(0)$$

And now we know we can everything is a C infinity function with compact support, T is a distribution so we can use the various commutative associative properties, so this is

$$= \left((T * \varphi^{\vee}) * \psi^{\vee}\right)(0) = (\left(T * \psi^{\vee}\right) * \varphi^{\vee})(0)$$

I have used both the commutativity and associativity properties.

$$= \int_{\mathbb{R}^N} (T * \varphi^{\vee})(-y)\,\psi^{\vee}(y)\,dy = \int_{\mathbb{R}^N} (T * \psi^{\vee})(-y)\,\varphi^{\vee}(y)\,dy$$

$$= \int_{\mathbb{R}^N} (T * \varphi^{\vee})(y)\,\psi^{\vee}(-y)\,dy = \int_{\mathbb{R}^N} (T * \psi^{\vee})(y)\,\varphi^{\vee}(-y)\,dy$$

$$= \int_{\mathbb{R}^N} (T\tau_y\varphi)\ \ \psi\ (y)\,dy = \int_{\mathbb{R}^N} (T\tau_y\psi)\ \ \varphi\ (y)\,dy$$

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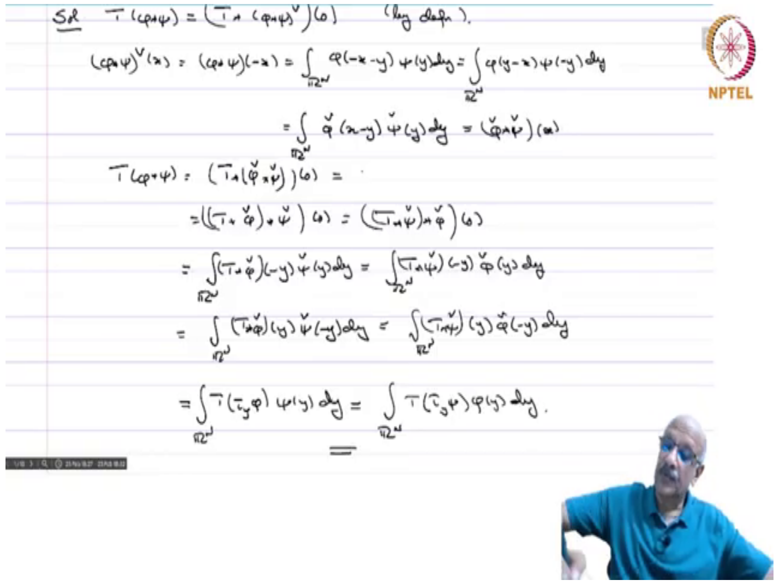

so that completes the solution of this exercise.

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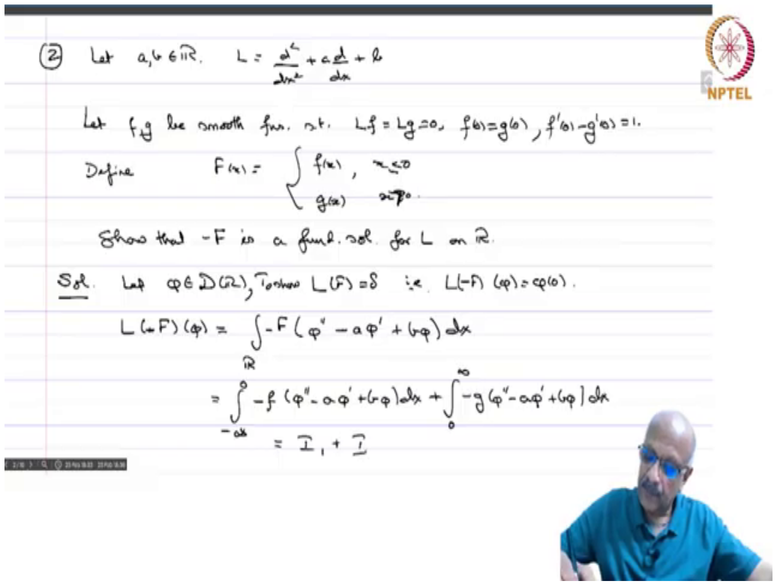

(2)

Second one, let $a, b \in \mathbb{R}$, $L = \frac{d^2}{dx^2} + a\frac{d}{dx} + b$

be a differential operator with constant coefficient is the usual standard second order differential operator. Let f, g be smooth functions such that

$$Lf = Lg = 0, \;\; f(0) = g(0), \;\; f'(0) - g'(0) = 1$$

Define

$$F(x) = f(x), \;\; if\; x \leq 0$$
$$= g(x), \;\; if\; x > 0.$$

does not matter where I put the equality, I have put it this way so this is what.

Show that $-F$ is a fundamental solution for L on $\mathbb{R}$.

Solution:

So, solution, so we just have to compute, so let

$\varphi \in D(\mathbb{R})$

so we have to show that $L(-F) = \delta$, *i.e.*, $L(-F)(\varphi) = \varphi(0)$.

