Sobolev Spaces and Partial Differential Equations
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Lecture 23
Fourier Inversion Formula

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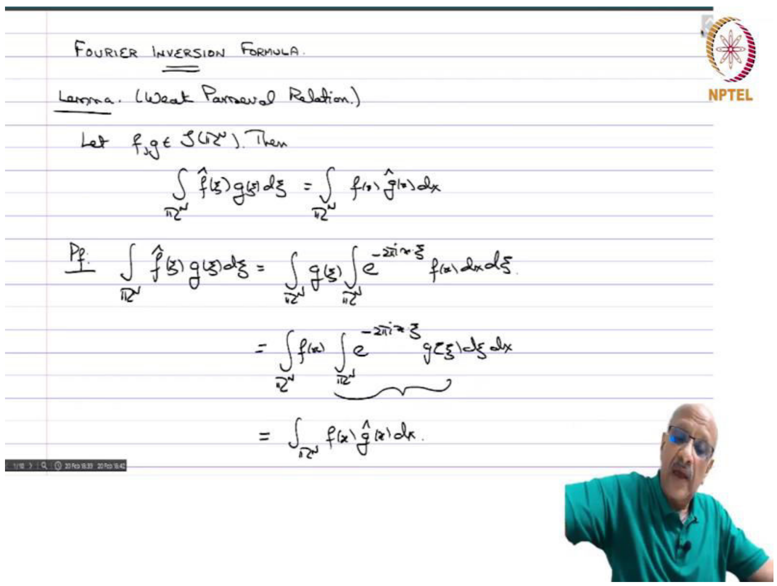

FOURIER INVERSION FORMULA:

Now, we will discuss the Fourier Inversion Formula. So, we have shown that the Fourier Transform maps S into S is a continuous linear map. And now we will show that in fact it is an isomorphism of the Schwartz space into itself. So,

Lemma:(Weak Parseval Relation)

So, Let $f, g \in S(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} \hat{f}(\xi) g(\xi)\, d\xi = \int_{\mathbb{R}^n} f(x) \hat{g}(x)\, dx$$

So, recall that if f and $g \in S(\mathbb{R}^n)$ so or $\hat{f}$ and $\hat{g}$ and $S \subset L^2$, so if you can always integrate the product of two functions, there is no problem. So,

proof. So, there is just an immediate consequence of Fubini's theorem, so you have

$$\int_{\mathbb{R}^n} \hat{f}(\xi) g(\xi)\, d\xi = \int_{\mathbb{R}^n} g(\xi) \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\, dx\, d\xi$$

Now, $|e^{-2\pi i x\cdot\xi}| \leq 1$, mod $|fg|$ are all L^1, L^2 and so on, so the product is R in L^1 and therefore the product is in integrable in the product measure and therefore by Fubini's theorem we can invert the order of the integration,

$$= \int_{\mathbb{R}^N} f(\xi) \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}\, g(x)\, dx\, d\xi$$

now this inner integral is nothing but the Fourier Transform of g in the variable x, that is all. So,

$$= \int_{\mathbb{R}^N} f(\xi)\widehat{g}(\xi)\, d\xi$$

and that completes the proof of this particular lemma. So, now we have the important theorem.

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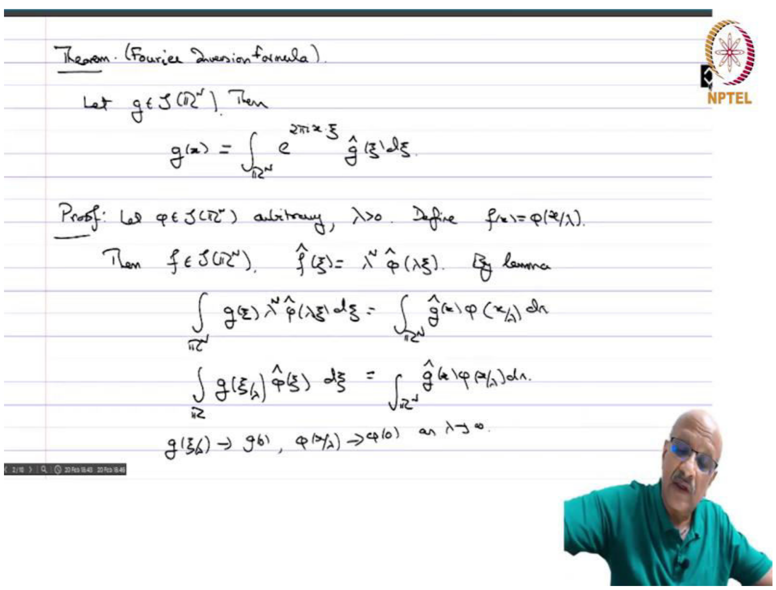

