Sobolev Space and Partial Differential Equations

Professor S Kesavan

Department of Mathematics

Institute of Mathematical Science

Fundamental Solutions

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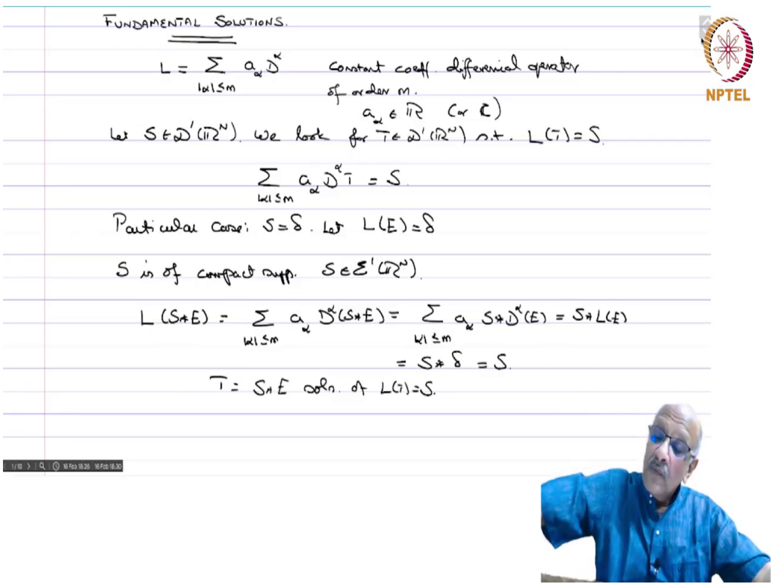

We will now discuss Fundamental Solutions. We are talking about solutions of differential equations. So, let us take differential operators, so L is a differential operator with let us say constant coefficients,

$$L = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

so this is called a constant coefficient differential operator of order m.

So, a_α are all constants $\mathbb{R}$ *or* $\mathbb{C}$if you are working with the complex numbers, so they are all constants, D^α the usual derivatives partial derivatives and you are looking at all multi-indices of order less than or equal to m and therefore, this a constant coefficient differential operator. So, let $S \in D^{'}(\mathbb{R}^{N})$

And we look for $T \in D^{'}(\mathbb{R}^{N})$ such that $L(T) = S$, so if it is a constant coefficient this well defined because $D^{\alpha}T$ is well defined. So that means,

$$\sum_{|\alpha| \leq m} a_{\alpha} D^{\alpha} T = S$$

so these are called distribution solutions of this differential equation.

In particular, if you take f to be a local integral function and you can take $S = T(f)$, then you will have the usual differential equation then 1 can look for all distribution solutions of this equation and decide if these distributions come from functions or not, that is a different investigation altogether.

So, while we saw in the beginning of this course, there are differential equations which you cannot have classical solutions after a very short time, so now you have you can look for solutions in the set of distributions.

So, particular case so $S = \delta$

so let us say $L(E) = \delta$, suppose S is of compact support that means $S \in \mathcal{E}'(\mathbb{R}^{N})$. Then let us look at $S * E$, so

$$L(S * E) = \sum_{|\alpha| \leq m} a_{\alpha} D^{\alpha}(S * E) = \sum_{|\alpha| \leq m} a_{\alpha}(S * D^{\alpha}E) = S * L(E) = S * \delta = S$$

therefore, $S * E$ is a solution of this differential equation $L(T) = S$, or $T = S * E$ solution of $L(T) = S$.

So, fundamental solutions can help you to find solutions of differential equations, they are the building blocks for finding solutions and also they give us a lot of information we will see that probably later about the solution itself, so given any solution you can from the fundamental solution you can predict the behaviour of some of the solutions even though you may or may not be able to solve.

