Sobolev Spaces and Partial Differential Equations
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The Fourier Transform

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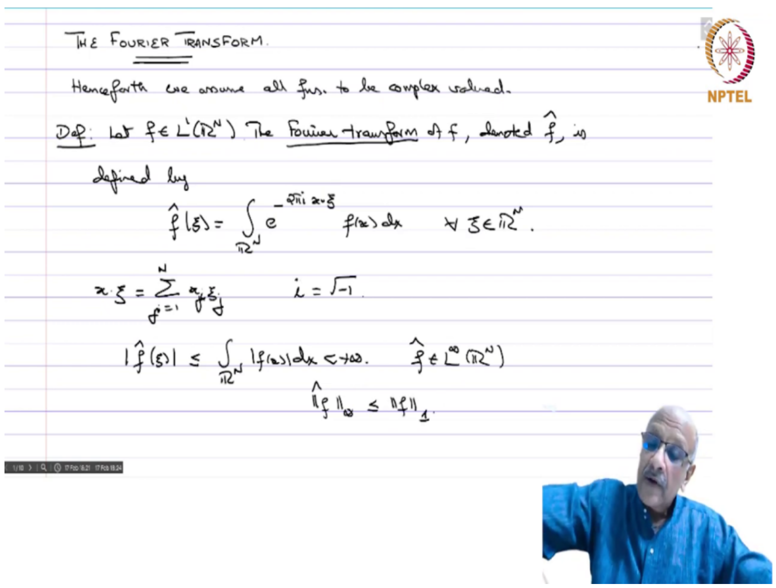

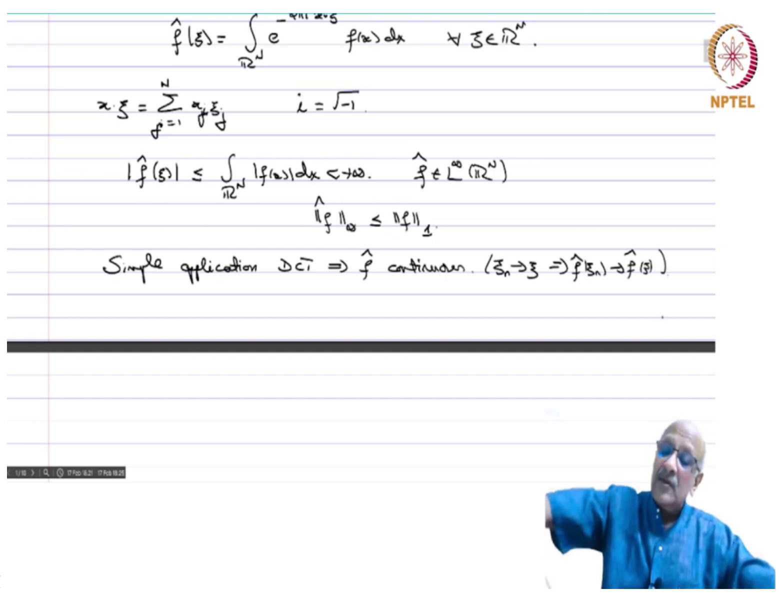

The Fourier Transform:

So, we will now discuss the Fourier transform. We will first discuss this for L^1 functions on $\mathbb{R}^N$ then see how we can extend this to a certain class of distributions. We cannot do it for all distributions. But there is a certain class where you can do it. So, henceforth we assume all functions to be complex values up to now it did not matter.

But now, the Fourier transform is particularly adapted for complex valued functions. So,

Definition:

Let $f \in L^1(\mathbb{R}^N)$, the Fourier transform of f denoted by $\hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} f(x)dx, \quad \forall \ \xi \in \mathbb{R}^N.$$

So, what is $x \cdot \xi$? $x \cdot \xi = \sum_{j=1}^{N} x_j \xi_j$ $i = \sqrt{-1}$

So, we this is the complex i and therefore, I do not want to use it for the index as well. So, this is the definition of the Fourier transform. So, to start with you have to check if it is well defined. So, if I take

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^N} |f(x)| \, dx < +\infty$$

So, you have that the Fourier transform is well defined and in fact,

$$\hat{f} \in L^{\infty}(\mathbb{R}^N)$$

$$||\hat{f}||_{\infty} \leq ||f||_1$$

So, simple application of the dominated convergence theorem implies $\hat{f}$ is in fact continuous. I will leave you to verify the details. So, you take

$$\xi_n \to \xi \Rightarrow \hat{f}(\xi_n) \to \hat{f}(\xi)$$

erges to f hat of xi. So, this is just direct and simple application of the dominated convergence theorem and you can do it as a very simple exercise.