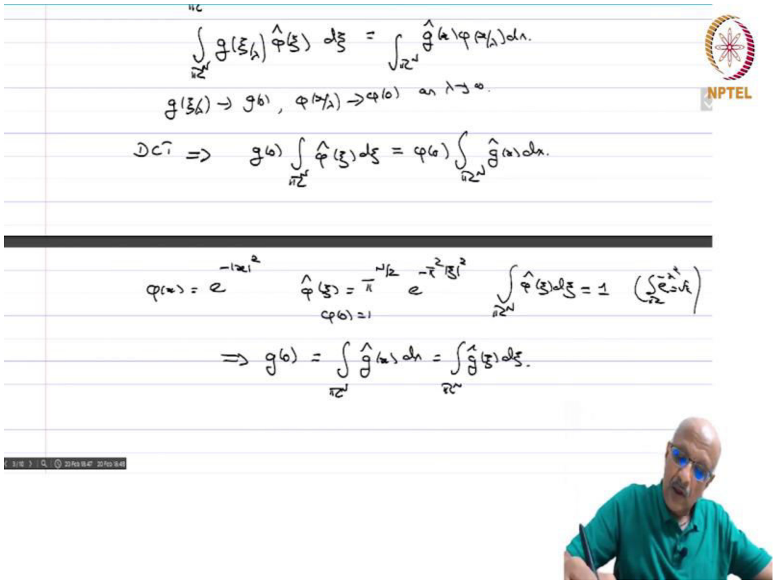

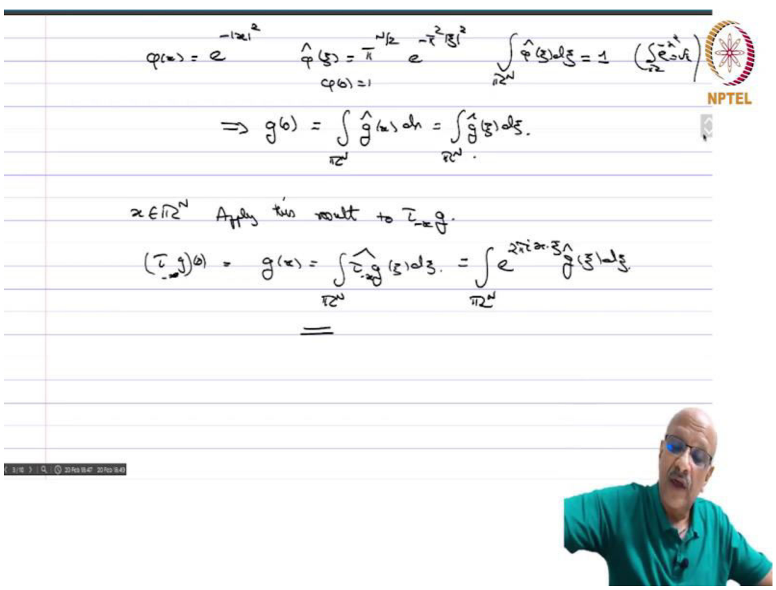

Theorem:(Fourier Inversion formula)

So, let $g \in \mathcal{S}(\mathbb{R}^N)$. Then

$$g(x) = \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\, \hat{g}(\xi)\, d\xi$$

can be expressed you can recover g from its Fourier Transform, so that is what it mean by Fourier inversion and that is a very beautiful Formula.