So, we have to show this

$$L(-F)(\varphi) = \int_{\mathbb{R}^N} -F\left(\varphi'' - a\varphi' + b\varphi\right) dx$$

$$= \int_{-\infty}^{0} -f\left(\varphi'' - a\varphi' + b\varphi\right) dx + \int_{-\infty}^{0} -g\left(\varphi'' - a\varphi' + b\varphi\right) dx = I_1 + I_2$$

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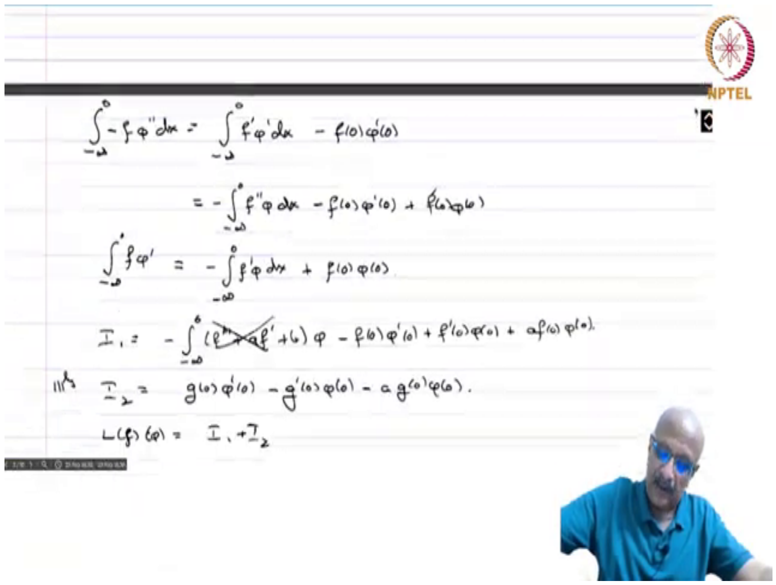

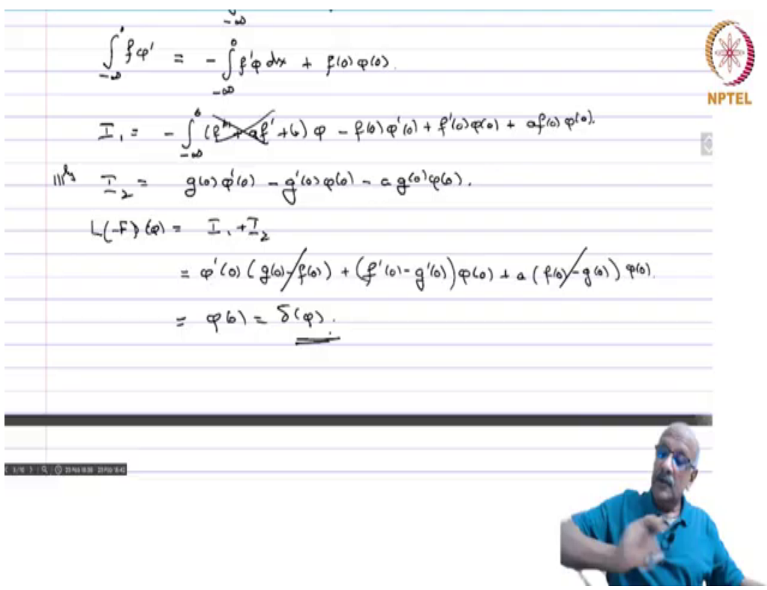

So, now let us compute each of these integrals and now so the first one

$$\int_{-\infty}^{0} - f\left(\varphi^{''}\right)\, dx = \int_{-\infty}^{0} f'\varphi'\, dx - f(0)\,\varphi^{'}(0)$$

$$= -\int_{-\infty}^{0} f^{''}\varphi\, dx - f(0)\,\varphi^{'}(0) + f^{'}(0)\varphi(0)$$

so it is just a question of integration by parts, this is minus infinity to 0 of f phi dash dx and then you have to take the boundary terms, there is no boundary term at minus infinity because phi has compact support and so do all its derivatives.

$$\int_{-\infty}^{0} f\,\varphi'\,dx = \int_{-\infty}^{0} f\,\varphi'\,dx + f(0)\,\varphi\ (0)$$

So, what is I_1? So, I_1 is the sum of these two, there is one more term with the b and therefore we will have to add all those three terms carefully. So, you have

$$I_1 = -\int_{-\infty}^{0} \left(f'' + af' + b\right)\varphi\ dx - f(0)\,\varphi'(0) + f'(0)\varphi(0) + af(0)\,\varphi\ (0)$$

$$= -\ f(0)\,\varphi'(0) + f'(0)\varphi(0) + af(0)\,\varphi\ (0)$$

Similarly, you go through the same rigmarole so you get

will be equal to only now the limit of the integration you have, the upper limit will give you nothing the lower limit will give you everything, so there will be a minus sign involved.

So, if you do the calculation you should get

$$I_2 = g(0)\,\varphi'(0) - g'(0)\varphi(0) - ag(0)\,\varphi\ (0)$$

So, $L(-\ F)(\varphi) = I_1 + I_2$

$$= \varphi'(0)(g(0) - f(0)) - (f'(0) - g'(0))\,\varphi(0) + a(\,f(0) - g(0))\ \varphi\ (0)$$

$$= \varphi(0) = \delta(\varphi)$$

which is δ so that proves.

