Sobolev Spaces and Partial Differential Equations
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The Schwarz space - Part 2

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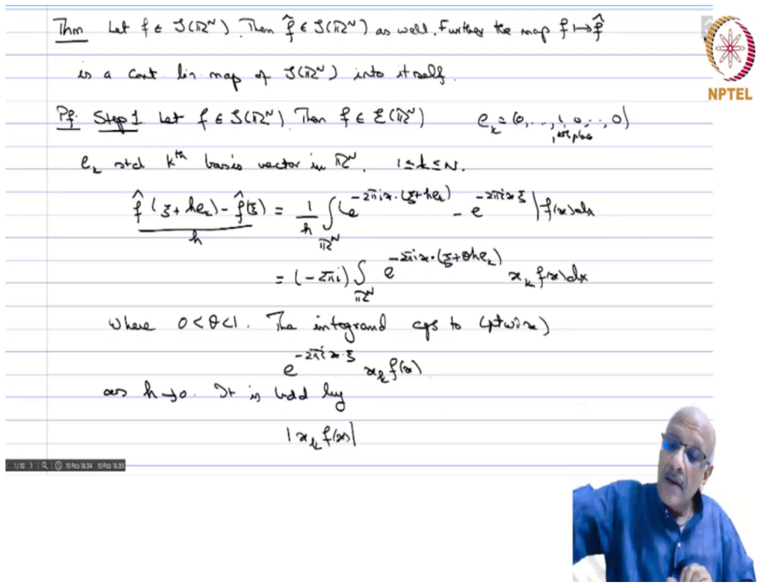

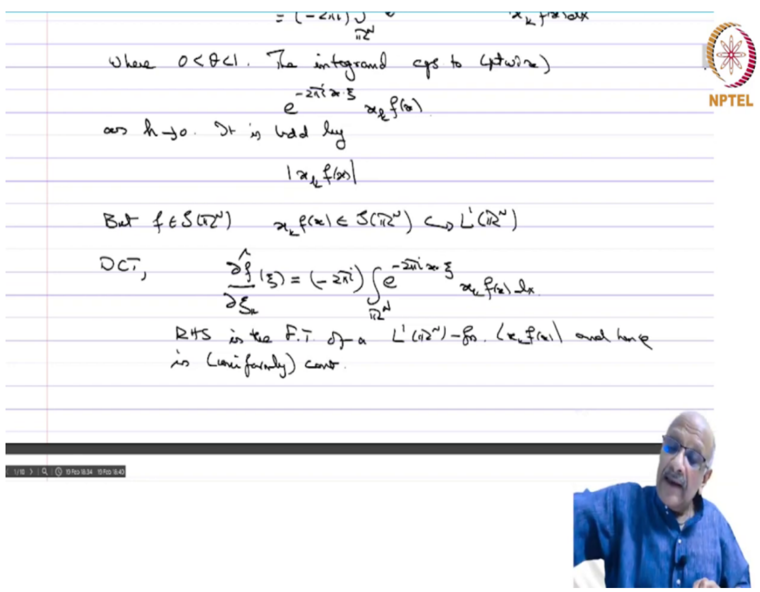

We were discussing the short space and then we said that it is very congenial with respect to the Fourier transform. So, let us prove the following theorem.

Theorem: Let $f \in S(\mathbb{R}^N)$ then $\hat{f} \in S(\mathbb{R}^N)$ as well, further the map $f \to \hat{f}$ is a continuous linear map of $S(\mathbb{R}^N)$ into itself. So,

proof. So, we will do it in some steps.