Proof:

Let $\varphi \in \square(\mathbb{R}^{\square})$ arbitrary and let $\lambda > 0$, Define

$$\square(\square) = \square(\frac{\square}{\square}).$$

Then $\square \in \square(\mathbb{R}^{\square})$.

no problem, you have only rescale the function, so it will continue to decrease rapidly at infinity and you have already seen what is a

$$\widehat{\square}(\square) = \square^{\square}\widehat{\square}(\square\square).$$

this is what we have already seen this was left as an exercise and we used it also later on.

So, now we apply the lemma, so by lemma namely the Weak Parseval Relation you have

$$\int_{\mathbb{R}^{\square}} \square(\square)\square^{\square}\widehat{\square}(\square\square)\;\square\square = \int_{\mathbb{R}^{\square}} \widehat{\square}(\square)\;\square(\frac{\square}{\square})\;\square\square$$

$$\int_{\mathbb{R}^{\square}} \square(\frac{\square}{\square})\widehat{\square}(\square)\;\square\square = \int_{\mathbb{R}^{\square}} \widehat{\square}(\square)\;\square(\frac{\square}{\square})\;\square\square$$

So,

$$\square(\frac{\square}{\square}) \to \square(0),\;\; \square(\frac{\square}{\square}) \to \square(0) \;\text{ as } \lambda \to \infty$$

So by the dominated convergence theorem, so once again the dominated convergence theorem implies that if you let

$$\varphi(\square) = \square^{-|\square|^2} \quad \widehat{\square}(\square) = \square^{\frac{\square}{2}}\square^{-\square^2|\square|^2} \qquad \int_{\mathbb{R}^{\square}} \widehat{\square}(\square)\;\square\square = 1 \qquad \int_{\mathbb{R}} \square^{-\square^2} = \sqrt{\square}$$

So, we have to you just use the expression and then make a change of variable here and then you will get this is true. So, if you apply all these things you get

$$\Rightarrow \square(0) = \int_{\mathbb{R}^{\square}} \widehat{\square}(\square)\;\square\square = \int_{\mathbb{R}^{\square}} \widehat{\square}(\square)\;\square\square.$$

So, now, what do you get you have to take for a if

$\square \in \mathbb{R}^{\square}$ then apply this result to $\tau_{-\square}\square$.

$$\tau_{-x} g(0) = g(x) = \int_{\mathbb{R}^n} \widehat{\tau_{-x} g}(\xi)\, d\xi = \int_{\mathbb{R}^n} e^{2\pi i x.\xi} \hat{g}(\xi)\, d\xi.$$

which again the properties of the Fourier Transform we saw how the Fourier Transform behaves with respect to translation. So, this completes the proof of the Fourier inversion formula.

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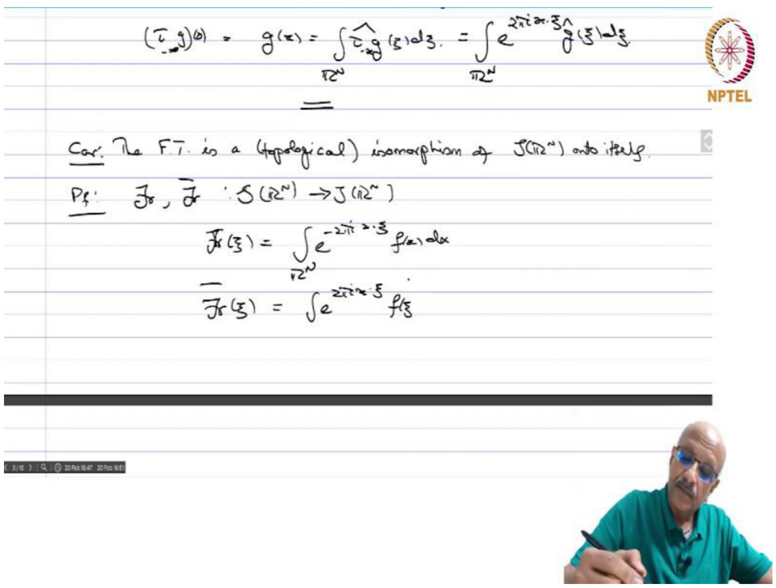