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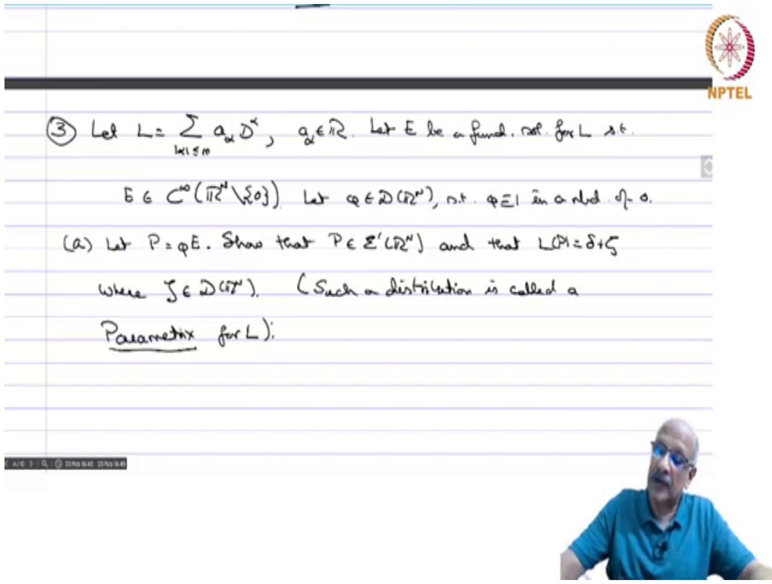

(3)

Let $L = \sum_{|\alpha| \leq m} a_{\alpha} D^{\alpha}, \quad a_{\alpha} \in \mathbb{R}.$

where a α are constants so this is a constant coefficient differential operator of order m. Let E be a fundamental solution for L such that E belongs to C infinity of $\mathbb{R}^N$. So, this E is a distribution but if you restrict the distribution to the open set which is the complement of the origin you know what that means then it coincides with that distribution generated by a C infinity function, this is what we mean by saying

$E \in C^{\infty}(\mathbb{R}^N \backslash \{0\})$. Let $\varphi \in D(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in a neighborhood of 0.

- **(a)** Then a, so the first part let $P = \varphi E$. Show that $P \in \mathcal{E}'(\mathbb{R}^N)$ and belongs that means it is a distribution with compact support and that $L(P) = \delta + \varsigma$ where $\varsigma \in D(\mathbb{R}^N)$ such a distribution which is, which when acted on by L gives you the Dirac plus a perturbation of the Dirac distribution by a C infinity function with compact support is called a parametrix for L. So, it is almost a fundamental solution but it is missing by means by the addition of a C infinity function with compact support.

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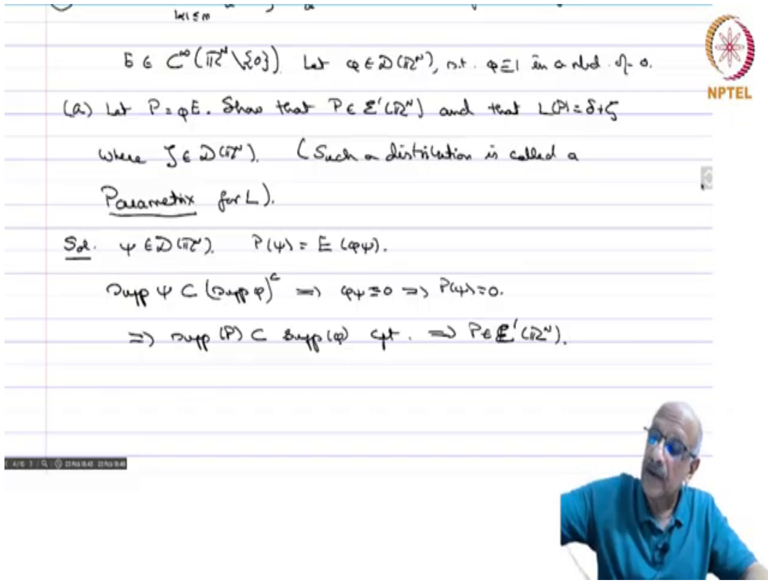

solution: so $\psi \in D(\mathbb{R}^N)$so $P(\psi) = E(\phi\psi)$ so if $supp(\psi) \subset (supp\ \varphi)^c \Rightarrow \varphi\psi = 0 \Rightarrow P(\psi) = 0$.. Therefore, $supp(P) \subset supp(\varphi)$ which is compact $\Rightarrow P \in \mathcal{E}'(\mathbb{R}^N)$ so that is complete.