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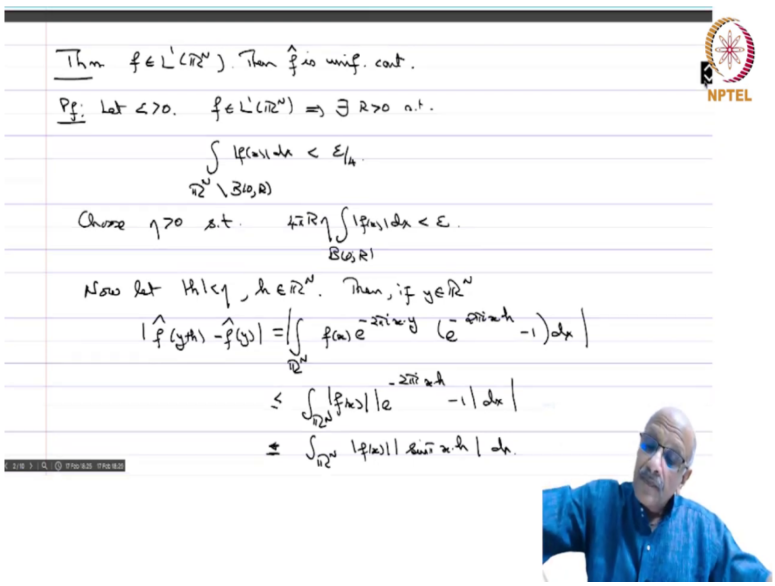

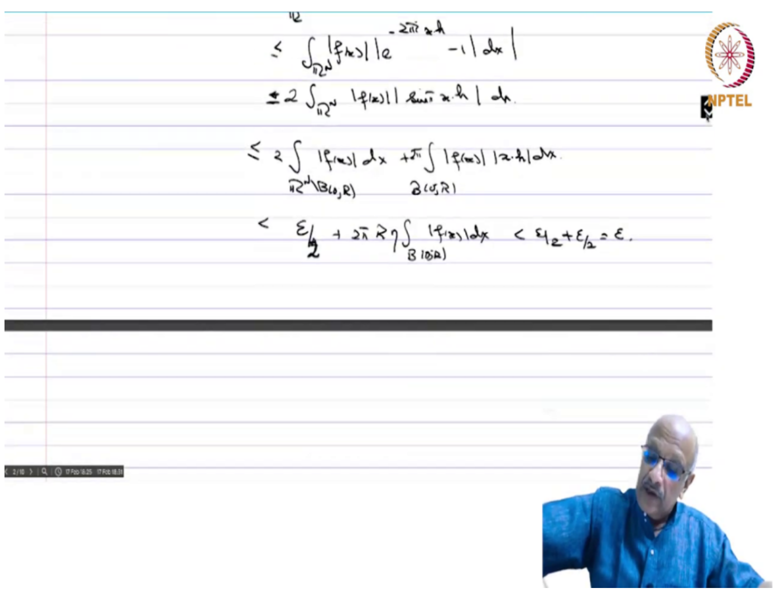

In fact, we have something much more which we will now prove, Theorem: $f \in L^1(\mathbb{R}^N)$. Then $\hat{f}$ is uniformly continuous

proof: So, let $\varepsilon > 0$ arbitrary small positive quantity and then $f \in L^1(\mathbb{R}^N) \Rightarrow \exists R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(0,R)} |f(x)|dx < \frac{\varepsilon}{4},$$ because integral mod f is a convergent integral.

Therefore, this is the tail of a convergent integral you are taking away a ball of radius R. So, as R goes becomes bigger and bigger. What is remaining outside the ball? The integral will be smaller and smaller is just the fact that $f \in L^1(\mathbb{R}^N)$ what.