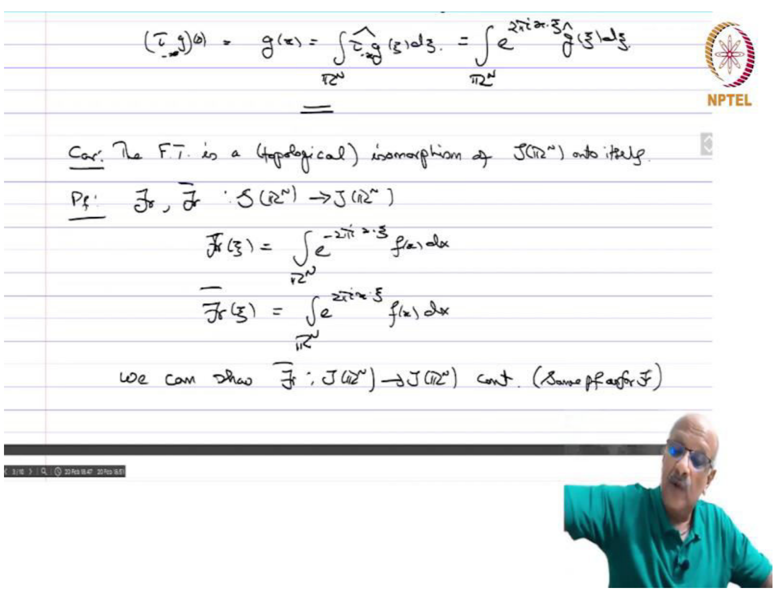

Corollary: the Fourier Transform is topological isomorphism of $\mathcal{S}(\mathbb{R}^n)$ on to itself. So, we have proof, we have two maps one is the Fourier Transform which is and the other is

$$\mathcal{F},\ \overline{\mathcal{F}}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x.\xi} f(x)\, dx$$

$$\underline{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x.\xi} f(x)\, dx$$

So, we can show again $\hat{f}, \underline{f}: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$

continuous, same proof as for f, f is the usual Fourier Transform we saw this that it maps $\mathcal{S}(\mathbb{R}^n)$ into itself there is nothing changes because of the sign change in the exponential, nothing at all changes that is still the modulus one, everything will go through, so identical proof there is no change in the proof, so both of them are continuous.

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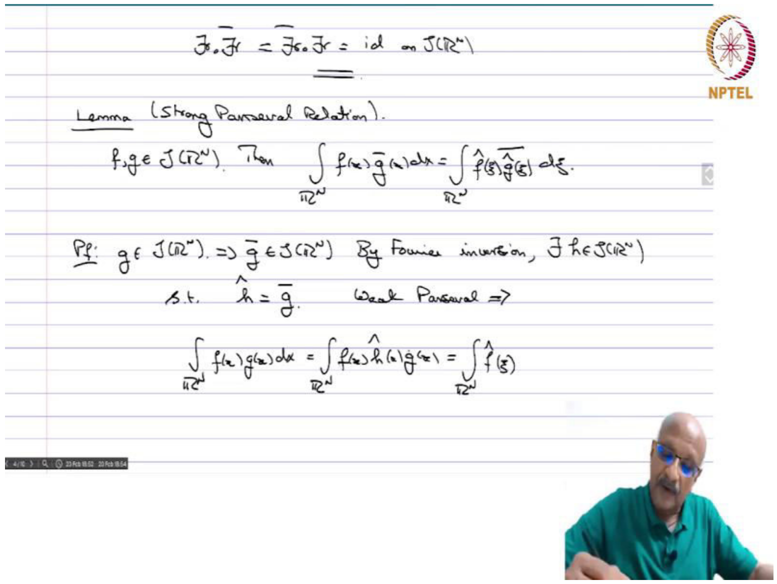

