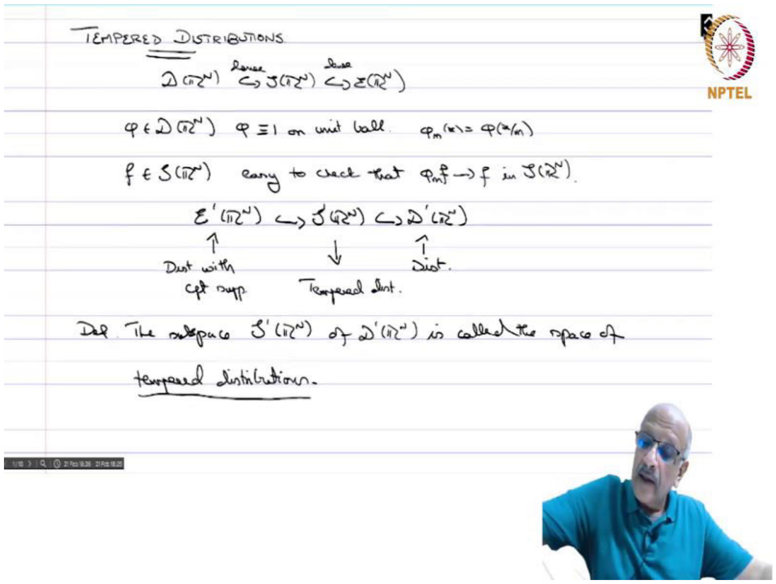
Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
Institute of Mathematical Science
Lecture 24
Tempered Distributions

(Refer Slide Time: 00:19)



Tempered distributions:

Now, we will talk about tempered distributions, this is the class of distributions to which we will extend the definition of the Fourier Transform. So, we saw that

$$\mathcal{D}(\mathbb{R}^N) \subset \mathcal{S}(\mathbb{R}^N) \subset \mathcal{E}(\mathbb{R}^N)$$

and this we know is dense, this is also dense.

$\mathcal{D}(\mathbb{R}^N)$ is also dense in $\mathcal{S}(\mathbb{R}^N)$ suppose you take

$\varphi \in \mathcal{D}(\mathbb{R}^N)$ $\varphi \equiv 1$ on the unit ball with centre at the origin of course, and you set

$$\varphi_m(x) = \varphi(\frac{x}{m}).$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$

it is easy to check that which is now C infinity function with compact support therefore it is in in $\mathcal{D}(\mathbb{R}^n)$

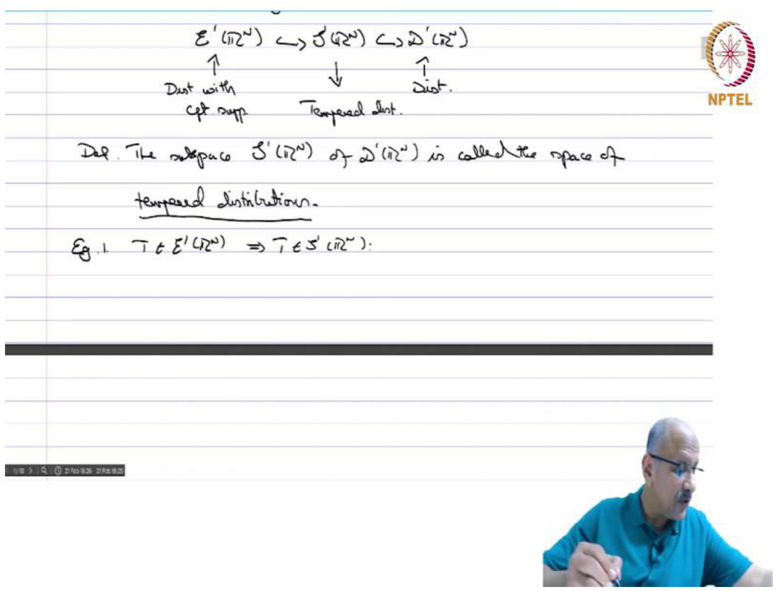
$$\varphi_{\square} \square \to \square \quad \mathcal{D}(\mathbb{R}^n)$$

which is now C infinity function with compact support therefore it is in

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

So, these are this is the space of all distributions, this is the base space of distributions with compact support and this we call as a space of tempered distributions. So, tempered definition this subspace $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{D}'(\mathbb{R}^n)$ called the space of tempered distributions. So, now let us look at examples of template distributions.