Step 1: let $f \in S(\mathbb{R}^N) \Rightarrow f \in \mathcal{E}(\mathbb{R}^N)$. So, as the first step is going to belong to $S(\mathbb{R}^N)$it better belong to $\mathcal{E}(\mathbb{R}^N)$ So, we are going to show first that if $\mathcal{E}(\mathbb{R}^N)$. So, $e_k$is the standard k basis vector in $\mathbb{R}^N$. So, what is e_k?

$$e_k = (0, 0, 0\ldots 1\ldots, 0, 0)\ ,\ 1\ in\ the\ kth\ position,\quad 1 \le k \le N.$$

Then

$$\frac{\hat{f}(\xi + he_k) - \hat{f}(\xi)}{h} = \frac{1}{h}\int_{\mathbb{R}^N}\left(e^{-2\pi i x\cdot(\xi + he_k)} - e^{-2\pi i x\cdot\xi}\right)f(x)\,dx$$

and that is equal to we apply the mean value theorem to this differentiable function.

So, what is this is can be taken as the derivative at some intermediary point so, what is the derivative? That is

$$= (-\ 2\pi i)\int_{\mathbb{R}^N} e^{-2\pi i x\cdot(\xi + \theta he_k)} x_k f(x)\,dx$$

So, this is just differentiated $e^{-2\pi i x\cdot\xi}x_k f(x)$ with respect to the k th values is x_k. So, I get the $e^{-2\pi i x\cdot\xi}x_k f(x)$ here and that is what we have here. Where $0 < \theta < 1$and it will depend on x and h.

So, now the integrand converges to that is point wise $e^{-2\pi i x\cdot\xi}x_k f(x)$ as $h \to 0$ and it is bounded by so, mod less of this exponential here is just 1 so by $|x_k f(x)|$.

But $f \in S(\mathbb{R}^N)$. So, $x_k f(x) \in S(\mathbb{R}^N)$ just multiplying by a monomial and that of course is continuously embedded in the $L^1(\mathbb{R}^N)$. So, this is integrable it converges point wise. So, by the dominated convergence theorem, you have that the limit.

So, if you take the limit on the left hand side that will give you

$$\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-2\pi i) \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} x_k f(x)\, dx$$

as we already saw belongs to $S(\mathbb{R}^N)$. So, that is in $L^1(\mathbb{R}^N)$. So, this is the Fourier transform of some function. So, RHS is the Fourier transform of L^1 function namely $x_k f(x)$ and hence is in fact uniformly that is not necessarily for the moment continuous.

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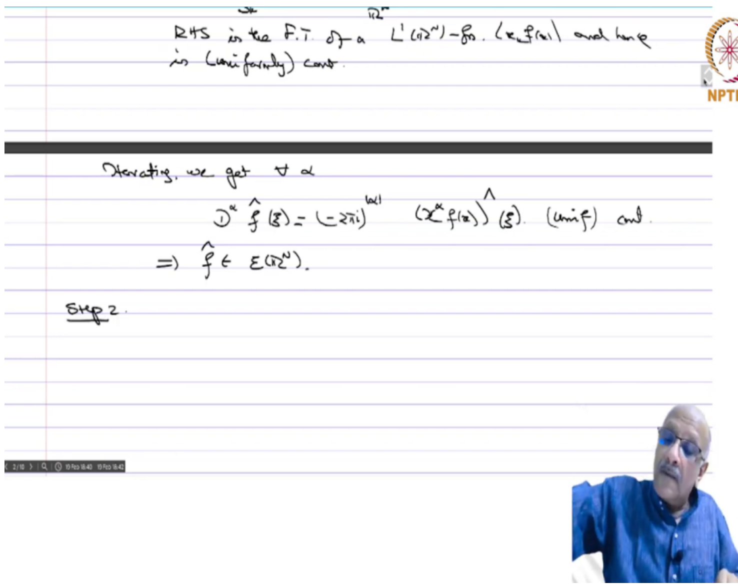