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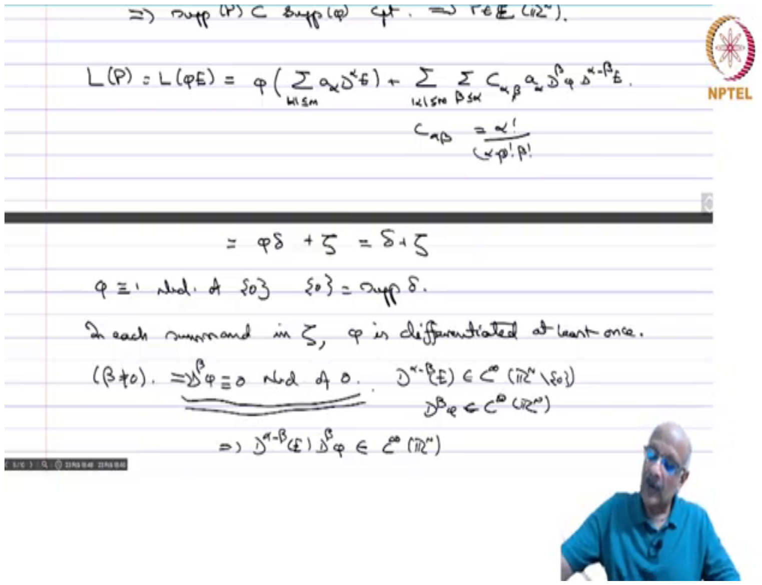

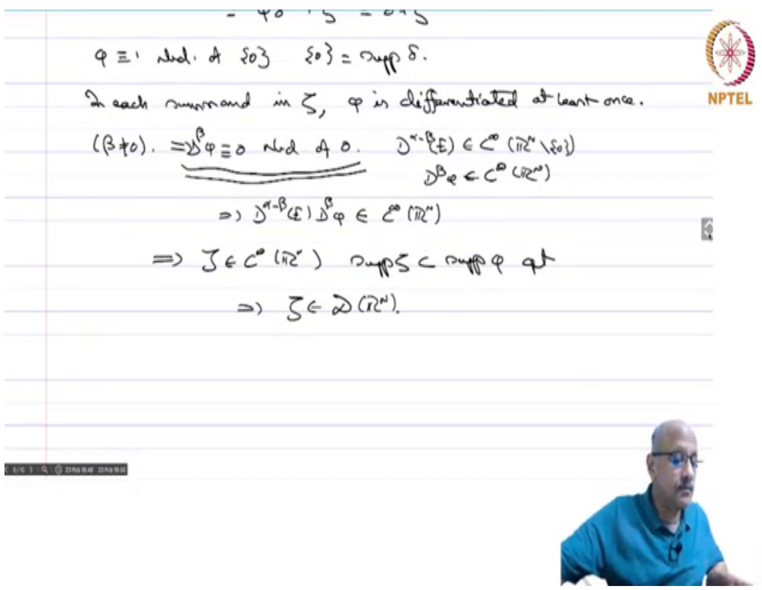

So, now we have to compute

$$L(P) = L(\varphi E) = \varphi\left(\sum_{|\alpha|\le m} a_\alpha D^\alpha E\right) + \sum_{|\alpha|\le m}\sum_{\beta\le\alpha} C_{\alpha,\beta} a_\alpha D^\beta \varphi D^{\alpha-\beta} E$$

$$C_{\alpha,\beta} = \frac{\alpha!}{(\alpha-\beta)!\,\beta!}$$

now we apply the Leibnitz formula. this is a common name, we have given this formula and therefore.

So, the here there are no derivatives of φ that term I have taken out separately and in the second term φ is differentiated at least once, so that is the important thing. Now, what is the first term so that is

is a fundamental solution of L, so that $= \varphi\delta + \varsigma = \delta + \varsigma$

$\varphi \equiv 1$ in the neighborhood of 0 and $\{0\} = supp\, \delta$.

And therefore, function $\delta = \varphi\delta$ that we have already seen earlier. So, this so we have this now we have to show that ς about the term. Now, in each term summoned in ς so each term you have φ is differentiated at least once that is

$\beta \neq 0 \Rightarrow D^{\beta}\varphi \equiv 0$ in the neighborhood of 0. So $D^{\alpha-\beta}E \in C^{\infty}(\mathbb{R}^{N}\backslash\{0\})$

$and\ D^{\beta}\varphi \in C^{\infty}(\mathbb{R}^{N}) \Rightarrow D^{\alpha-\beta}(E)\, D^{\beta} \in C^{\infty}(\mathbb{R}^{N})$

$\Rightarrow \varsigma \in C^{\infty}(\mathbb{R}^{N})\ \ supp(\varsigma) \subset supp\, \varphi$

$\Rightarrow \varsigma \in D(\mathbb{R}^{N})$

so we have shown that part of the exercise.