(Refer Slide Time: 03:25)



Example 1:

$$T \in \mathcal{E}'(\mathbb{R}^n) \Rightarrow T \in \mathcal{S}'(\mathbb{R}^n)$$

as we have just shown, so in particular the Dirac distribution is a tempered distribution.

(Refer Slide Time: 03:45)

Eg. 2 Let μ be a meas on $\mathbb{R}^N$ which is <u>slowly increasing</u>

i.e. $\exists$ a non-neg int. $k \geq 0$ s.t.

$$\int_{\mathbb{R}^N} \frac{d\mu(x)}{(1+|x|^2)^k} \;\; \text{of}$$

Eg. 2 Let μ be a meas on $\mathbb{R}^N$ which is <u>slowly increasing</u>

i.e. $\exists$ a non-neg int. $k \geq 0$ s.t.

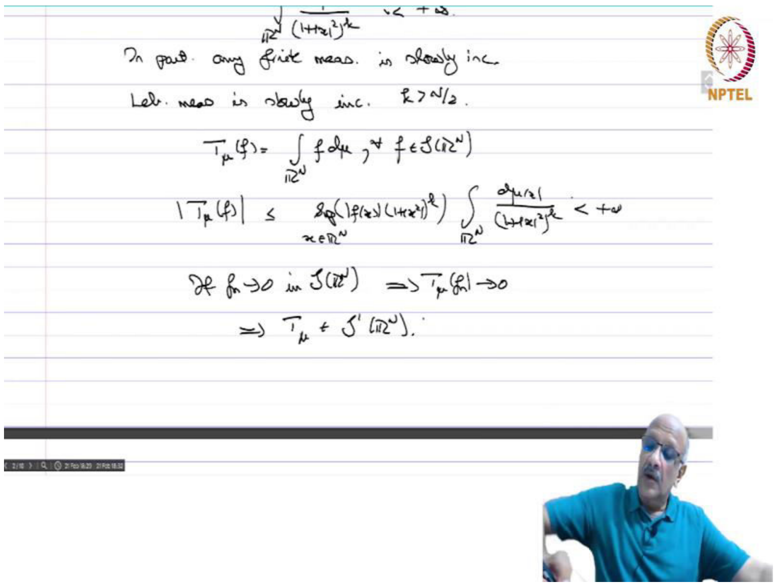
$$\int_{\mathbb{R}^N} \frac{d\mu(x)}{(1+|x|^2)^k} < +\infty.$$

In part. any finite meas. is slowly inc

Leb. meas is slowly inc. $k > N/2$.

$$T_\mu(f) = \int_{\mathbb{R}^N} f \, d\mu, \; \forall f \in \mathcal{S}(\mathbb{R}^N)$$

$$|T_\mu(f)| \leq \sup_{x \in \mathbb{R}^N} \left(|f(x)|(1+|x|^2)^k\right) \int_{\mathbb{R}^N} \frac{d\mu(x)}{(1+|x|^2)^k}.$$



Example 2:

Let μ be a measure on $\mathbb{R}^N$ which is slowly increasing, what does this mean? That is there exists a positive integer k such that you have

$$\int_{\mathbb{R}^N} \frac{d\mu(x)}{(1+|x|^2)^k} < +\infty$$

So, in particular any finite measure because k can be taken as 0 is slowly increasing, Lebasque measure is slowly increasing, because you know $k > \frac{N}{2}$. So, now if such a measure which is slowly increasing you define

$$T_\mu(\varphi) = \int_{\mathbb{R}^N} f\, d\mu,\ \forall f \in \mathcal{S}(\mathbb{R}^N)$$

$$|T_\mu(\varphi)| = \sup_{x\in\mathbb{R}^N} \left(|f(x)|(1+|x|^2)^k\right) \int_{\mathbb{R}^N} \frac{d\mu(x)}{(1+|x|^2)^k} < +\infty$$

