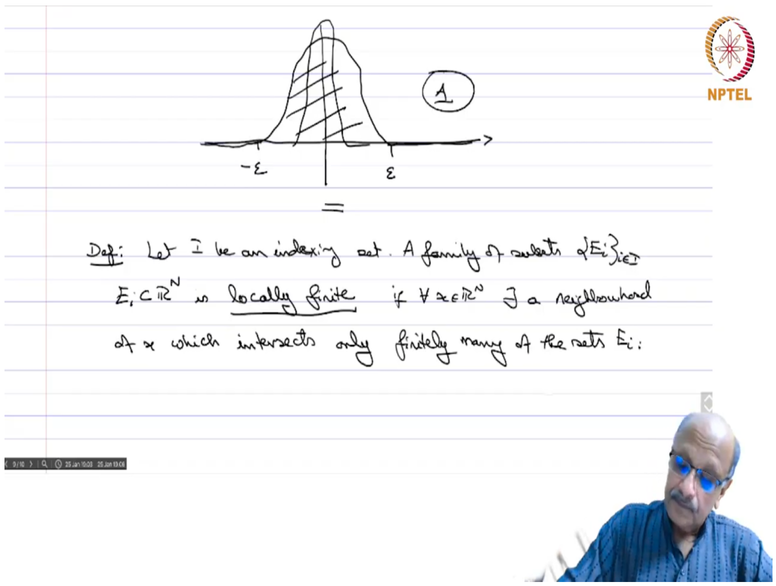
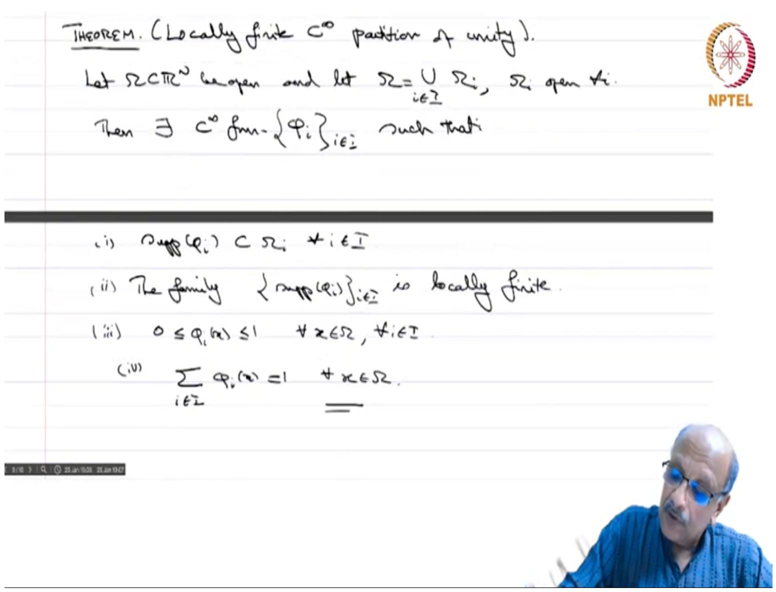
Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics, IMSc
Indian Institute of Technology, Madras
Lecture - 2
Test Functions – Part 2

(Refer Slide Time: 00:17)





So, we need the following notion.

Definition: Let I be an indexing set (it may be finite, countable or uncountable). A family of sets $\{E_i\}_{i\in I}$, where $E_i \subset \mathbb{R}^N$, is locally finite if $\forall\, x \in \mathbb{R}^N$, there exists a neighborhood of x which intersects only finitely many of the sets E_i.

We now state a theorem. I will not be proving this. The proof can be found in the appendix of the book which I cited earlier; and it is a standard theorem which follows from the paracompactness of the Euclidean space. Let me first state the theorem.

Theorem: Let $\Omega \subset \mathbb{R}^N$ be open and let $\Omega = \cup_{i\in I} \Omega_i$, where Ω_i is open for all i. Then there exists C^∞ functions $\{\phi_i\}_{i\in I}$, such that

(i) $\text{supp}(\phi_i) \subset \Omega_i$, for all i.

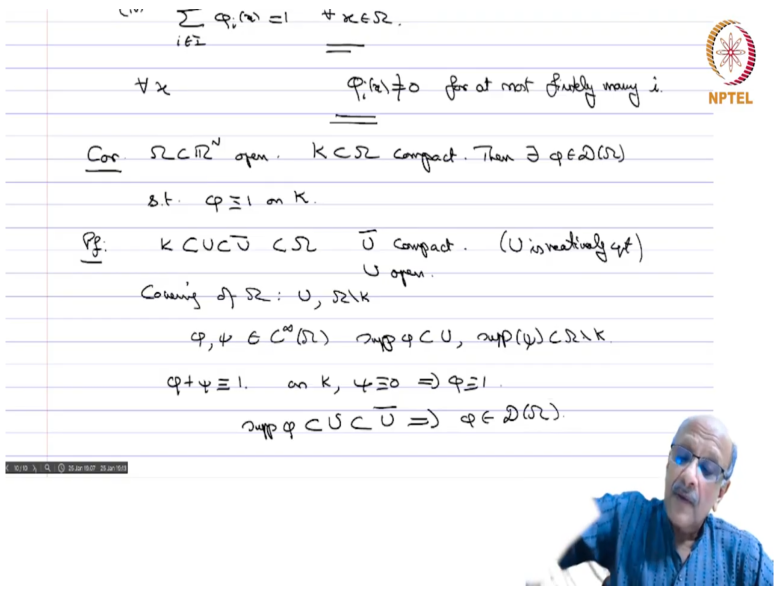
(ii) the family $\{\text{supp}(\phi_i)\}_{i\in I}$ is locally finite.

