Sobolev Spaces and Partial Differential Equations
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Lecture 22
Examples – Part 1

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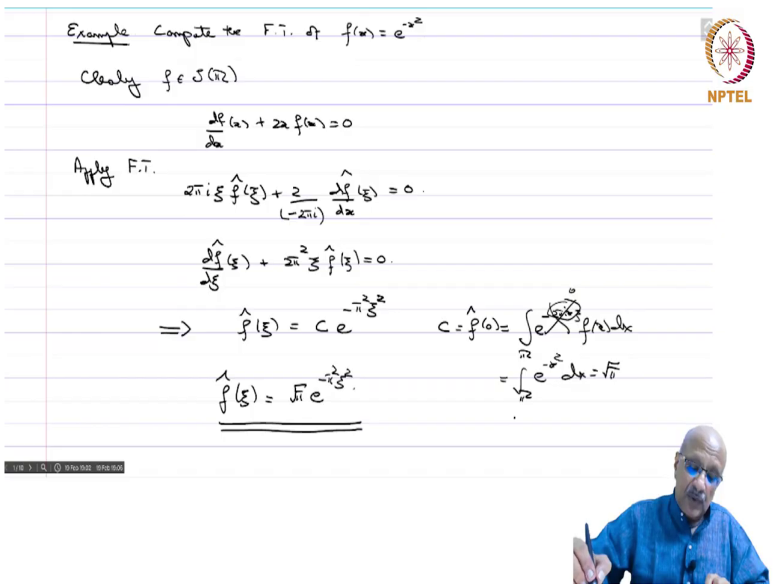

So, let us apply this theorem which we proved on the Schwartz space, so first we will start, so **Example:**

Compute the Fourier Transform of $f(x) = e^{-x^2}$?

so this is what we left last time as incomplete namely we I brought it to the count to the integral and ask you to do it and promise you that I will do it in a different way, so now we will do it.

So clearly, we have already seen $f \in S(\mathbb{R})$, this is a standard prototype example we have, now

$$\frac{df}{dx} + 2xf(x) = 0$$

because if you just differentiate it, you get $\frac{df}{dx} = -\ 2xe^{x^2}$

so, $\frac{df}{dx} + 2xf(x) = 0$. So, $f \in S(\mathbb{R})$, $xf \in S(\mathbb{R})$, so you can now apply the Fourier Transform, apply Fourier Transform.

So, what we get? I am differentiating, which means I must get

$$2\pi i\xi\hat{f}(\xi) + \frac{2}{(-2\pi i)}\frac{d\hat{f}}{d\xi}(\xi) = 0$$

the Fourier Transform I mean derivative of the Fourier Transform,

$$\frac{d\hat{f}}{d\xi}(\xi) + \quad 2\pi^2\xi\hat{f}(\xi) = 0$$

So, from this, we can easily evaluate, we know how to solve this linear differential equation multiply by the integrating factor. So

$$\hat{f}(\xi) = Ce^{-\pi^2\xi^2} \qquad C = \hat{f}(0) = \int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$$

$$\Rightarrow \hat{f}(\xi) = \sqrt{\pi}e^{-\pi^2\xi^2}.$$

So, this is a very simple application of the previous theorem which we proved to compute the thing.

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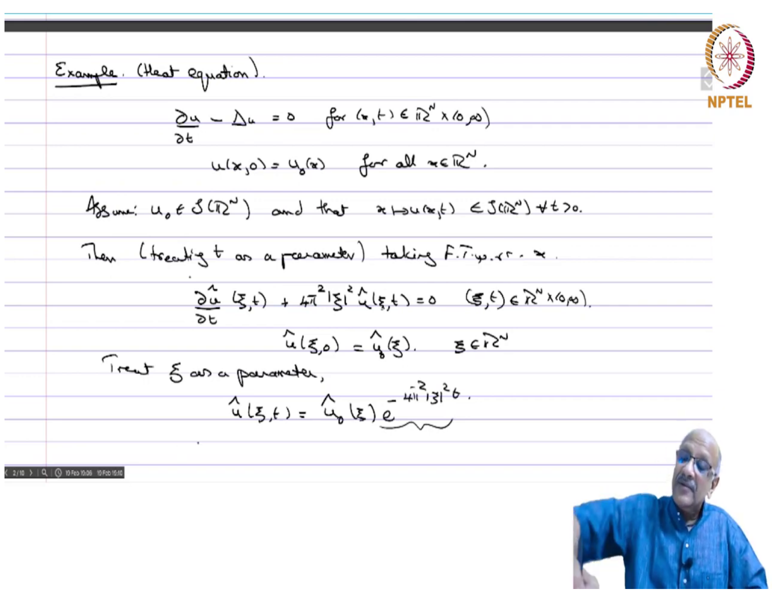