And therefore, this is finite and also it shows that if

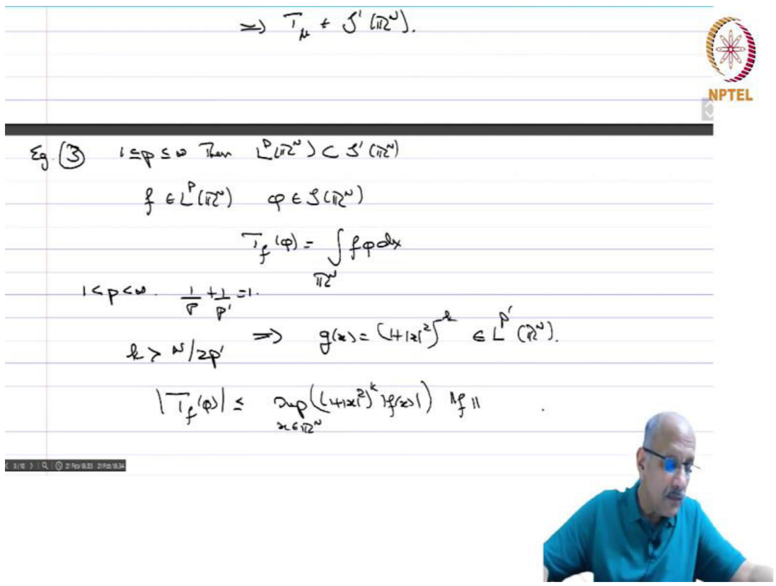
$f_n \to 0$ in $\mathcal{S}(\mathbb{R}^N) \Rightarrow T_\mu(f_n) \to 0.$

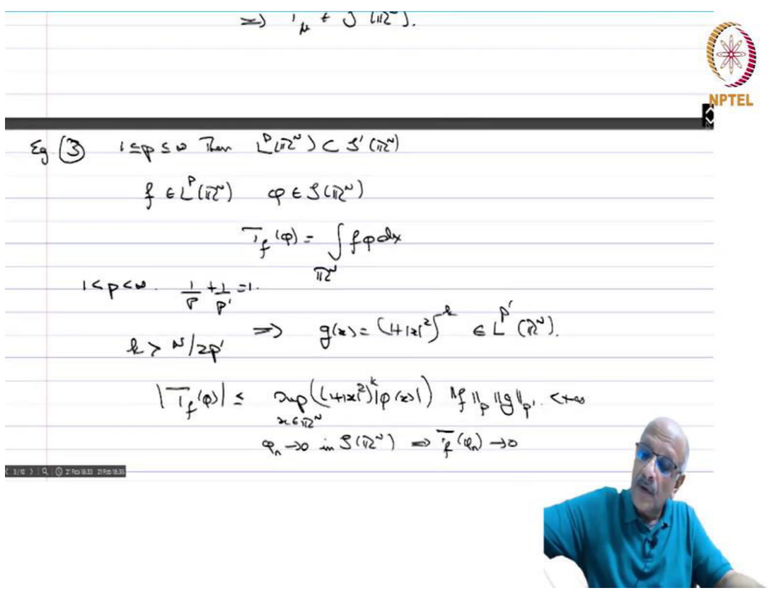
so this implies that

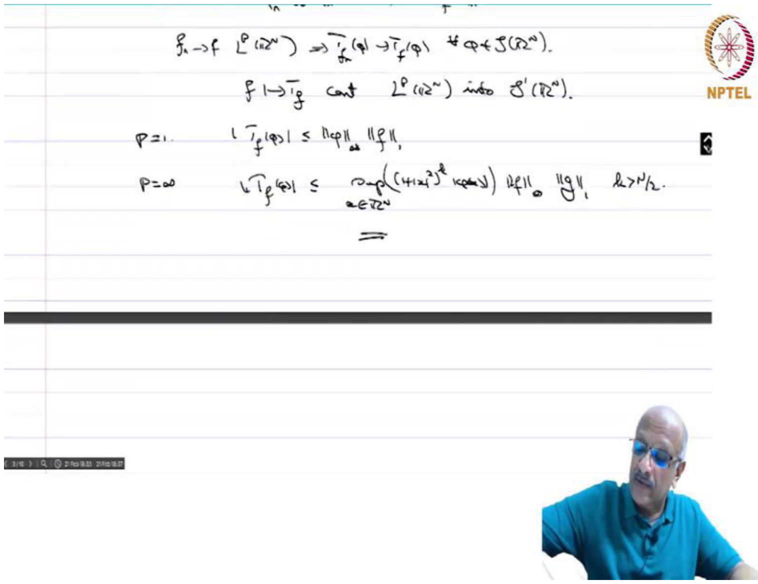
$$\Rightarrow T_\mu \in \mathcal{S}'(\mathbb{R}^N)$$

So, any measure which produces this.