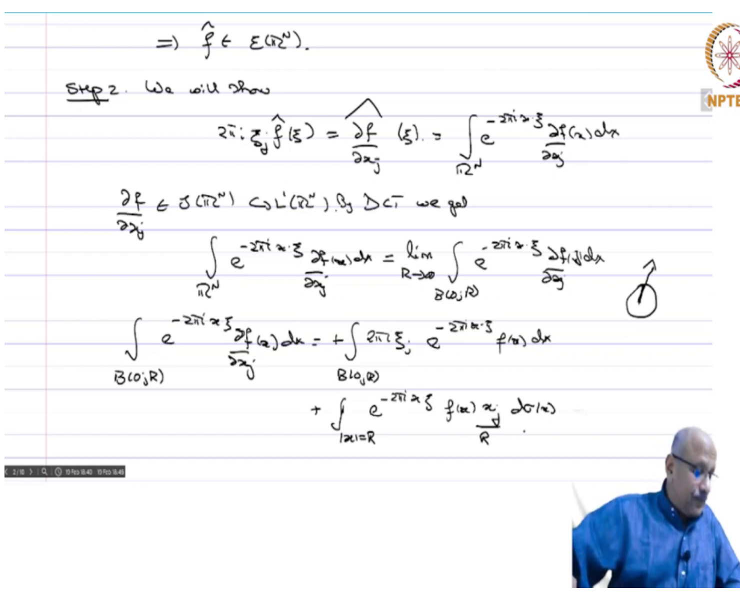

So, we can iterate this, iterating we get for all

$\forall\alpha$ multi index you have $D^{\alpha}\hat{f}(\xi) = (-\,2\pi i)^{|\alpha|} \quad \widehat{(x^{\alpha}f(x))}\,(\xi)$

So, if you want to differentiate the Fourier transform, you first multiply the function by a corresponding monomial and then take the Fourier transform that is now $\mathcal{E}(\mathbb{R}^N)$, we have the dual. So, this implies and this is uniformly continuous to this $\Rightarrow \hat{f} \in \mathcal{E}(\mathbb{R}^N)$.

Step 2, so we will show

$$2\pi i \xi_j \hat{f}(\xi) = \widehat{\frac{\partial f}{\partial x_j}}(\xi)$$

So, this is nothing which is by definition, because it is just the $\frac{\partial f}{\partial x_j}(x)$ function. So, this is again

$$= \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} \frac{\partial f}{\partial x_j}(x)\, dx$$

So, you see the duality here. So, when you want to differentiate the Fourier transform you multiply the function by a monomial and then take the Fourier transform.

If you want to differentiate the function and take the Fourier transform then you take the Fourier transform multiply by appropriate monomials. So, this is the duality between these two properties and so. Now, this is essentially a consequence of Greens theorem or integration by parts. So, we have

$$\frac{\partial f}{\partial x_j}(x) \in S(\mathbb{R}^N) \subset L^1(\mathbb{R}^N).$$

And so, we have by the dominated convergence theorem we get, so by the dominated convergence theorem, so you get the

$$\int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} \frac{\partial f}{\partial x_j}(x)\, dx = \lim_{R\to\infty} \int_{B(0;R)} e^{-2\pi i x\cdot\xi} \frac{\partial f}{\partial x_j}(x)\, dx$$

Because you think of this function as integral over $\mathbb{R}^N$ with $\chi_{B(0;R)}$multiplied here. Then point wise it converges to is it is bounded by a integrable function.

So, by the dominated convergence theorem, you get this. So, we just have to look at this bounded domain integral and see what the limit is going to be. So,

$$\int_{B(0;R)} e^{-2\pi i x\cdot\xi} \frac{\partial f}{\partial x_j}(x)\, dx = \int_{B(0;R)} e^{-2\pi i x\cdot\xi} \frac{\partial f}{\partial x_j}(x)\, dx$$

so I am going to differentiate with respect to. So, we have to differentiate.

So, you get a minus sign first when you changing the derivative on the side, but when you bring back the when you differentiate you get $2\pi i$ and you have differentiate with respect to x_j.

$$=+\int_{B(0;R)} 2\pi i \xi_j e^{-2\pi i x\cdot\xi} f(x)\, dx + \int_{|x|=R} e^{-2\pi i x\cdot\xi} f(x) \frac{x_j}{R}\, dx$$

So, this is what you get. So, that is what you will get? When you do the plus you have a boundary term. Now, what is the boundary term?