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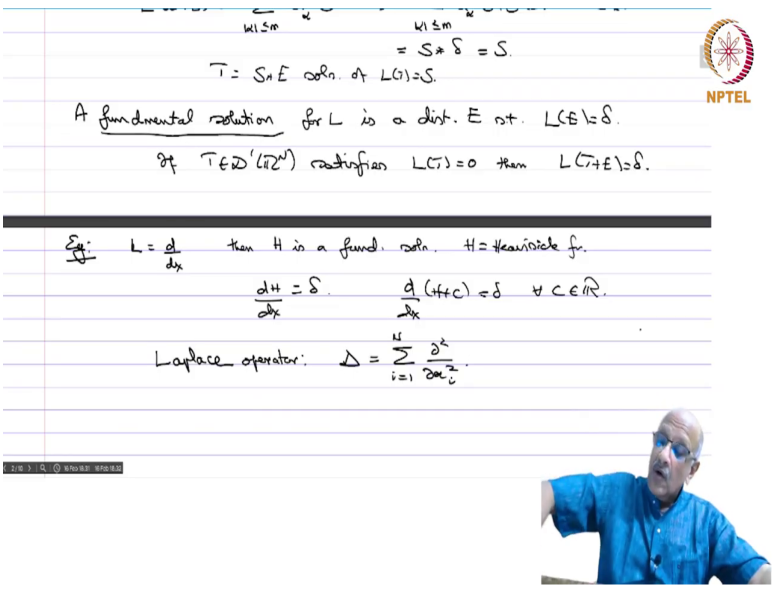

So, we saw a fundamental solution for L is distribution E such that $L(E) = \delta$, I have used a and not the because the fundamental solution is not unique, $T \in D^{'}(\mathbb{R}^{N})$ satisfies $L(T) = 0$ then by linearity $L(T + E) = \delta$,therefore you do not have uniqueness of the fundamental solution, so let us take the first example, we have already seen this example.

So, example, so if you take $L = \frac{d}{dx}$ then $H$is a fundamental solution H=heavy side function, because we know the $\frac{dH}{dx} = \delta$ also we have $\frac{d(H+C)}{dx} = \delta, \ \forall\ C \in \mathbb{R}$ because, $\frac{d(C)}{dx} = 0,$ the, so now we will look at the Laplace operator a very important operator in PDE theory operator.

So, Laplacian operator: $\Delta = \sum_{i=1}^{N} \frac{{}^{2}}{x_i^{2}}$

and we want to calculate the fundamental solution of the Laplace operator, so this is a good example a good exercise in computing distribution derivatives verifying what is the distribution, derivative, etc.

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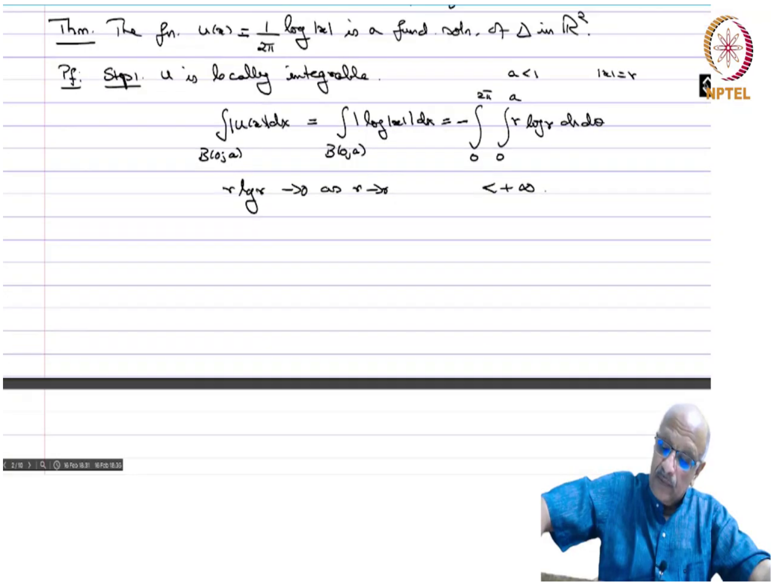