Choose $\eta > 0$ such that you have $4\pi R\eta \int_{B(0,R)} |f(x)|dx < \varepsilon$, because this is now a finite quantity once R is fixed constant.

So, I can choose η small enough such that you can bring it less. So, now, let

$|h| < \eta$ and $h \in \mathbb{R}^N$. Then if $y \in \mathbb{R}^N$

What do you have?

$$\left|\hat{f}(y + h) - \hat{f}(y)\right| = \left|\int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot y}(e^{2\pi i x.h} - 1)dx\right|$$

$$\leq \int_{\mathbb{R}^N} |f(x)||e^{2\pi i x.h} - 1|dx$$

$$= \int_{\mathbb{R}^N} |f(x)||sin\, \pi\, x.h|\, dx$$

So, now that is less than or equal to I am going to split the integral into two parts.

$$\leq \int_{\mathbb{R}^N \setminus B(0,R)} |f(x)|\, dx + 2\pi R\eta \int_{B(0,R)} |f(x)|\, |x.h|\, dx < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

So, this proof, the uniform continuity of the Fourier transform.

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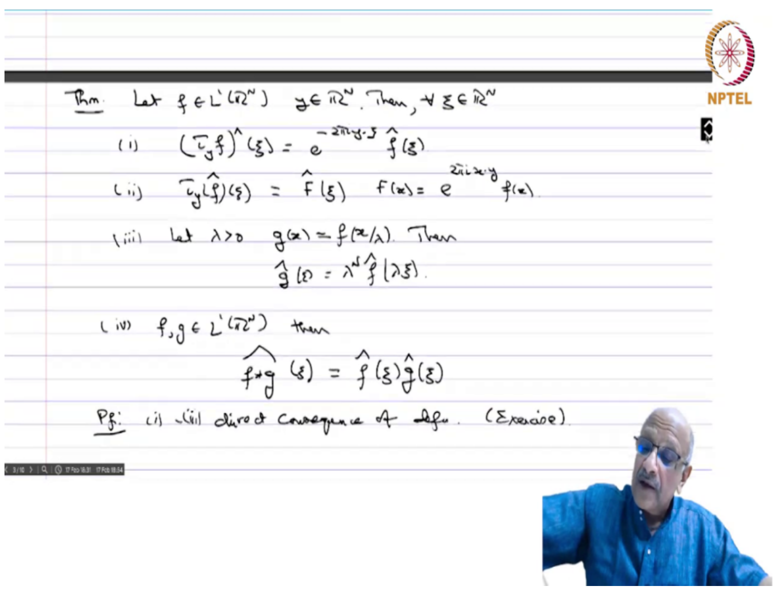

So, the following proposition is very, theorem is very useful in all calculations involving the Fourier transform.

Theorem:

So, let

$f \in L^1(\mathbb{R}^N)$ $y \in \mathbb{R}^N$. Then $\forall \xi \in \mathbb{R}^N$

(i) $(\hat{\tau_y f})(\xi) = e^{-2i\pi y\cdot\xi}\, \hat{f}(\xi)$

(ii) $\hat{\tau_y(f)}(\xi) = \hat{F}(\xi),\ F(x) = e^{2i\pi x\cdot y}f(x)$

(iii) Let $\lambda > 0\ \ g(x) = f(\frac{x}{\lambda}).\ Then\ \hat{g}(\xi) = \lambda^N \hat{f}(\lambda\xi)$

(iv) Let $f, g \in L^1(\mathbb{R}^N),\ then$

$$\widehat{f \star g}(\xi) = \hat{f}(\xi)\ \hat{g}(\xi)\ .$$

So, here they are dual t to each other. If you take the translation and then take the Fourier transform, then you get multiplied by an exponential if you want to translate the Fourier transform.

Then you have to first multiply the function with an exponential and then take the Fourier transform. So, it is again the L^1 function. I can take the Fourier transform, this is a beautiful property, this is just $\hat{f}(\xi)\ \hat{g}(\xi)$. So, the convolution product on taking the Fourier transform becomes the algebraic product. So, this convolution together with Fourier transform becomes a very powerful tool in the study of partial differential equations, essentially, because of this property.

Proof: So, (i), (ii), (iii) direct consequence of definition. So, it is an exercise to familiarize yourself with the formula and all there.