(Refer Slide Time: 07:11)







Then 3, this very important,

Example 3:

$$1 \le p \le \infty$$

then $L^p(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ in fact it is a continuous inclusion also. So, if you take f that is first take, so f you have

$$f \in L^p(\mathbb{R}^n) \;\; \varphi \in S(\mathbb{R}^n)$$

so it is a locally integral function as a distribution,

$$T_f(\varphi) = \int_{\mathbb{R}^n} f\varphi \, dx, \; \forall \varphi \in S(\mathbb{R}^n)$$

the definition $T_f(\varphi)$ is of course which extends the definition of distribution is integral $f \varphi \, dx$ overall.

So, let us assume that $1 < p < \infty$ and

$$\frac{1}{p} + \frac{1}{p'} = 1$$

$$k > \frac{n}{2p'} \Rightarrow g(x) = (1 + |x|^2)^{-k} \in L^{p'}(\mathbb{R}^n)$$

So, by (())(08:29) inequality you have

$$|□_□(□)| \leq □□□_{□\in\mathbb{R}^□} \left((1+|□|^2)^□ |□(□)|\right) ||□||_□ \, ||□||_{□'}$$

So, this shows that this is finite and

$\varphi_□ \to 0, □□ \; □ \; (\mathbb{R}^□) \Rightarrow □_□(□_□) \to 0 \cdot$

So, this also shows that f and if

$□_□ \to □ \; □□ \; □^□(\mathbb{R}^□) \Rightarrow □_{□_□}(□) \to □_□(□) \;\; \forall \; □ \in □(\mathbb{R}^□)\cdot$

And this is the sequential continuity which we will use for the different for the linear maps as we already said and on the short space and therefore you have that

$□ \to □_□$ is continuous from $□^□(\mathbb{R}^□)$ into $□'(\mathbb{R}^□)$, so you have that all $□^□$ functions are in fact in $□'(\mathbb{R})$ Now, if

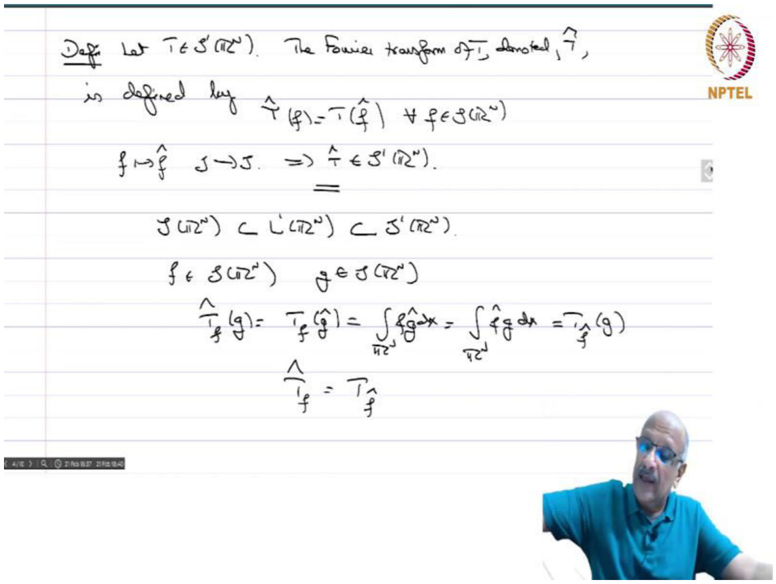
$$□ = 1, \;\; |□_□(□)| \leq ||□||_\infty \, ||□||_1$$