That is the normal outward normal in the j direction, but this is a ball center origin and therefore the normal is just the radius vector and you have to divide it, to make it the unit normal divided by the length of the radius. So, this is what you will get when you write Greens theorem.

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$|x|=R \qquad R$

$$\left| \frac{1}{R} \int_{|x|=R} e^{-2\pi i x\cdot\xi} f(x)\, x_j \, d\sigma(x) \right| \le \frac{1}{R} \int_{|x|=R} |x_j|\, |f(x)| \, d\sigma(x)$$

$$\le \frac{M_k}{R} \int_{|x|=R} |x_j| (1+|x|^2)^{-k} \, d\sigma(x)$$

$$|f(x)|(1+|x|^2)^k \le M_k. \quad (f \in \mathcal{S}(\mathbb{R}^N))$$

But
$$\frac{M_k}{R} \int_{|x|=R} |x_j| (1+|x|^2)^{-k} d\sigma(x) = \frac{M_k R^{N-1}}{(1+R^2)^k} \underbrace{\int_{|y|=1} |y_j|\, d\sigma(y)}$$
$\frac{x}{R} = y$

$\longrightarrow 0$ as $R \to \infty$ if k large enough.

$\longrightarrow 0$ as $R \to \infty$ if k large enough.

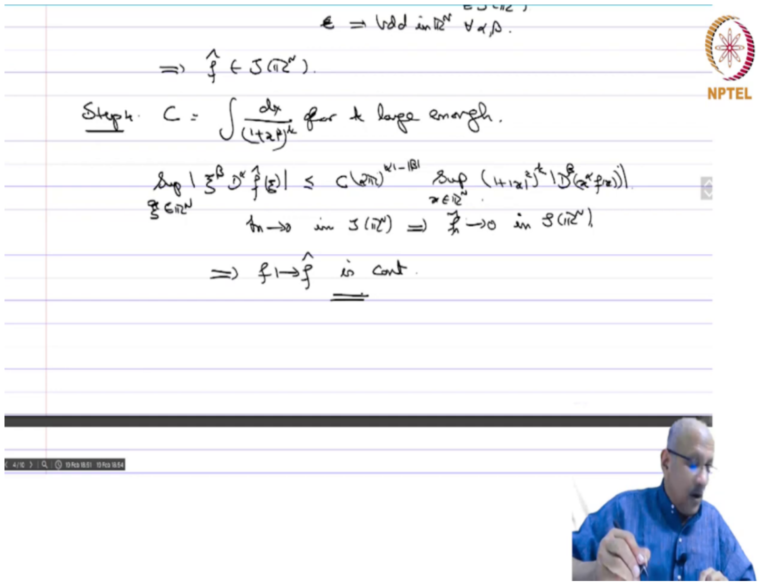
$$\Rightarrow \quad 2\pi i\, \xi_j \hat{f}(\xi) = \widehat{\frac{\partial f}{\partial x_j}}(\xi).$$

<u>Step 3.</u> Iterating
$$(2\pi i \xi)^\beta \hat{f}(\xi) = \widehat{D^\beta f}(\xi)$$

Combining this with step 1.

$$\xi^\beta D^\alpha \hat{f}(\xi) = \frac{(-2\pi i)^{|\alpha|}}{(2\pi i)^{|\beta|}} \underbrace{\left(D^\beta (x^\alpha f(x)) \right)}_{\in \mathcal{S}(\mathbb{R}^N)}{}^{\wedge}(\xi).$$

$\Rightarrow$ bdd in $\mathbb{R}^N$ $\forall\, \alpha, \beta$.