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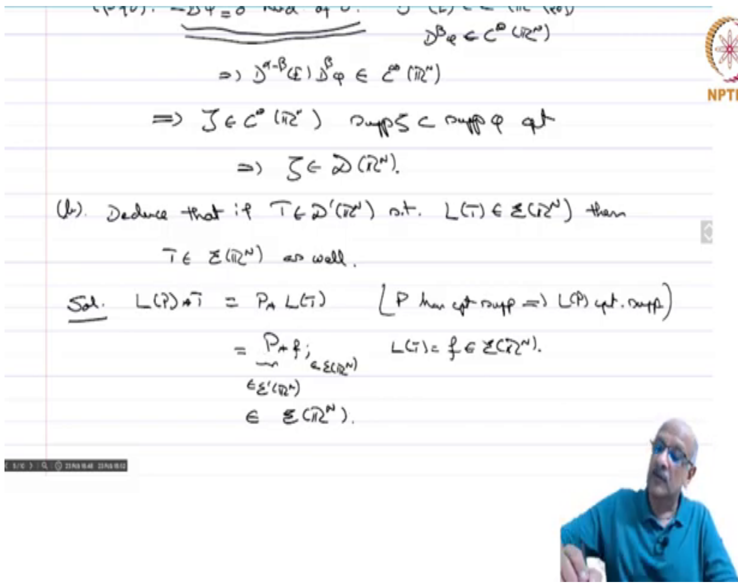

Now, **(b)** deduce that if $T \in D'(\mathbb{R}^N)$ such that $L(T) \in \mathcal{E}(\mathbb{R}^N)$ it means $L(T)$ is given by a distribution which is the distribution generated by a C^∞ function. Then $T \in \mathcal{E}(\mathbb{R}^N)$ as well. So, we have a distribution $L(T) = f$ solution of this differential equation if the data is C^∞, namely f is C^∞then you have T the solution distribution solution is automatically a C^∞ function, so that is the power of this here. So, this comes because $E \in C^\infty(\mathbb{R}^N \backslash \{0\})$

solution: $L(P) * T = P * L(T)$, P has compact support.

by the properties of the convolution and so P has compact support implies $L(P)$ also has compact support, and therefore $L(P) * T$ is well defined. So, we can write this, this is no problem that is equal to

$= P * f$ where $L(T) = f \in \mathcal{E}(\mathbb{R}^N)$. Now, $P \in \mathcal{E}'(\mathbb{R}^N)$, $f \in \mathcal{E}(\mathbb{R}^N)$ and therefore this belongs$\mathcal{E}(\mathbb{R}^N)$. You have a distribution with compact support convolved with a C^∞ function and therefore that is well defined and you have this thing.

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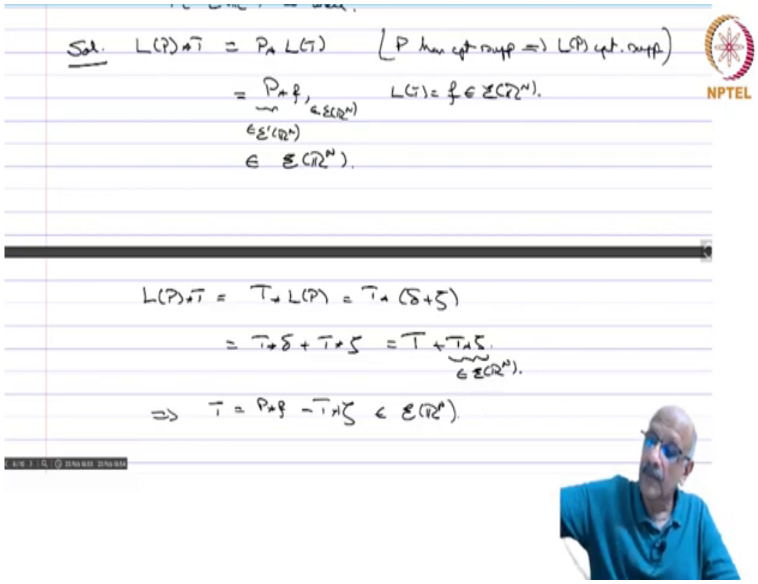

On the other hand, you have

$$L(P) * T = T * L(P) = T * (\delta + \varsigma),$$

$$= T * \delta + T * \varsigma = T + T * \varsigma$$

$$\Rightarrow T = P * f - T * \varsigma \in \mathcal{E}(\mathbb{R}^N)$$

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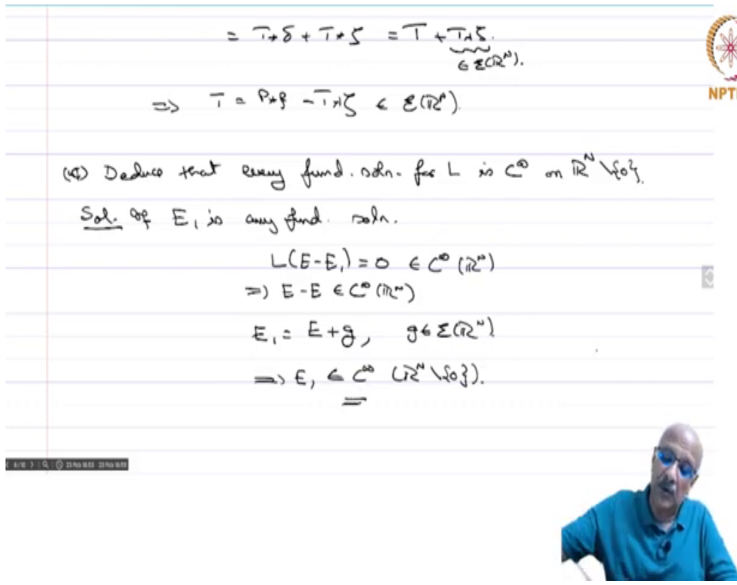

(c) deduce that every fundamental solution for f, for L is C^{∞} on $\mathbb{R}^{N}\backslash\{0\}$.