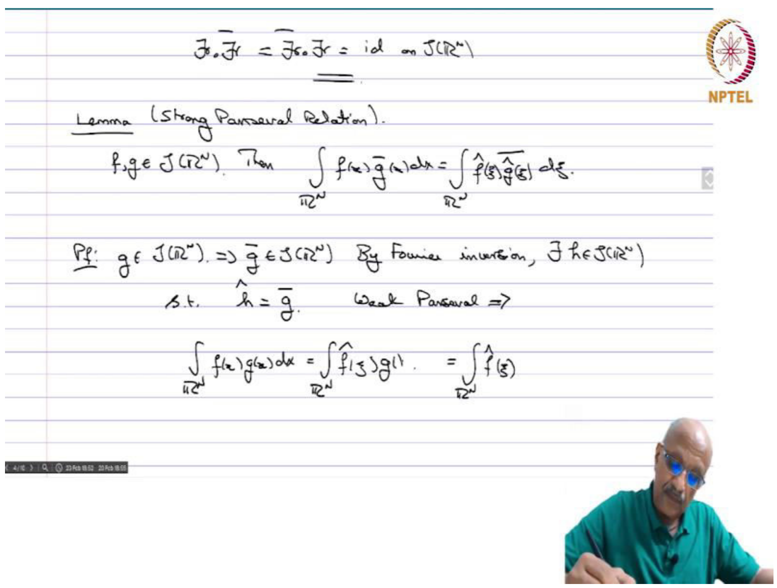

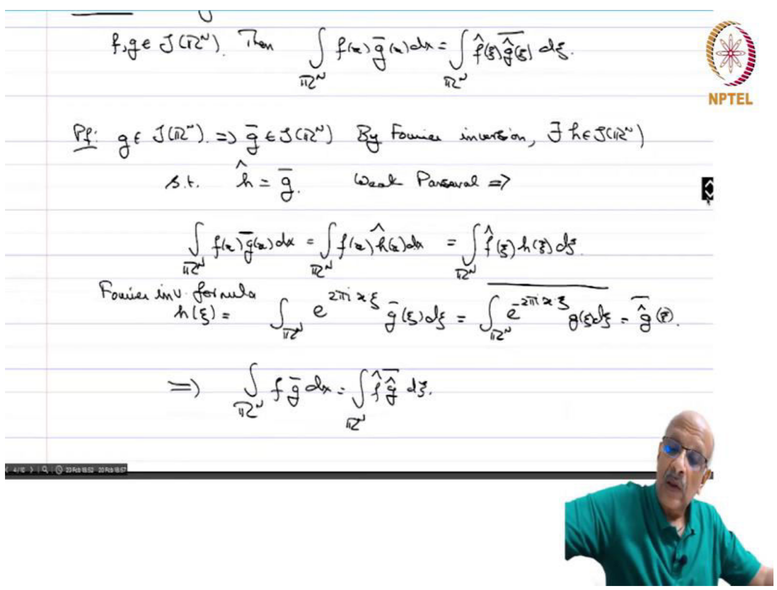

And also, we have that $\mathcal{F} \circ \overline{\mathcal{F}} = \overline{\mathcal{F}} \circ \mathcal{F} = I$ and that proves the theory composed with F bam, so this completes the proof. Now, we can strengthen this colliery further.

Lemma:(Strong Parseval Relationship)

$f, g \in \mathcal{S}(\mathbb{R}^n)$.

then $\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$.

proof. So, $g \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \overline{g} \in \mathcal{S}(\mathbb{R}^n)$.

and nothing changes in the decreasing decay properties of that function and therefore by Fourier inversion there exists a

$\exists h \in \mathcal{S}(\mathbb{R}^n)$, such that $\hat{h} = \overline{g}$.

then by the Weak Parseval, what does it imply? That says the

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} f(x)\widehat{h(x)}\, dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\, h(\xi)\, d\xi.$$

But, what is $h(\xi)$? By the definition of the Fourier Transform

$$h(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x.\xi}\,\overline{g}(x)\, dx = \overline{\int_{\mathbb{R}^n} e^{-2\pi i x.\xi} g(x)\, dx} = \overline{\hat{g}(\xi)}.$$

and that is what is under the conjugate sign is nothing but the Fourier Transform of g, And therefore, you have

$$\Rightarrow \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$$

so that proves the Strong Parseval Relation.