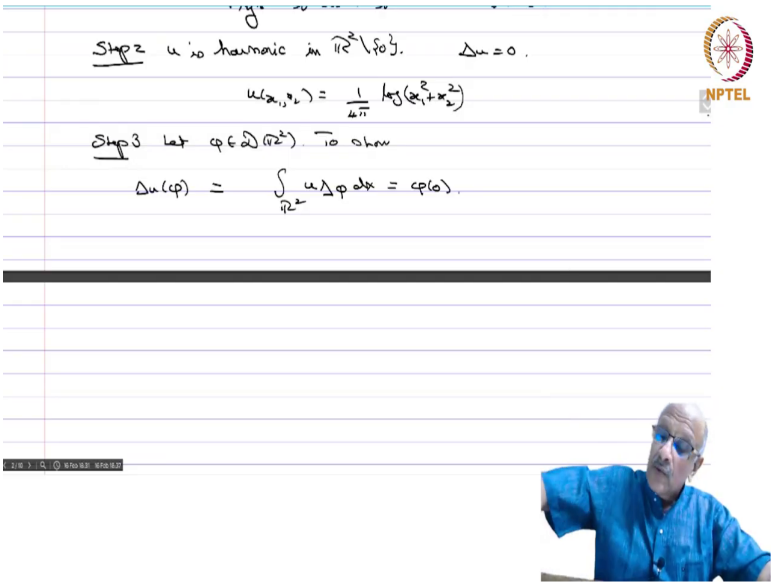

So, Theorem: The function $u(x) = \frac{1}{2\pi} log|x|$

is a fundamental solution of the Laplace operator in $\mathbb{R}^2$, so we will prove this in several steps, so proof step 1: First of all u is locally integral. Away from the origin $log|x|$ is a nice continuous function therefore, you have no problems, so only have to look at the integrability over a compact set in a neighbourhood of omega.

So we want to consider integral $\int_{B(0;a)} |u(x)|dx = \int_{B(0;a)} log|x|dx =- \int_0^{2\pi}\int_0^a rlog\, rdrd\theta$

Now, if you convert this into, so let us take a to be less than 1 so that $log|x|$is nothing but $-\ log|x|$, so it is a negative number, so you have the modulus will be, minus log r is mod log x because r is less than 1.

So$|x| = r$, and then you have the $rdrd\theta$ which is coming from the polar coordinates, now

$$rlogr \to 0\ \ as\, r \to \infty$$

and therefore, this function is a good function it is a nice continuous function and therefore, this is finite, so this integral is finite. So, this defines this locally integral function and therefore defines distribution.

So, Step 2: so the function u is harmonic in $\mathbb{R}^2 - \{0\}$ that means$\Delta u = 0$, so this is just a routine check, so you take

$$u(x_1, x_2) = \frac{1}{4\pi} log(x_1^2 + x_2^2)$$

because this root of $\sqrt{x_1^2 + x_2^2} = r = |x|$ and therefore, I take, so for x small am taking this you have not, so it is not minus let us say it is just equal to this and therefore, you now straightforward computation.

You just compute Δu, $\frac{^2u}{x_i^2}$ and of the origin and you will find that it is a harmonic function and actually you do not have to worry whether they are doing the distribution or classical derivative, here the classical derivative will do but that happens to also be the distribution derivative because you are talking of a very smooth function in the compliment of the origin and therefore, you do not have to worry.

So, Step 3: you have to show let $\varphi \in D(\mathbb{R}^2)$ so to show

$$\int_{\mathbb{R}^2} u\Delta\varphi dx = \varphi(0)$$

so this is what we need to show. So, now, let us go ahead and do that.

