Sobolev Spaces and Partial Differential Equation
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The Schwarz space – Part 1

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SCHWARTZ SPACE:

Today we will discuss the Schwarz space. So if

$$f \in L^1(\mathbb{R}^N).$$

We define

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi I x\cdot\xi} f(x)dx \Rightarrow \hat{f} \in L^\infty(\mathbb{R}^N)$$

And in fact, it was a uniformly continuous function.

So and it need no it is bounded, but it need not be integrable. So it does not, we are now looking because ultimately, we want to define the Fourier transform of a distribution. So we want some subset of $L^1(\mathbb{R}^N)$ which would be stable under the Fourier transform that means if you take a function in that space, the Fourier transform must also belong to that space.

Now, the first space which comes to our mind is our favourite space, namely the space of test functions, but $\mathcal{D}(\mathbb{R}^N)$ of is not stable under the Fourier transform. So let us prove the following theorem. So, there is a very particular case, a partial result of a big theorem known as the Payley–Wiener theorem. Which is a more comprehensive and precise result, we are going to just through a portion of it.

Theorem(Payley Wiener): Let $f \in C_c(\mathbb{R})$. we are just in one dimension, then $\hat{f}$ is analytic.

Proof: So even if you have a continuous function with compact support, then the Fourier transform is analytic. And if it is analytic function, it cannot have compact support, because if it is if it vanishes on an open set, then analytic function has to vanish everywhere.

So, this shows that $C_c(\mathbb{R})$ so in particula $\mathcal{D}(\mathbb{R})$ is not stable under the Fourier transform. And this, the Paley–Wiener theorem essentially extends all this to N dimensions and in fact gives you a kind of growth conditions on the Fourier transform also. Anyway, we will prove a very simple result here. So, we have $f \in C_c(\mathbb{R})$, which is obviously contained in $L^1(\mathbb{R})$. Because continuous functions with compact support are of course integral.

So,

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi I x\xi} f(x)dx$$

$$\hat{f}(\xi) = \int_{\mathbb{R}} \left[1 + \sum_{n=1}^{\infty} \frac{(-2\pi i x\xi)^n}{n!}\right] f(x)dx$$

So, now, you have the let us assume that support of f is contained in some big interval say

$$supp(f) \subset [-A, A],\ \ |f| \leq M$$

And since f is on of compact support, we have the F is bounded as well.

$$\hat{f}(\xi) = \int_{-A}^{A} \left[1 + \sum_{n=1}^{\infty} \frac{(-2ix\pi)^n}{n!}\xi^n\right] f(x)dx$$

So, to get the we would like to do a term by term integration of this for that we have to do some work. So, let us look at that. So, what can we say about $\left|\frac{(-2\pi i x\xi)^n}{n!}\right|$ So,

$$\left|\frac{(-2\pi i x)^n}{n!} f(x)\right| \leq M \frac{(2\pi i A)^n}{n!} = M_n$$

So, M is a common factor and therefore, you have $\sum M_n < \infty$ is finite because it is just an exponential series. So, it always converges and therefore this is finite. So, you have that the infinite series

$$\left[1 + \sum_{n=1}^{\infty} \frac{(-2ix\pi)^n}{n!} \xi^n\right] f(x) \text{ converges uniformly on } [-A, A].$$

This is called the Weierstrass M-test. So, you uniformly bound all the terms by terms of a convergent series then the term goes uniformly.

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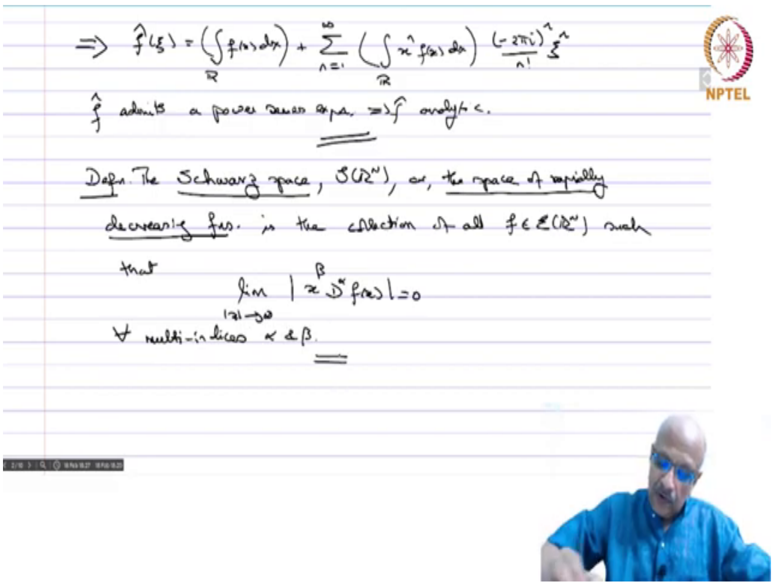