(iii) $0 \le \phi_i(x) \le 1$, for all $x \in \Omega$, for all $i \in I$.

(iv) $\sum_{i\in I} \phi_i(x) = 1$, for all $x \in \Omega$.

So, why do we call it a locally finite C infinity partition of unity? Of course all the functions are C^∞ functions and they are locally finite, because the supports of these functions are locally finite. Now, the last condition needs some explanation; because I told you that I can be a finite, countable or uncountable set. So, how do we define the sum here?

(Refer Slide Time: 05:02)



Now, for given x, there exists a neighborhood which intersects only a finite number of the sets $\mathrm{supp}(\phi_i)$. That means, for every x, $\phi_i(x) \neq 0$ for at most finitely many i and therefore the sum is in fact a finite sum; because you have a neighborhood which will intersect the supports for finitely many of these functions ϕ_i.

So, for all those for which it does not intersect the $\mathrm{supp}\,\phi_i(x)$, is automatically 0; and therefore, you have that this is nonzero only for finitely many. And therefore, this sum makes sense and therefore you can define it, and it is always equal to 1. And because you have taken the constant function 1 and broken it up as a sum of C^∞ functions, you call it as C^∞ partition of unity. So, this is the meaning of the statement.

Now, we have a simple corollary.

Corollary: Let $\Omega \subset \mathbb{R}^N$ be open and $K \subset \Omega$ be compact. Then $\exists\ \phi \in D(\Omega)$ such that $\phi \equiv 1$, on K.

proof: By the separation properties of $\mathbb{R}^N$ (i.e., T_3, T_4 and so on), we can find U such that

$$K \subset U \subset \overline{U} \subset \Omega \text{ and } U \text{ is open, } \overline{U} \text{ is compact (i.e., } U \text{ is relatively compact).}$$

Now we consider the following covering of omega, namely U and $\Omega\setminus K$. Here K is compact, so it is closed. Therefore $\Omega\setminus K$ is open. Also U is open. So, together these two cover the whole set, so, the C infinity function partition of unity, well we do not have to worry about locally finite; because we have only two sets, which is finite.

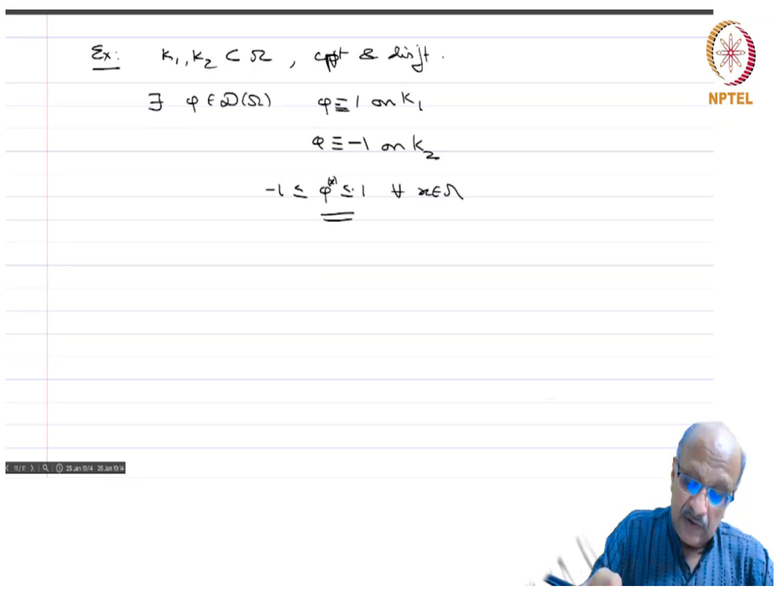
So, we can find functions satisfying

$$\phi,\ \psi \in C^{\infty}(\Omega),\ \text{supp}(\phi) \subset \Omega\ ,\ \text{supp and}\ \phi + \psi \equiv 1.$$

This means that on K, $\psi \equiv 0 \Rightarrow \phi \equiv 1$ on K. Further, $\text{supp}(\phi) \subset U \subset \overline{U}$ and $\overline{U}$ is compact. Therefore, $\text{supp}(\phi)$ is also compact. Thus $\phi \in D(\Omega)$; so, this proves the theorem.

As an exercise, you can try the following.

(Refer Slide Time: 09:32)



Exercise: If you have $K_1,\ K_2 \subset \Omega$ and they are compact and disjoint, then there exists

$\phi \in D(\Omega)$ such that $\phi \equiv 1$ on K_1 , $\phi \equiv -1$ on K_2 and $-1 \leq \phi(x) \leq 1$, for all $x \in \Omega$.

One can use the last corollary to do this.

So, you see that the function space $D(\Omega)$ is very rich. Now, we want to put a topology on $D(\Omega)$. And then we will take the dual, which will make it a topological vector space and then we will take the dual space of that. So that is our next object.