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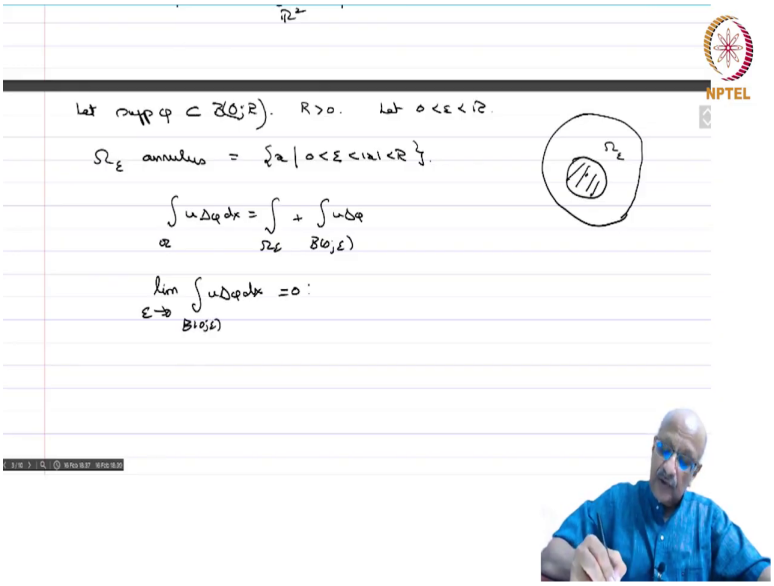

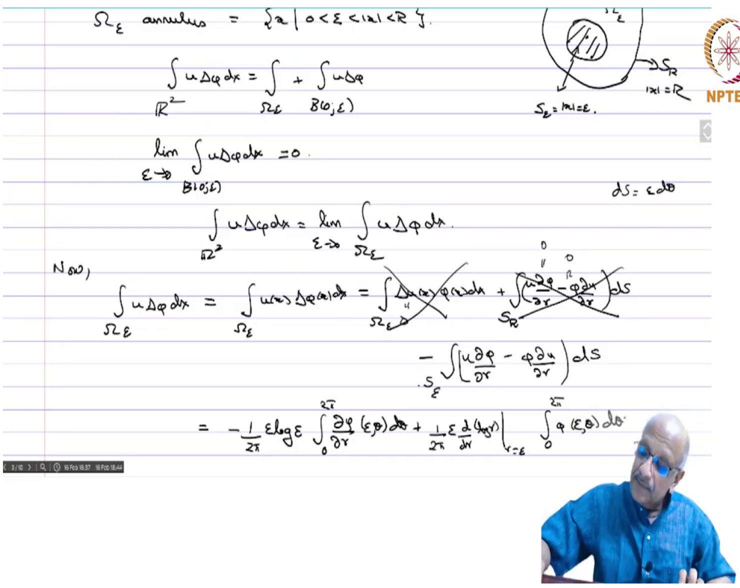

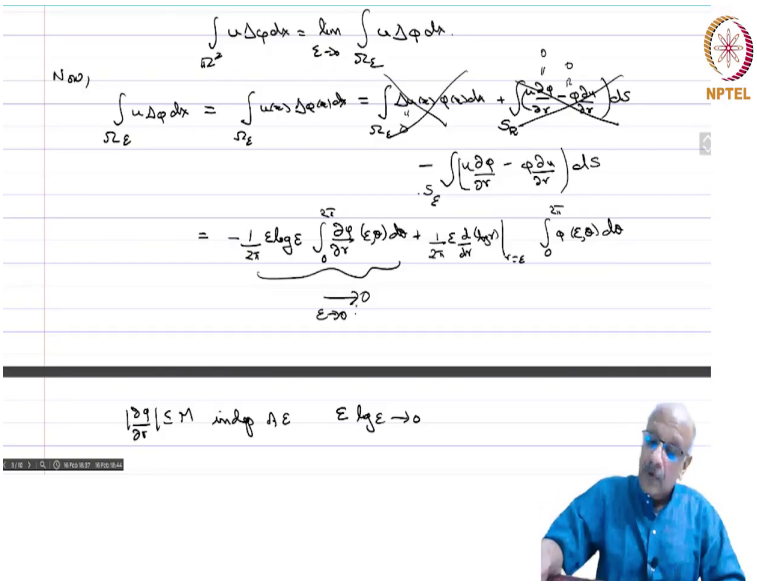

So, let $supp\ \varphi \subset B(0, R)$, R>0,

which is compact, so it can be put in some big balls, so it is contained in $B(0, R)$. So,