And therefore, you can do once you have uniform convergence you can allow term by term integration. So, you have

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)dx + \sum_{n=1}^{\infty} \left(\int_{\mathbb{R}} x^n f(x)dx\right) \frac{(-2\pi i\xi)^n}{n!} \xi^n$$

So, $\hat{f}$ admits a power series expansion implies if $\hat{f}$ is analytic and therefore, you see $\mathcal{D}(\Omega)$ is not also going to work. So, we want to find a space which is between the two in some sense so, then that is space which we call the Schwarz space this is called the space of uniformly rapidly decreasing functions at infinity. That is they go to zero along with all derivatives faster than any polynomial.

Schwartz Space:

Definition: formal definition the Schwarz space $S(\mathbb{R}^N)$, or the space of rapidly decreasing functions is the collection of all $f \in \mathcal{E}(\mathbb{R}^N)$ C infinity, C infinity functions such that

$$\lim_{|x| \to \infty} \left|x^{\beta} D^{\alpha} f(x)\right| = 0 \quad , \forall\, multi - index\ \alpha\ \&\ \beta$$

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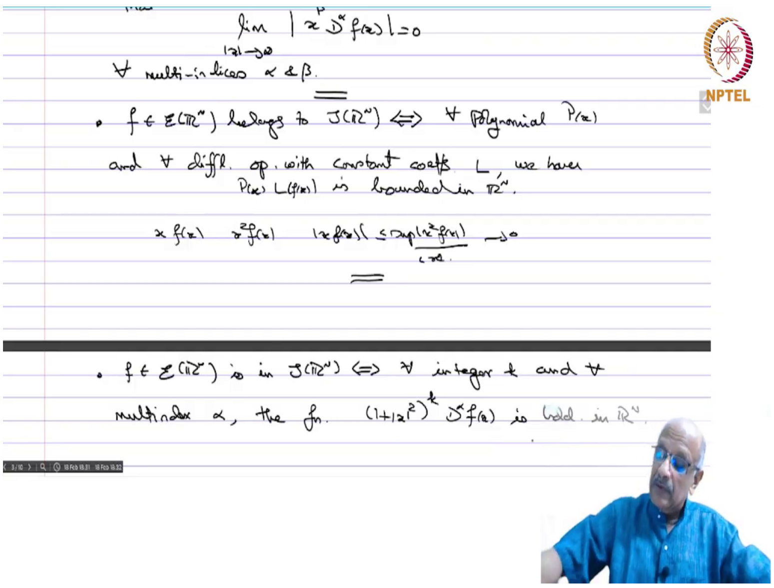

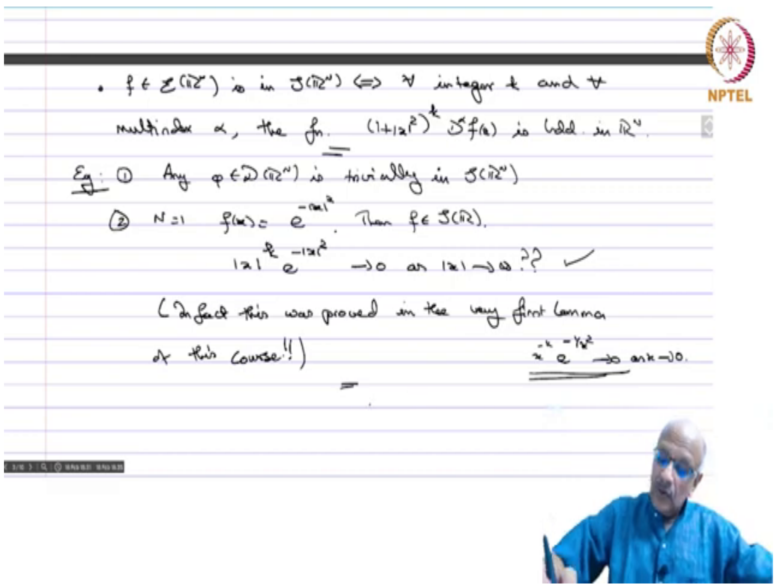

So, the following facts follow immediately from the definition. So,

$f \in \mathcal{E}(\mathbb{R}^N)$ belongs to $f \in S(\mathbb{R}^N) \Leftrightarrow \forall$ For every polynomial $P(x)$ and for every differential operator with constant coefficients L, we have $P(x)L(f(x))$ is bounded in $\mathbb{R}^N$. So, you can so, bounded for every polynomial multiplication they all derivatives is the same as saying that they all go to zero.