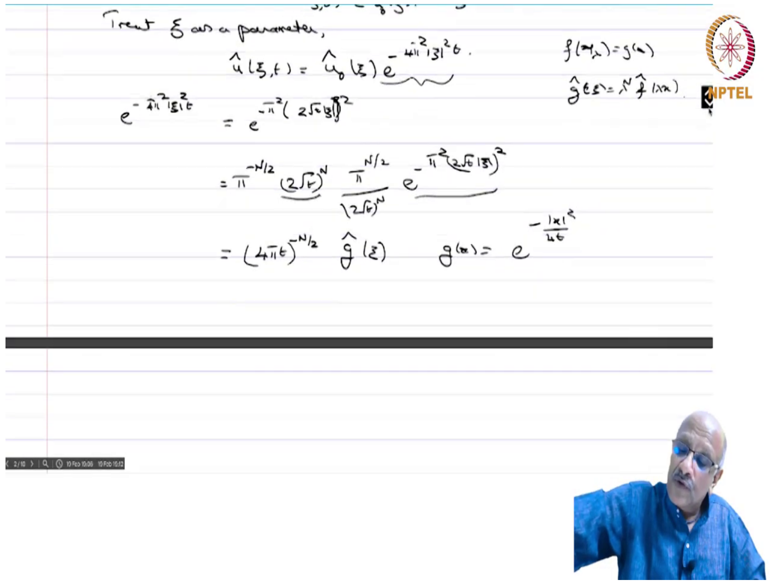

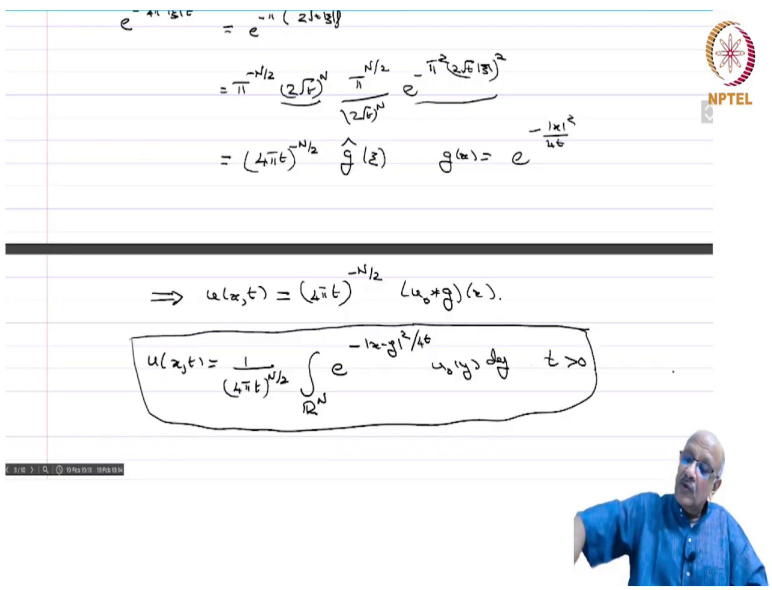

We will do one more example, and this is the heat equation.

Example (Heat equation)

So, we consider the following initial value problem

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{for } (x, t) \in \mathbb{R}^N \times (0, \infty)$$

$$u(x, t) = u_0(x)$$

So, this tells you if you have an infinite bar which is kept initially at temperature at u_0 and then the heat is allowed to diffuse according to the law of diffusion then the time temperature at any time t in the bar in the infinite bar will give you the is given by the solution to this problem.

So, let us assume, so assume $u_0 \in S(\mathbb{R}^N)$, the initial data is very good, and that

x going to $u(x, t) \in S(\mathbb{R}^N) \quad \forall t > 0$

So, then you can then we get treating t as a parameter taking the Fourier Transform the

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) + 4\pi^2|\xi|^2\hat{u}(\xi, t) = 0 \quad (x, t) \in \mathbb{R}^N \times (0, \infty)$$

So, t is just a dummy here, so you just get that plus so you have Laplacian, so if you use law for the differentiate Fourier Transform of a derivative each time you pick up a

$$\hat{u}(\xi, 0) = \hat{u}_0(\xi) \quad \xi \in \mathbb{R}^N.$$

So, now you just solve this treat t as a parameter and just solve this equation I mean treat xi as a parameter, so you and take here and taking Fourier Transform with respect to x. Now, your treat ξ as a parameter, then what do you get you have a ordinary differential equation with a certain initial value as far as

$$\hat{u}(\xi, t) = \hat{u}_0(\xi)e^{-4\pi^2|\xi|^2 t} \quad \xi \in \mathbb{R}^N.$$

$$e^{-4\pi^2|\xi|^2 t} = e^{-\pi^2(2\sqrt{t}|\xi|\ )^2} \qquad f(\tfrac{x}{\lambda}) = g(x); \quad \hat{g}(\xi) = \lambda^N \hat{f}(\lambda\xi)$$