Solution, if E_1 is any fundamental solution then we have

$$L(E - E_1) = 0$$

$$\Rightarrow E - E \in C^{\infty}(\mathbb{R}^{N})$$

$$E_1 = E + g,\ g \in \mathcal{E}(\mathbb{R}^{N})$$

$$\Rightarrow E_1 \in C^{\infty}(\mathbb{R}^{N}\backslash\{0\})$$

So, this is a very nice result what we have proved is a partial, one way I mean we have proved the following fact.

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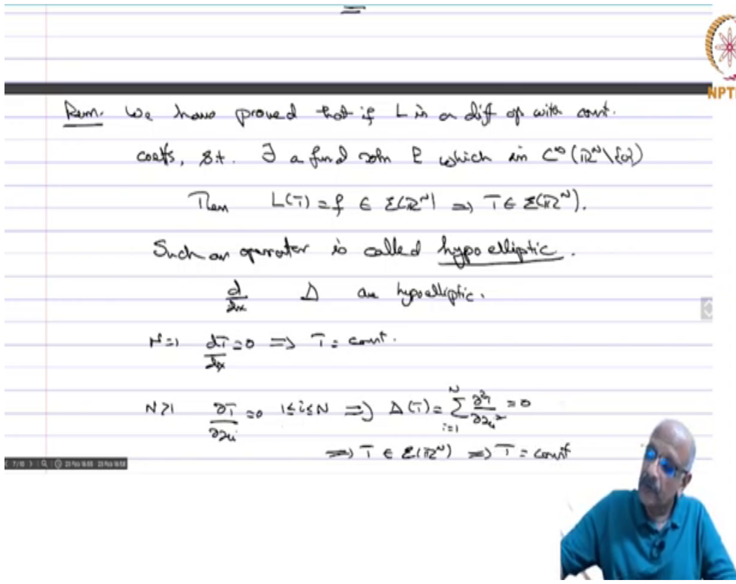

So, **remark** we have proved that if L is a differential operator with constant coefficients such that $\exists$ a fundamental solution E which is in $C^{\infty}(\mathbb{R}^N\backslash\{0\})$ then $L(T)$=f$\in$ $\mathcal{E}(\mathbb{R}^N) \Rightarrow T \in \mathcal{E}(\mathbb{R}^N)$. So, we have shown some kind of regularity theorem.

So, such an operator, converse is anyway true because if you, if this property holds or in fact it is true for any omega that we can, we have not shown that, it is true if wherever the f is C^{∞}, T will be C^{∞} there, so if it is true, if it is in $E(\Omega)$ where Ω is contained in $\mathbb{R}^N\backslash\{0\}$ then T will be also $C^{\infty}(\Omega)$ a so we have not shown that portion, we have only shown it for whole of $\mathbb{R}^N$.

So, such an operator is called hypo elliptic. So, examples

$\frac{d}{dx}$ is a hypo elliptic operator, what is its fundamental solution, heavy side function which except at the origin is a piecewise constant and therefore it is C^{∞} . What about the Laplacian? Again, either it is $log\ |x|$ which has the only singularity at the origin or $\frac{1}{|x|}$ to the $\frac{N}{2}$, N-2 which again is singular only at the origin, and therefore are all hypo elliptic.

And now we have, we showed that in

$N = 1,$ if $\frac{dT}{dx} = 0 \Rightarrow T\ is\ a\ constant.$ So, if in general N how do you do that?

$N > 1$, Suppose $\frac{\partial T}{\partial x_i} = 0,\ 1 \leq i \leq N \Rightarrow \Delta(T) = \sum_{i=1}^{N} \frac{\partial^2 T}{\partial x_i^2} = 0$

$\Rightarrow T \in \mathcal{E}(\mathbb{R}^N) \Rightarrow T\ is\ constant.$

So, that is how you prove it in higher dimensions and you use this particular result.

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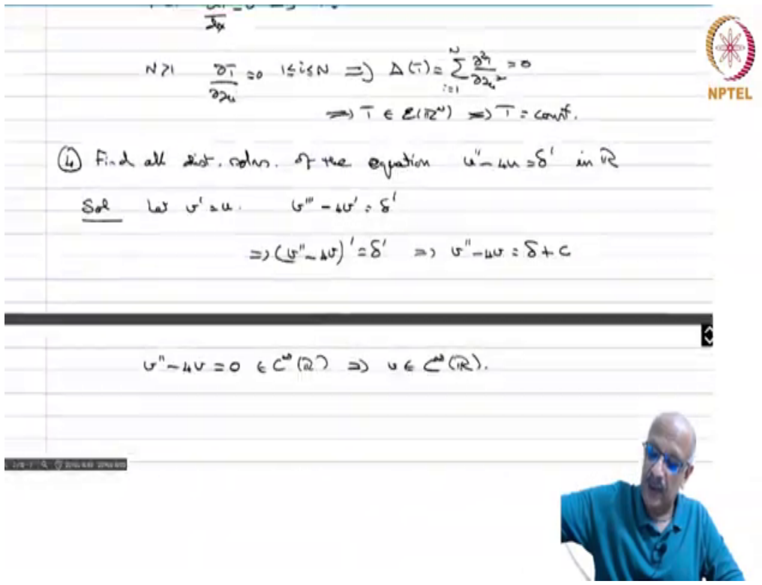