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So, we will just proof (iv).

$$h = f \star g \in L^1(\mathbb{R}^N)$$

$$\hat{h}(\xi) = \int_{\mathbb{R}^N} e^{-2i\pi x\cdot\xi}\left(\int_{\mathbb{R}^N} f(x-y)g(y)dy\right)dx$$

$$= \int_{\mathbb{R}^N} g(y) \int_{\mathbb{R}^N} e^{-2i\pi x\cdot\xi} f(x - y)\, dx\, dy$$

$$= \int_{\mathbb{R}^N} e^{-2i\pi y\cdot\xi}\, g(y) \left(\int_{\mathbb{R}^N} e^{-2i\pi x\cdot\xi} f(x - y)\, dx \right) dy$$

$$= \int_{\mathbb{R}^N} e^{-2i\pi y\cdot\xi}\, g(y) dy\, \hat{f}(\xi) = \hat{f}(\xi)\, \hat{g}(\xi)$$

So modules will all be an integrable function, and therefore, you can interchange the order of the integration by Frobenius theorem. you just get the… So, this is a very useful property.

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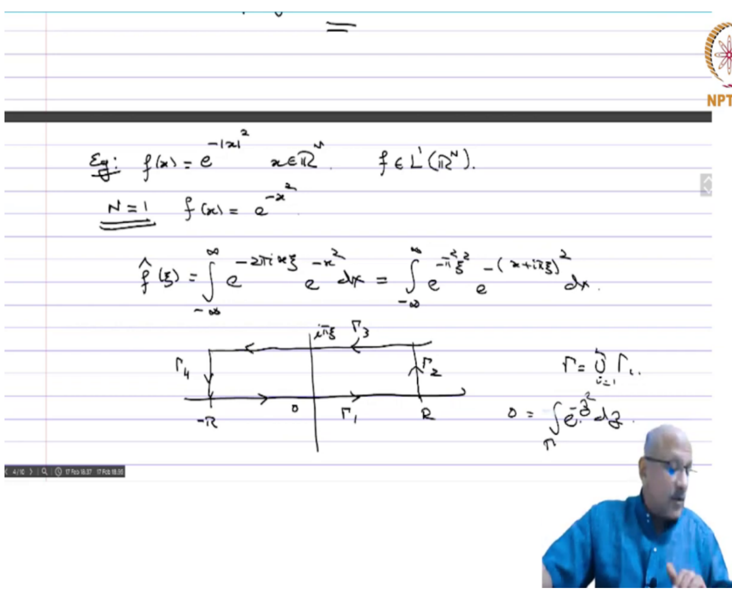