So, now we want to recall two facts, one is if you have the product of two Fourier Transform that is a Fourier Transform of the convolution, so we will and secondly, we know that the exponential. So, now we are trying to write try to write this as the Fourier Transform of a suitable function. So, let us do that.

$$= \pi^{-\frac{N}{2}}(2\sqrt{t})\,\frac{\pi^{\frac{N}{2}}}{(2\sqrt{t})}\,e^{-\pi^2(2\sqrt{t}|\xi|\ )^2}$$

$$= (4\pi t)^{-\frac{N}{2}}\hat{g}(\xi) \qquad g(x) = e^{-\frac{|x|^2}{2t}}$$

So, hence so you have this as the convolution product of two convolutions, so from this I mean product of two Fourier Transforms and that is the Fourier Transform the convolution and therefore you have

$$\Rightarrow u(x, t) = (4\pi t)^{-\frac{N}{2}}(u_0 \star g)(x)$$

u must be the convolution of these two functions. So

$$u(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}\int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy \ \ , t > 0 \qquad \text{.... (*).}$$

this is a very familiar formula you might have already seen when studying PDE somewhere else,

So, this is the formula for the solution of the heat equation assuming of course that you have u_0 is an S and u(x,t) is also in S, so and you do have a solution, therefore if you have u_0 in S you know u can be written in S and this is the solution to this.

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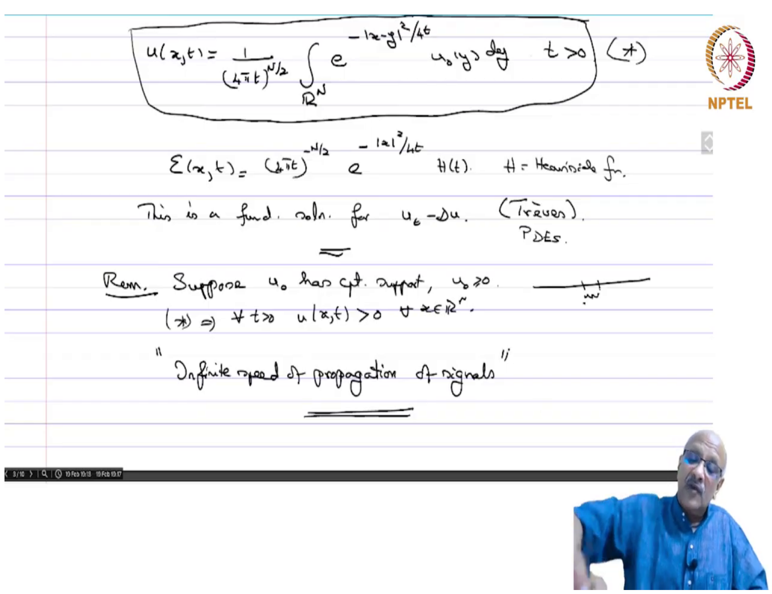

Now, if you consider

$$E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} H(t)$$

this is a fundamental solution for ut minus delta u, the heat operator, that can be so proof can be found in the book of (())(13:05) partial differential equations. So, now this is somewhat so if you take the convolution of this, so here we are taking only for t>0, and therefore you had $H(t) = Heavy\, side\, function$.

Remark: suppose u_0 has compact support, and say $u_0 \geq 0$, then so call the (*),

$$(*) \Rightarrow \forall t > 0,\ u(x,t) > 0,\ \ \forall x \in \mathbb{R}^N$$

so this is called, so you have this suppose you were in real line and you have u_0 is positive and it has support here, instantaneously the heat is felt throughout the rod, to seems counterintuitive, but it is true, of course one can show that it is in fact it be very small as you go far away, but still it will be non-negative.

And so instantaneously the heat diffuses throughout the thing, so this is called the infinite speed of propagation of signals, there is a technical term NPD, unlike the wave equation which will have a finite speed that means if you have initial data of compact support the solution at any time t will also have compact support.

So, you if you switch on the light, it will not become bright everywhere, it will only be bright inside the light cone, we will see that later on. So, but in case of the heat operator, the moment you have heating instantaneously for every positive time, you can feel the heat everywhere, of course not very hot, but still you can feel it. So now we will continue with the Fourier Transform and how one can generalize it to other class of functions.