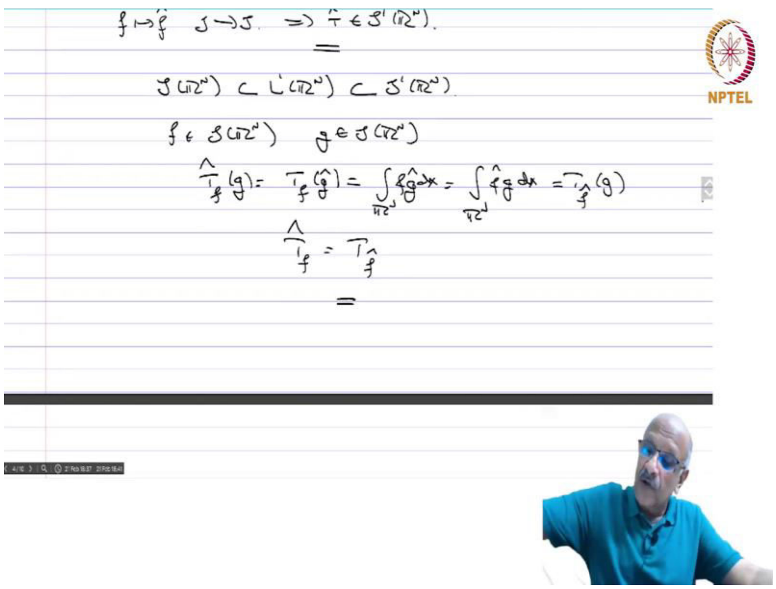
so there is nothing to do, so its automatically we chose the continuity, and

$$□ = \infty, \;\; |□_□(□)| \leq □□□_{□\in\mathbb{R}^□} \left((1+|□|^2)^□ |□(□)|\right) ||□||_\infty \, ||□||_1, \;\; □ > \frac{□}{2}$$

And this again tells you that this is in $□'$. So, this completely establishes our claim that $□^□$ for all $1 \leq □ \leq \infty$1 is in fact embedded in $□'(\mathbb{R}^□)$ continuously.

(Refer Slide Time: 11:56)





Definition:

So, now we are going to define definition let

$T \in \mathcal{S}'(\mathbb{R}^n)$ then the Fourier Transform of T denoted $\hat{T}$ is defined by the

$$\hat{T}(f) = T(\hat{f}) \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

and therefore this makes sense, so this definitely makes sense and we know that f going to f hat is a continuous from

$$f \to \hat{f} \quad \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \quad \Rightarrow \hat{T} \in \mathcal{S}'(\mathbb{R}^n)$$

So, it is a tempered distribution automatically. So, this is the definition, so this is what we could not do because of the Paley Wiener theorem for all distributions, because you had when you want to make this definition $\hat{\varphi}$ refused to be inside the space of continuous function C infinity functions with compact support, whereas this Schwartz space is like that and therefore we can use this definition.

$$\square(\mathbb{R}^{\square}) \to \square^{1}(\mathbb{R}^{\square}) \to \square'(\mathbb{R}^{\square})$$

So, we seem to have two definitions of the Fourier Transform, one is the definition because

$$\square(\mathbb{R}^{\square}) \subset \square^{1}(\mathbb{R}^{\square})$$

gets a definition of Fourier Transform from $\square^{1}$, but $\square(\mathbb{R}^{\square})$ is also contained in $\square'(\mathbb{R}^{\square})$ the and therefore $\square'(\mathbb{R}^{\square})$ gives you by the above definition (())(14:14) this, so we want to know that these two are not different.

$$\square \in \square(\mathbb{R}^{\square}) \quad \square \in \square(\mathbb{R}^{\square})$$

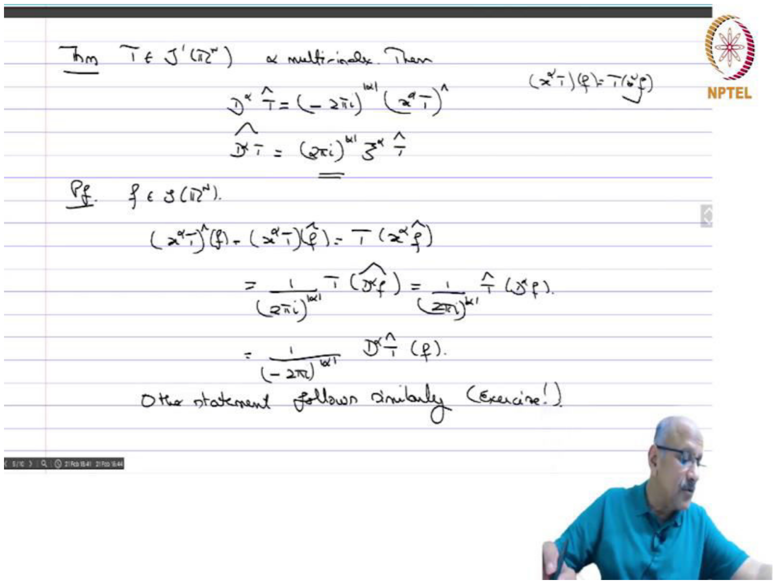
but we have proved the Weak Parseval Relation for functions in $\square(\mathbb{R}^{\square})$ this is equal to