Because if for instance in one dimensions if I want if of

$$xf(x) \quad x^2 f(x) \quad |xf(x)| \le sup\left|\frac{x^2 f(x)}{x}\right| \to 0 \text{ as } x \to \infty$$

And therefore, you see you can always by taking higher and by higher powers being bounded you can show that all the previous ones go to zero so, this these two are equivalent.

Another the equivalent way of looking at it is

$f \in \mathcal{E}(\mathbb{R}^N)$ is in $f \in S(\mathbb{R}^N)$ $\Leftrightarrow$for every integer k and $\forall$multi index α the function $(1 + |x|^2)^k D^\alpha f(x)$ is bounded in $\mathbb{R}^N$.

Why it is true? You can show that any monomial $|x|$ to the $|\alpha|$ or x^α any monomial is bounded ten the α, x can be bounded by the supremum of a thing of such a function for high enough k.

And therefore, automatically we come in to the previous one. So, that is the reason and then once you do it for every monomial it is true for every polynomial because it is just a finite linear combination. So, these two conditions are equivalent definitions of the Schwarz space. So, we will remember these things.

So, just think about it and try to convince yourself by your own arguments that these are true. So, now, let us look at some examples. So,

(1) N=1, Any $\varphi \in \mathcal{D}(\mathbb{R}^N)$ is trivially in $S(\mathbb{R}^N)$.

because after the complexity everything vanishes and therefore, everything is bounded all derivatives. All multiplication by any polynomial nothing matters.

(2) Second example; let us take N=1 and $f(x) = e^{-|x|^2}$.Then $f \in S(\mathbb{R}^N)$.

So, this is a prototype of functions which are in the Schwarz space. So, in fact, all you have to do is look at $|x|^k e^{-|x|^2} \to 0\ as\ |x| \to \infty$. So, in fact, this was proved in the very first lemma of this course.

What did we have to that we looked at $|x|^{-k} e^{-1/|x|^2} \to 0\ as\ |x| \to 0$. This was the lemma which we proved. So, that to prove the continuity of the function which is zero on the left and which is equal to $e^{-1/|x|^2}$ on the right. we wanted show that it was a C infinity function and this is what we showed and this result is exactly the same as the result which we have here and therefore, you have this belongs to us.

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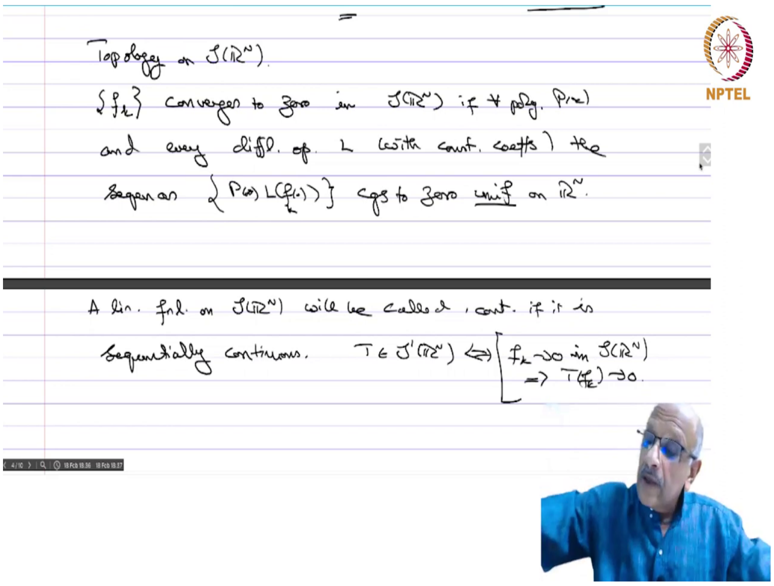