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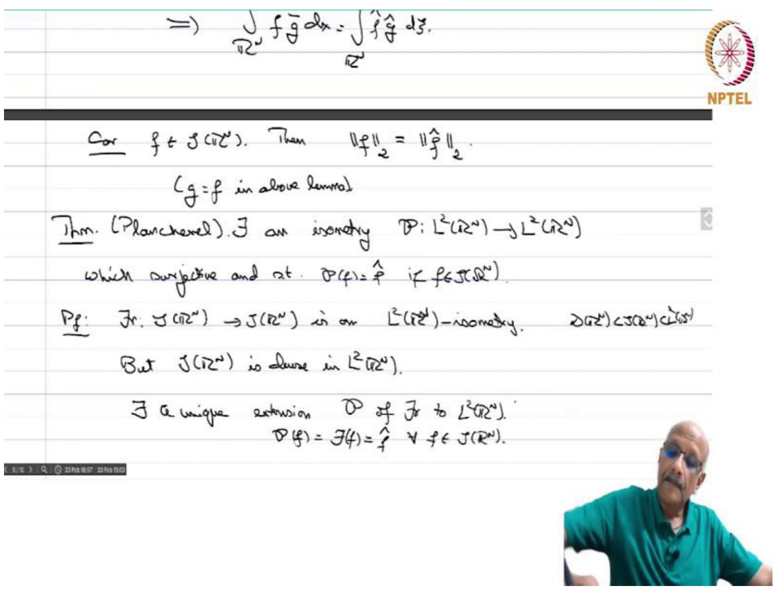

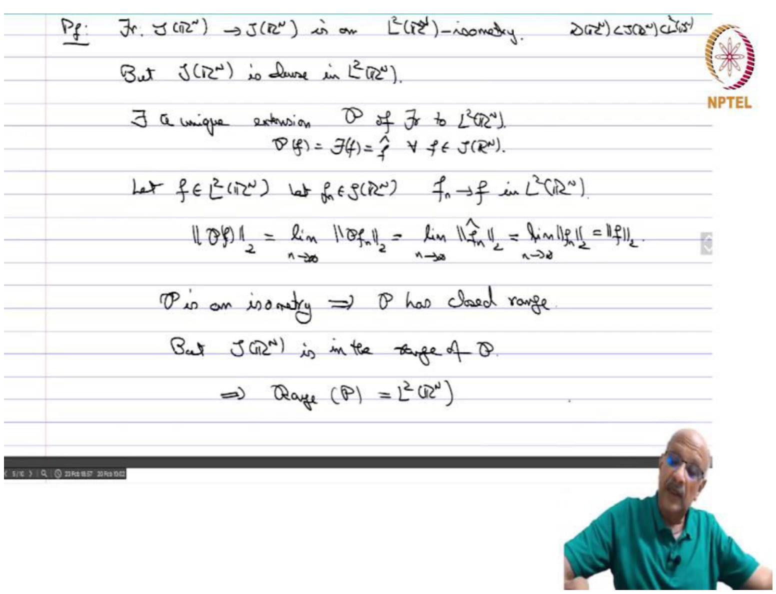

Corollary:

□ ∈ □(ℝ$^{□}$), Then $||□||_2 = ||\hat{□}||_2$

that is obvious put □ = □ in above lemma. So, $||□||_2 = ||\hat{□}||_2$ this leads to the following important theorem. So,

Theorem (Plancherel's theorem): so there exists an isometry, isometry means a linear continuous linear map which preserves a norm, norm

□: □^{2}(ℝ$^{□}$) → □^{2}(ℝ$^{□}$)

which surjective and such that □(□) = $\hat{□}$ if □ ∈ □(ℝ$^{□}$)

So, you can extend the Fourier Transform to □^{2} norm. So, this is what this theorem does. So, we could prove only for □^{1} to be defined the Fourier Transform for □^{1}, then we went to the Schwartz space we prove this Corollary above it says that the Fourier Transform is □^{2} isometry on the Schwartz space and from that we are now going to show that you can extend it to a isometry on L. So, this we call the Fourier Transform on □^{2}.