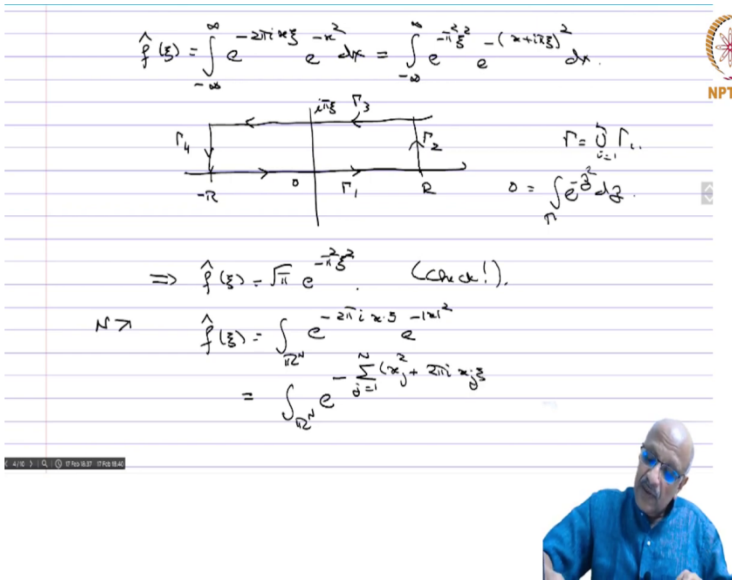

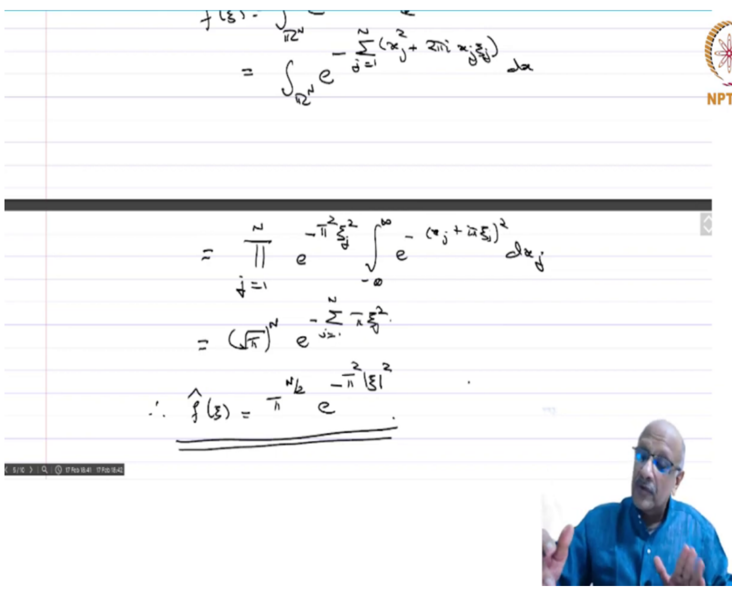

So, now, let us conclude this section with an example.

Example:

So, let us take $f(x) = e^{-|x|^2}$, $x \in \mathbb{R}^N$ $f \in L^1(\mathbb{R}^N)$

and we want to compute this Fourier transform. So, this f belongs to in fact all $g \in L^1(\mathbb{R}^N)$ So, let us first of all that $N = 1$. So, $f(x) = e^{-x^2}$.

So, then what is

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-2\pi x\xi} f(x)dx = \int_{-\infty}^{+\infty} e^{-2\pi x\xi} e^{-x^2} dx = \int_{-\infty}^{+\infty} e^{-\pi^2 x^2} e^{-(x+i\pi x)^2} dx.$$

So, now, we have to evaluate this integral. So, we will use Cauchy theorem for contour integrals. So, this is some -R this is +R and then this is the line $-i\pi\xi$ and then you have this is the origin. So, you have this take this contour and you integrate efs this is Γ. So, Γ_1, Γ_2, Γ_3 and Γ_4.

So, $\Gamma = \bigcup_{i=1}^{4} \Gamma_i$

and then you look at

$$0 = \int_{\Gamma} e^{-\xi^2} d\xi$$

and then you evaluate it around this contour let R tend to infinity and so on and then you will get final e. So, I will leave the contour integration because anyway I will find the Fourier transform by another method which is more elegant later on.

So, this contour the integral like you can do it as an exercise for yourself. So, you will get. So, this will imply that

$$\hat{f}(\xi) = \sqrt{\pi} e^{-\pi^2 \xi^2}$$

checked.

So, if $N > 1$ so, then you have

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} e^{-|x|^2} dx$$

$$= \int_{\mathbb{R}^N} e^{-\sum_{j=1}^{N}(x_j^2+2\pi i x_j\xi_j)} dx$$

$$= \prod_{j=1}^{N} e^{-\pi^2\xi_j^2} \int_{-\infty}^{\infty} e^{-\sum_{j=1}^{N}(x_j+i x_j\xi_j)^2} dx_j$$

$$= (\sqrt{\pi})^N e^{-\sum_{j=1}^{N}(\pi^2\xi_j^2)}$$

Consequently, you have that.

So, therefore,

$$\hat{f}(\xi) = (\sqrt{\pi})^N e^{-(\pi^2|\xi|^2)}$$

So, you see $e^{-|x|^2}$, the Fourier transform looks very much the same; it is almost the same up to some scaling factors. And this is some kind of Eigen function for this operator. But anyway, that is I am just a passing remark. So, we do not have to take it too seriously.

So, we will see, but we will derive this in a completely different way using the properties. So, next properties is to. So, L^1 is taken to L^∞ by the Fourier transform, we are we want to look at some space which is stable under the Fourier transform. So, that we can use it to define some function space which is stable under the Fourier transform.

That means, the function belongs to the space the Fourier transform also belongs to the space and, once we identify that space, then we will see that we can extend the definition of the Fourier transform to certain classes of distributions.