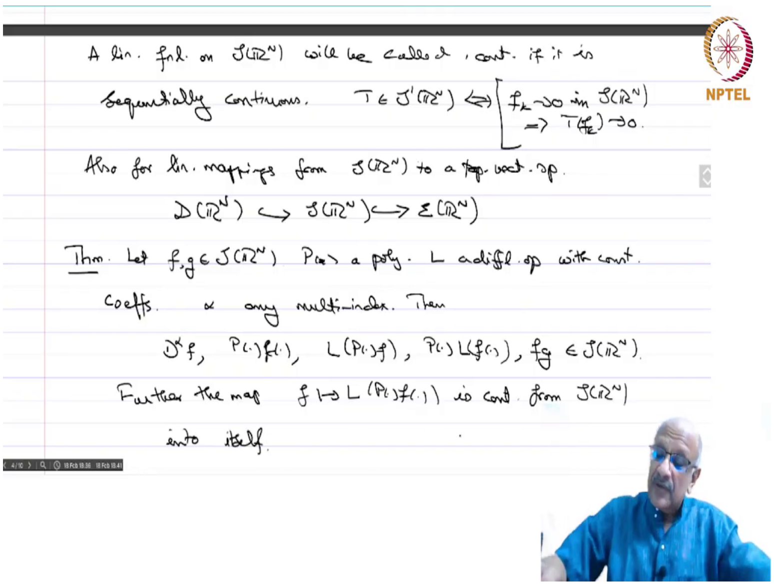

So, now, we want to put a topology on S. Again we will content ourselves by saying what is a convergent sequence. So,

$\{f_k\} \to 0$ in $S(\mathbb{R}^N)$ if for every polynomial $P(x)$ and every differential operator L with constant coefficients the sequences $\{P(x)L(f_k(x))\}$ converges to 0 uniformly on $\mathbb{R}^N$. So, this is the…

So, a linear functional on $S(\mathbb{R}^N)$ of will be continuous if will be called continuous if it is sequentially continuous. There is

$$T \in S'(\mathbb{R}^N) \Leftrightarrow f_k \to 0 \; in \; S(\mathbb{R}^N) \Rightarrow T(f_k) \to 0.$$

So, this is the definition of a continuous linear function. So, now in linear so, this is true for linear mappings also.

So, linear also for linear mappings from $S(\mathbb{R}^N)$ to a topological vector space, so we identify continuity and sequential continuity in this case. So, if you look at $\mathcal{D}(\mathbb{R}^N)$, this is a subspace of $S(\mathbb{R}^N)$ and that is of course, a subspace of $\mathcal{E}(\mathbb{R}^N)$ these are all inclusion. Now, if something goes to zero and $\mathcal{D}(\mathbb{R}^N)$ obviously, all the $P(x)DL(f\ (x)) \in P(x)L(f_k(x))$ then they will all go to 0 in $\mathcal{D}(\mathbb{R}^N)$ and by the topology of $\mathcal{D}(\mathbb{R}^N)$ and therefore, they will all go uniformly all over the in k.

The fixate compact set which contains all the supports and in particular it goes in $\mathbb{R}^N$ itself. So, $\mathcal{D}(\mathbb{R}^N)$ is continuously embedded in the $S(\mathbb{R}^N)$. And the $S(\mathbb{R}^N)$ is continuously embedded in $\mathcal{E}(\mathbb{R}^N)$ because, all derivatives will all go to zero uniformly in $\mathbb{R}^N$ itself, and therefore in every complex therefore, trivially we have these three inclusions.

Theorem: So, next theorem again it is obvious let $f, g \in S(\mathbb{R}^N)$, $P(x)$ a polynomial, L is a differential operator with constant coefficients and α any multi index.

Then $D^\alpha f,\ P(.)f(.),\ P(.)L(f(.)),\ fg \in S(\mathbb{R}^N)$. Further the map

$f \to L(P(.)f(.))$ *is continuous from* $S(\mathbb{R}^N)$ into itself. And these are all just obvious statements coming from the definition. So, that is everything is just from the definition.

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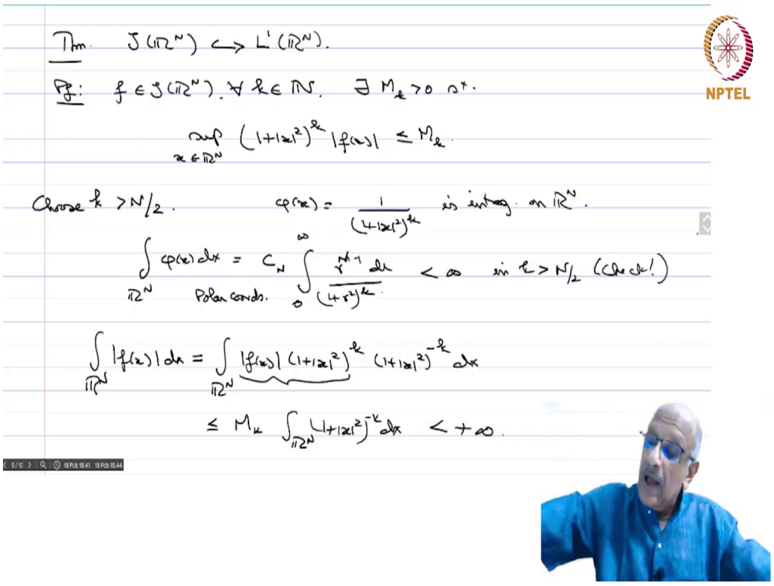