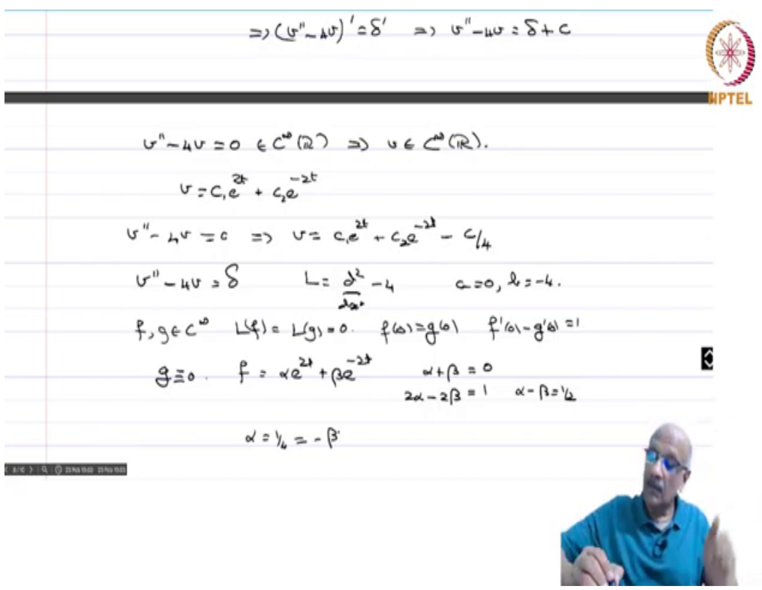

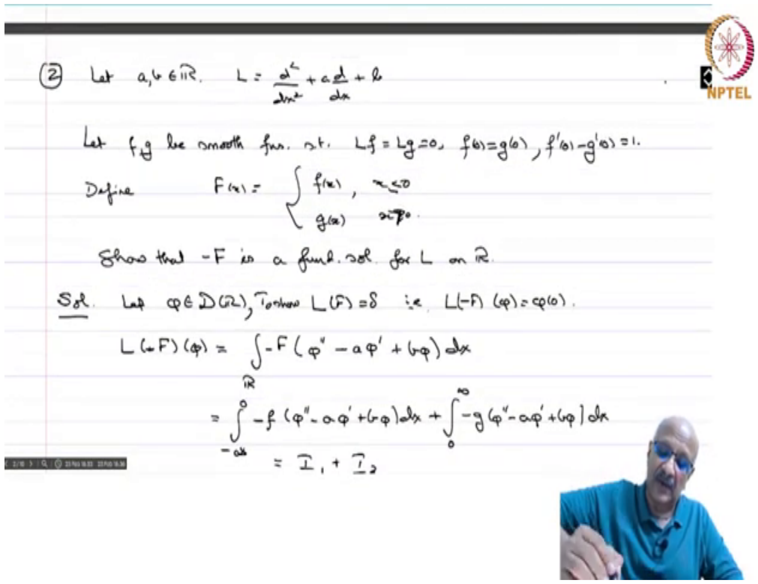

So, then we will do the next

exercise 4, find all distribution solutions of the equation

$$u'' - uv = \delta' \text{ in } \mathbb{R}$$

Solution, so we have to find all solutions of this equation. So,

let $v' = u$, $v''' - 4v' = \delta'$

$$\Rightarrow (v'' - 4v)' = \delta' \Rightarrow v'' - 4v = \delta + c$$

$$\Rightarrow (v'' - 4v) = 0 \in C^{\infty}(\mathbb{R}) \Rightarrow v \in C^{\infty}(\mathbb{R})$$

because the fundamental solution of this operator is C^{∞} outside the origin, you know what it is from the previous exercise for this particular operator. And this is a particular case of this excise, so this function other than at the origin it is C^{∞} and so this hypo elliptic operator and therefore if you have anything equal to C^{∞} then the solution will also be a C^{∞} function. So, we are just going to apply that particular function. So, this means that we know how to solve that then.

$$\Rightarrow v = C_1 e^{2t} + C_2 e^{-2t}$$

$$\Rightarrow (v^{''} - 4v) = 0 \Rightarrow v = C_1 e^{2t} + C_2 e^{-2t} - \frac{C}{4}$$

if you solve the differential equation the classical way so that is all you get.