So, now, we will estimate this boundary term. So,

$$\left|\frac{1}{R}\int_{|x|=R} e^{-2\pi i x\cdot\xi} f(x)x_j\, d\sigma(x)\right| \le \frac{1}{R}\int_{|x|=R} |x_j|\,|f(x)|\; d\sigma(x)$$

Now, we are going to write this is less than or equal to

$$\le \frac{M_k}{R}\int_{|x|=R} |x_j|\,(1\,+\,|x|^2)^{-k}\, d\sigma(x)$$

What am I done I multiplied by $(1\,+\,|x|^2)^k$ and $(1\,+\,|x|^2)^{-k}$ and we have that

$$|f(x)|(1\,+\,|x|^2)^k \le M_k$$

This is the Schwarz space properties since $f\in S(\mathbb{R}^N)$. and you can choose any k you like. So, this constant k number k is independent. Now, we can choose whatever we like and therefore, you have. But what do you know about

$$\frac{M_k}{R}\int_{|x|=R}|x_j|\,(1+|x|^2)^{-k}\,d\sigma(x)=\frac{M_kR^{N-1}}{(1+R^2)^k}\int_{|y|=1}|y_j|\;\;d\sigma(y)\;\;\text{put}\;\;\frac{x}{R}=y$$

So, now this is a constant $|y_j|$ is integrable function and then this will

$$\to 0\quad as\quad R\to\infty\;\; if\;\;\; k\; large\; enough,$$

and we have the freedom to choose k and therefore, we choose k as large as you need. So, this boundary term goes to 0 and then therefore you get, therefore you have proved this result here.

$$2\pi i\xi_j\hat{f}(\xi)=\widehat{\frac{\partial f}{\partial x_j}}(\xi).$$

So, we as R tend to infinity we have that.

Step 3. So, iterating we get

$$(2\pi i\xi)^{\beta}\;\hat{f}(\xi)=\widehat{D^{\beta}f}(\xi)$$

So, combining this with step 1, so you get the

$$\xi^{\beta}D^{\alpha}\hat{f}(\xi)=\frac{(-2\pi i)^{|\alpha|}}{(2\pi i)^{\beta}}\;\;\widehat{(D^{\beta}(x^{\alpha}f(x)))}(\xi)$$

And this again is in $S(\mathbb{R}^N)$ and therefore, this is

$\Rightarrow$bonded in $\mathbb{R}^N$, $\forall\ \alpha,\beta$ and this proofs. So, this

$$\hat{f}\in S(\mathbb{R}^{N})$$

Step 4:

$$C=\int\frac{dx}{(1+|x|^2)^k}\;\;\text{for k large enough.}$$

$$sup_{\xi \in \mathbb{R}^N} \left| \xi^{\beta} D^{\alpha} \hat{f}(\xi) \right| \leq C(2\pi i)^{|\alpha| - |\beta|} \; sup_{x \in \mathbb{R}^N} (1 + |x|^2)^k \left| D^{\beta} (x^{\alpha} f(x)) \right|$$

and therefore,

$$\Rightarrow f_n \rightarrow 0 \;\; in \;\; S(\mathbb{R}^N) \;\; \Rightarrow \hat{f}_n \rightarrow 0 \;\; in \;\; S(\mathbb{R}^N)$$

$\Rightarrow f$ map to $\hat{f}$ is continuous.

So, that proves that theorem continuous.

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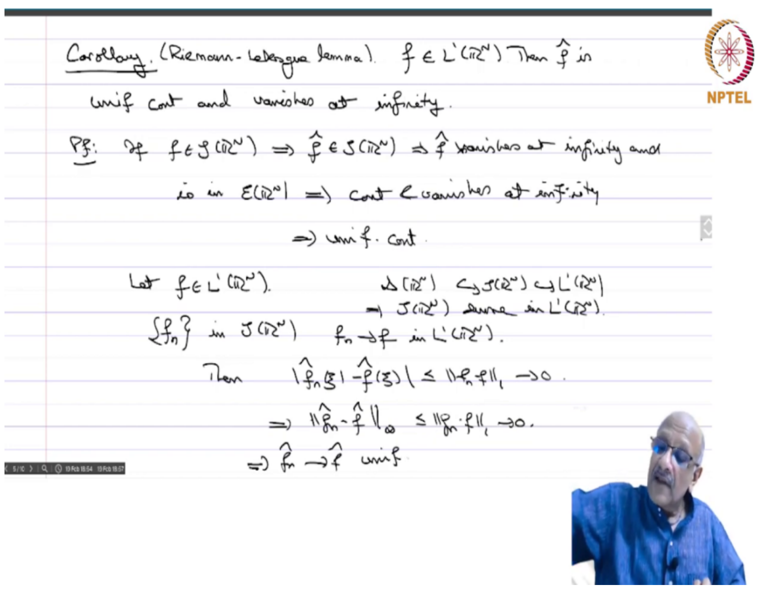