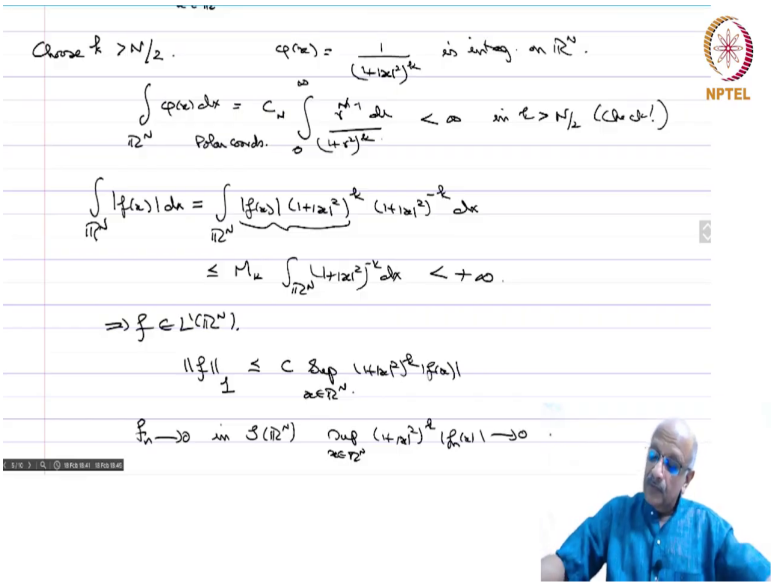

So, the first important result about the space,

Theorem: so theorem: $S(\mathbb{R}^N)$ is continuously included in $L^1(\mathbb{R}^N)$. So, that we can define the Fourier transform.

Proof: let $f \in S(\mathbb{R}^N)$. Take k any positive integer. Then there exists integer $M_k > 0$ such that

$$sup_{x \in \mathbb{R}^N} (1 + |x|^2)^k |f(x)| \leq M_k$$

This is characterization which we gave for the thing. Now, you choose. So, you will choose this is true for all k. So, therefore, you take $k > \frac{N}{2}$

Then what do we know about this function $\varphi(x) = \frac{1}{(1+|x|^2)^k}$ is integrable on $\mathbb{R}^N$.

Where is to solve because if you use so,

$$\int_{\mathbb{R}^N} \varphi(x)dx = C_N \int_0^{\infty} \frac{r^{N-1}}{(1+r^2)^k} dr < \infty \ in \ k > \frac{N}{2}$$

check. So, this is a elementary exercise in lab integration. So, if you write it in polar, so, this is polar coordinates. So, therefore, this integral is finite. So, now, if you choose such a $k > \frac{N}{2}$, let us evaluate

$$\int_{\mathbb{R}^N} |f(x)|dx = \int_{\mathbb{R}^N} |f(x)|(1 + |x|^2)^k(1 + |x|^2)^{-k} dx \leq M_k \int_{\mathbb{R}^N} (1 + |x|^2)^{-k} dx <+ \infty$$

So, you have $f \in L^1(\mathbb{R}^N)$.

so, this implies $f \in L^1(\mathbb{R}^N)$ that Further norm so

$$||f||_1 \leq C\, sup_{x\in\mathbb{R}^N} (1 + |x|^2)^k |f(x)|$$

So, if $f_n \to 0 \ in \ S(\mathbb{R}^N) \quad sup_{x\in\mathbb{R}^N} (1 + |x|^2)^k |f_n(x)| \to 0$

That is what we mean by the topology in $S(\mathbb{R}^N)$. It shaved all of these should go to zero uniformly and that means the supremum or the L infinity norm goes to zero, this means

$$\Rightarrow f_n \to 0 \text{ in } L^1(\mathbb{R}^N)$$

And therefore, the inclusion is continuous. So exercise

Exercise: proof $S(\mathbb{R}^N)$ is continuously embedded in the $L^p(\mathbb{R}^N)$, $\forall 1 < p < \infty$.

So, it is true for all $\forall 1 < p < \infty$. p equals infinity is obvious, because everything goes to zero uniformly that so, the topology is defined itself, and everything is bounded. And p=1 we have just proved so prove it for $1 < p < \infty$.. You only have to choose this k appropriately and then do it. So, it is a very small and simple exercise.

So, the usefulness of the Schwarz space will now be shown, which we will do next is to show that the stability of the Fourier transform. So, the Fourier transform behaves very well on the Schwarz space and is very useful so that we will see next.