Now, we are looking at the fundamental solution

$$(v^{''} - 4v) = \delta \quad L = \frac{d^2}{dx^2} - 4 \quad \text{a=0, b=-4}$$

in the previous solution, so let $f, g \in C^{\infty}(\mathbb{R})$ $L(f) = L(g) = 0$ $f(0) = g(0)$, $f^{'}(0) = g^{'}(0) = 1$

$g \equiv 0$ and then you take , $f = \alpha e^{2t} + \beta e^{-2t}$, $\alpha + \beta = 0$, $2\alpha - 2\beta = 1$

$$\alpha - \beta = \frac{1}{2} \ \alpha = \frac{1}{4} = -\beta$$

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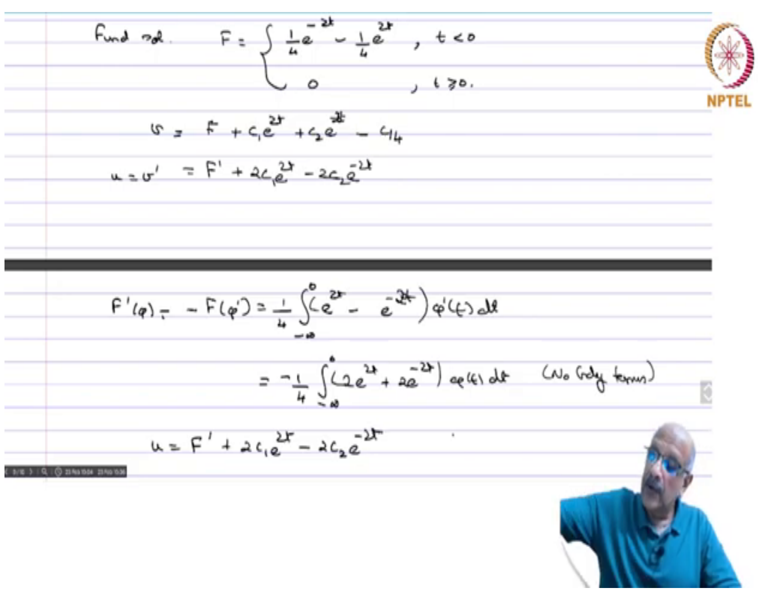

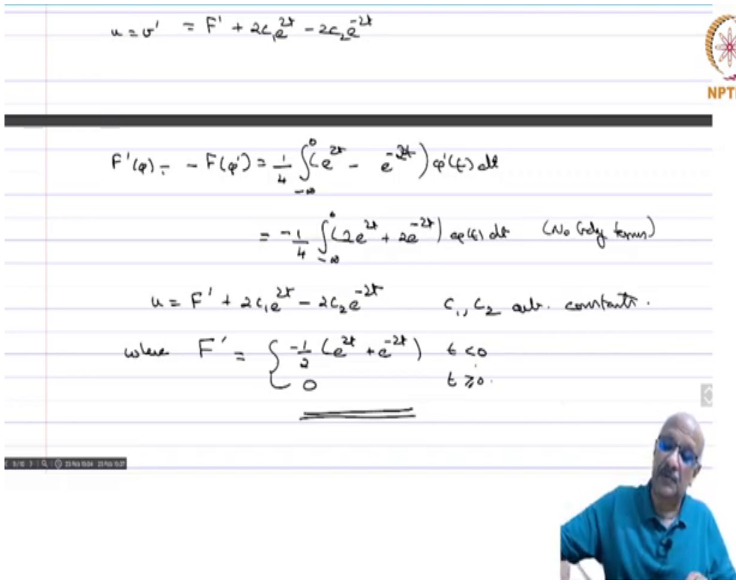

So, what is the fundamental solution? So, fundamental solution is given by

$$F = \frac{1}{4}e^{-2t} - \frac{1}{4}e^{2t}, \quad t < 0$$

$$= 0 \qquad , \quad t \geq 0$$

$v = F + C_1 e^{2t} + C_2 e^{-2t} - \frac{C}{4}$, And therefore, you get

$$u = v' = F' + 2t\, C_1 e^{2t} - 2t\, C_2 e^{-2t}$$

So, now we just only have to compute what is F'.

So, $F'(\varphi) = - F(\varphi) = \frac{1}{4}\int_{-\infty}^{0} (e^{2t} - e^{-2t})\varphi'(t)\ dt$

$$= -\frac{1}{4}\int_{-\infty}^{0} (2e^{2t} + 2)\varphi\ (t)\ dt$$

And then there will be no boundary terms because you just have to evaluate it at 0 $\varphi(0)$ is there this is also give you 1, this will give you -1 and therefore that will get cancelled, so there are no boundary terms, no boundary terms. So, I have just done the integration by parts. So, you have that u equal to, so all the solutions,

$$u = F' + 2C_1 e^{2t} - 2C_2 e^{-2t}\ ,\ C1,\ C2\ arbitrary\ constants\,.$$

$$F' = -\frac{1}{2}(e^{2t} + e^{-2t}),\quad t < 0$$

$$= 0 \qquad\qquad ,\ t \geq 0.$$

so this completes all the solutions of that differential equation. So, with this we will close this chapter, we have come to the end of the theory of distributions as I, as much as I wanted to present. So, we will next start the chapter on Sobolev spaces.