So, the Fourier Transform, so

proof,

□: □(ℝ$^{□}$) → □(ℝ$^{□}$) is an □^{2} isometry. □(ℝ$^{□}$) ⊂ □(ℝ$^{□}$) ⊂ □^{2}(ℝ$^{□}$)

isometry, but □(ℝ$^{□}$) is dense in □^{2}(ℝ$^{□}$) again this is because □(ℝ$^{□}$) is contained in the □(ℝ$^{□}$) which is contained in □^{2}(ℝ$^{□}$), therefore this is dense we know, so the □(ℝ$^{□}$) is automatically dense □^{2}(ℝ$^{□}$). So, let so implies there exists a unique extension □ □□ □ □□ □^{2}(ℝ$^{□}$).

So, this means what? That means,

□(□) = □(□) = $\hat{□}$ ∀□ ∈ □(ℝ$^{□}$).

Let □ ∈ □^{2}(ℝ$^{□}$) let $□_{□}$ ∈ □(ℝ$^{□}$) $□_{□}$ → □ in □^{2}(ℝ$^{□}$)

$$||\square(\square)||_2 = \underset{\square\to\infty}{\square\square\square}||\square\square_{\square}||_2 = \underset{\square\to\infty}{\square\square\square}||\widehat{\square_{\square}}||_2 = \underset{\square\to\infty}{\square\square\square}||\square_{\square}||_2 = ||\square||_2$$

So, P is an isometry and it implies that P has closed range, but $\square(\mathbb{R}^{\square})$ is in the range of $\square$ because $\square$ is a isometry from a isomorphism $\square(\mathbb{R}^{\square})$ into and $\square$ restricted $\square(\mathbb{R}^{\square})$ is nothing but $\square$ and therefore $\square(\mathbb{R}^{\square})$ is in the range of $\square$ and therefore you have that

$$\Rightarrow \square\square\square\square\square(\square) = \square^{2}(\mathbb{R}^{\square})$$

It is closed range and consist contains a dense subspace and therefore it has to be the whole space. So, this completes the theorem so it is a surjective map and so an isometric.

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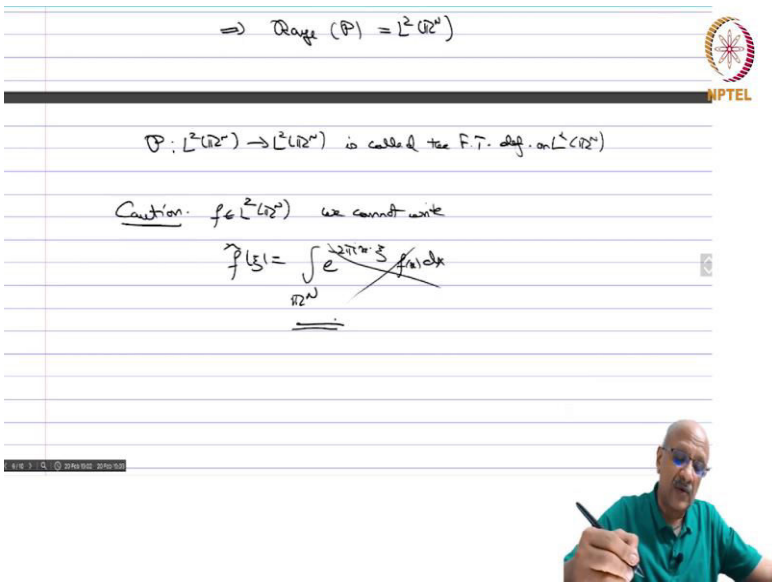

So, $\mathcal{P}: L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$

is called the Fourier Transform defined on it. So,

Caution:

If $f \in L^2(\mathbb{R}^N)$ we cannot write

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \xi} f(x)\, dx$$ because this integral

is not well defined, because if $f \in L^2(\mathbb{R}^N)$ it is only defined if $f \in L^1(\mathbb{R}^N)$. But we have extended it by density from $\mathcal{S}(\mathbb{R}^N)$ into a uniquely to the whole of $L^2(\mathbb{R}^N)$ as an isometry, so that you have to know. So, we have our next thing is to now use all these results to extend it to a class of distributions.