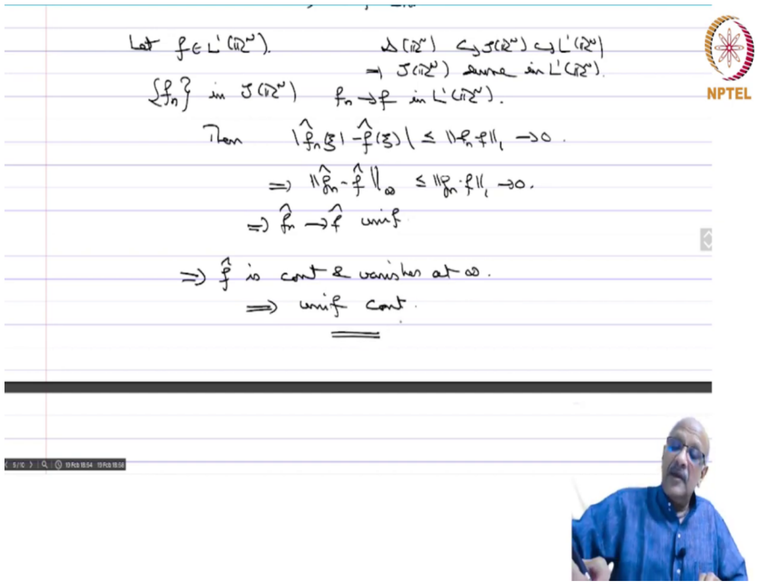

So, now we have the following corollary.

Corollary:(Rieman Lebesgue lemma): This is called the Riemann Lebesgue lemma, and there are several versions of this Riemann Lebesgue lemma. So, this is one way of stating it. So,

$f \in L^1(\mathbb{R}^N)$ then $\hat{f}$ is uniformly continuous this we already saw and vanishes set infinity. So,

Proof: So, let if $f \in S(\mathbb{R}^N)$this we have just shown that

$\hat{f} \in S(\mathbb{R}^N) \Rightarrow \hat{f}$ vanishes at infinity and is in $\mathcal{E}(\mathbb{R}^N)$ So, it is in particular continuous and vanishes at infinity, implies continuous and vanishes at infinity implies uniformly continuous. So, now $f \in L^1(\mathbb{R}^N)$

so we already know, $\mathcal{D}(\mathbb{R}^N) \subset S(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$ is dense and consequently so

$$f_n \in S(\mathbb{R}^N),\ f_n \to f \text{ in } L^1(\mathbb{R}^N)$$

Then, we have already seen

$$|\hat{f}_n(\xi) - \hat{f}(\xi)| \leq ||f_n - f||_1 \to 0$$

$$\Rightarrow f_n - f||_{\infty} \leq f_n - f||_1 \to 0$$

$\hat{f}_n \to \hat{f}$ uniformly.

So, if it converges uniformly that means, $\hat{f}$ is continuous and vanishes at infinity because see the space of continuous functions which vanish at infinity is a Bernard space for the sup.

Now and therefore, $\hat{f}_n$ at is continuous and vanishes at infinity because it is in $S(\mathbb{R}^N)$ and consequently $\hat{f}$ is also continuous vanishes at infinity in place it is uniformly continuous. So, this is the Riemann Lebesgue lemma.