$$\widehat{\square_{\square}}(\square) = \square_{\square}(\hat{\square}) = \int_{\mathbb{R}^{\square}} \square\, \hat{\square}\, \square\square = \int_{\mathbb{R}^{\square}} \hat{\square}\, \square\, \square\square = \square_{\hat{\square}}(\square).$$

So, $\widehat{\square_{\square}} = \square_{\hat{\square}}$

namely if $\square \in \square(\mathbb{R}^{\square})$ considered as a tempered distribution, then the Fourier Transform of it is nothing but the Fourier Transform generated by the usual Fourier Transform from $\square^{1}$, therefore there is no ambiguity both definitions coincide and therefore, that is fine.

(Refer Slide Time: 15:25)



Theorem:

So, now let us prove the following theorem which generalizes the properties of Fourier Transform of functions, so let $□ \in □'(\mathbb{R}^□)$

and let α be a multi index, then

$$□^□\widehat{□} = (-2□□)^{|□|}(\widehat{□^□□})$$

So, $(□^□□)$ is again a same tempered distribution, because if you have $(□^□□)$ acting on any f this is equal to $□(□)\ □^□$, this is the distribution definition for the multiplication base infinity function, but multiplication by monomials will still keep f into itself, so this makes sense and of course you have the convergence properties, therefore this defines a temporal distribution.

So, $(□^□□)$ is again a tempered power distribution and therefore $□^□□$, so exactly like when you want to differentiate you multiply the distribution by a pronominal and then take the Fourier Transform and put an appropriate multiple of $-2□□$. And (())(16:43) you will have

$$\widehat{□^□□} = (2□□)^{|□|}\ □^□\ \widehat{□}$$

$\widehat{□}$ is now our tempered distribution you can always multiply by means of monomials. So, let us prove this,

Proof:

so let $\square \in \square(\mathbb{R}^{\square})$ then

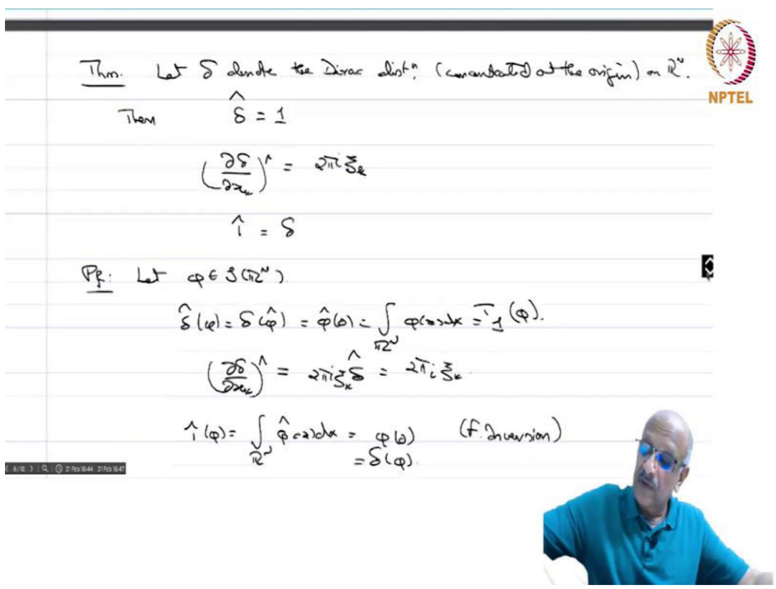
$$\widehat{\square^{\square}\square}(\square) = (\square^{\square}\square)(\widehat{\square}) = \square(\square^{\square}\widehat{\square})$$

$$= \frac{1}{(2\square\square)^{|\square|}}\square(\widehat{\square^{\square}\square}) = \frac{1}{(-2\square\square)^{|\square|}}\widehat{\square}(\square^{\square}\square)$$

$$= \frac{1}{(-2\square\square)^{|\square|}}\square^{\square}\widehat{\square}(\square)$$

And therefore, from this you cross multiply you get your answer, other statement follows similarly exercise. So, you can just check it, it is a trivial thing, so you.

(Refer Slide Time: 18:48)



Theorem:

Now, theorem again, so we know that the Dirac distribution is very special and so

Let δ denote the Dirac distribution concentrated at the origin on $\mathbb{R}^{\square}$, then you have

$$\hat{\delta} = 1$$

1 is a constant function constant function is slowly increasing, so we integral see we know that that is slowly increasing function, so therefore this is again in template distribution, so $\hat{\delta} = 1$ Now,

$$\widehat{\frac{\partial\delta}{\partial □_{□}}} = 2□□□_{□}$$ is again a polynomial so it is fine, so then

$$\hat{1} = □$$

Proof:

So, proof let $\varphi \in □(\mathbb{R}^{□})$

$$\hat{□}(□) = □(\hat{□}) = \hat{□}(0) = \int_{\mathbb{R}^{□}} □(□)\, □□ = □_{1}(□)$$

$$\widehat{\frac{\partial\delta}{\partial □_{□}}} = 2□□□_{□}\hat{□} = 2□□□_{□}$$

$$\hat{1}(□) = \int_{\mathbb{R}^{□}} \hat{□}(□)\, □□ = □(0) = □(□)$$

by the Fourier inversion formula and then that is equal to delta phi and therefore we have all these with nice results. So, finally we conclude this section with one more question of consistency.

(Refer Slide Time: 21:43)

So, you have $□(\mathbb{R}^{□})$, we already saw had two definitions one from $□'$ and one from $□^{1}$, and we saw the dose two definitions of the Fourier Transform were consistent. Now, you have

$$□(\mathbb{R}^{□}) \subset □^{2}(\mathbb{R}^{□}) \subset □'(\mathbb{R}^{□})$$

you have two definitions of the Fourier Transform, one considering it as $□'$, the other is got by density namely the isometry P which we defined last time, I am using the Fourier Inversion formula and therefore, we want to know if these two are the same.

So, you have let $□ \in □^{2}(\mathbb{R}^{□})$ $□_{□} \in □(\mathbb{R}^{□})$ $□_{□} \to □\ □□\ □^{2}(\mathbb{R}^{□})$

$□^{2}(\mathbb{R}^{□}) \subset □'(\mathbb{R}^{□})$ $□_{□} \to □\ □□\ □'(\mathbb{R}^{□})$ $\int_{\mathbb{R}^{□}} □_{□}□\ □□ \to \int_{\mathbb{R}^{□}} □□\ □□$, $\varphi \in □(\mathbb{R}^{□})$

because Fourier Inversion formula so you can think of

$$\Rightarrow \widehat{\square_{\square}} \to \hat{\square} \text{ in } \square'(\mathbb{R}^{\square}) \quad \int_{\mathbb{R}^{\square}} \square_{\square}\hat{\square}\ \square\square \to \int_{\mathbb{R}^{\square}} \square\hat{\square}\ \square\square,\ \varphi \in \square(\mathbb{R}^{\square})$$

$$\Rightarrow \int_{\mathbb{R}^{\square}} \widehat{\square_{\square}}\square\ \square\square \to \int_{\mathbb{R}^{\square}} \hat{\square}\square\ \square\square,$$

$$\square(\square_{\square}) \to \square(\square) \text{ in } \square^{2}(\mathbb{R}^{\square})$$

$$\Rightarrow \square(\square_{\square}) \to \square(\square)\ \square\square\ \square'(\mathbb{R}^{\square})$$

But $\square(\square_{\square})$ is what? because this is the Fourier Transform

$$\Rightarrow \square(\square_{\square}) = \widehat{\square_{\square}} \to \square(\square) \Rightarrow \square(\square) = \hat{\square}.$$

So, that settles it. So, we will stop here, we will now do some more exercises before coming to an end in this chapter.