let 0 $0 < \varepsilon < R$ and you take $\Omega_{\varepsilon} = \{x \quad | \quad 0 < \varepsilon < |x| < R\}$

is the annulus, so this is Ω_{ε} is the annulus. so you have the ball radius R and then you have the ball radius ε and therefore, this is omega epsilon.

Now,

$$\int_{\mathbb{R}^2} u\Delta\varphi dx = \left(\int_{\Omega_{\varepsilon}} + \int_{B(0;\varepsilon)}\right) u\Delta\varphi dx$$

but since the Lebesgue integral u is locally integrable, therefore it is integrable on $B(0; \varepsilon)$, $\Delta\varphi$ is a C infinity function with compact support. So, $u\Delta\varphi$ is a integrable function and therefore, by the absolute continuity of the Lebesgue integral you have the

$$\lim_{\varepsilon \to 0} \int_{B(0;\varepsilon)} u\Delta\varphi = 0$$

And therefore, you have that

$$\int_{\mathbb{R}^2} u\Delta\varphi dx = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} u\Delta\varphi dx$$

so this is the limit which we need to compute. So, let us compute that integral, so integral on omega, so now

$$\int_{\Omega_\varepsilon} u\Delta\varphi dx = \int_{\Omega_\varepsilon} u(x)\Delta\varphi(x)dx$$

$$= \int_{\Omega_\varepsilon} \Delta u(x)\varphi(x)dx + \int_{S_R} (u\frac{\varphi}{r} - \varphi\frac{u}{r})ds - \int_{S_\varepsilon} (u\frac{\varphi}{r} - \varphi\frac{u}{r})ds$$

so I am going to use the everything is a smooth function Ω_ε is away does not contain the origin, so u is a smooth function $\Delta\varphi$ is a smooth function.

Therefore, we are going to use Green's theorem which is the higher dimensional generalisation of the integration by parts formula and therefore.

So, let us call this outer boundary as S_R,

so $S_R = \{x \mid |x| = R\}$ and $S_\varepsilon = \{x \mid |x| = \varepsilon\}$

So, you have these 2 surfaces and therefore circles, Now, I should write $u\frac{\varphi}{r}$is a normal derivative, but the outer normal derivative is same as derivative in the radial direction where the ball is concerned and went centred at the origin and then similarly, for the inner ball, you will have

Now, by step 2, you delta u is harmonic, so this is equal to 0, so this integral disappears. Now phi a C infinity function with compact support, so all its function and all the derivatives vanish outside the compact set which is contained inside r, so on the ball of radius r all these functions are 0 and therefore, this integral also disappears. So, you have this is equal to… Now, I just have to write this one this is s ε.

So, I am going to write the integral on ds, S is the circle remember that $ds = rd\theta$, so the integral is nothing but$ds = \varepsilon d\theta$, so that is the integral for ds, ε is the radius of the circle. So you have

$$= -\frac{1}{2\pi}\varepsilon\, log\varepsilon \int_{0}^{2\pi} \frac{\varphi}{r}(\varepsilon, \theta)d\theta + \frac{1}{2\pi}\varepsilon\, \varepsilon \frac{d}{r}(logr)|_{r=0} \int_{0}^{2\pi} \varphi(\varepsilon, \theta)d\theta$$

Now, $|\frac{\varphi}{r}| \leq M$ independent of ε

because, φ is C infinity functions with compact support and $\varepsilon log\, \varepsilon \rightarrow 0$, so this whole integral here goes to 0, $\varepsilon \rightarrow 0$ so now we are giving…

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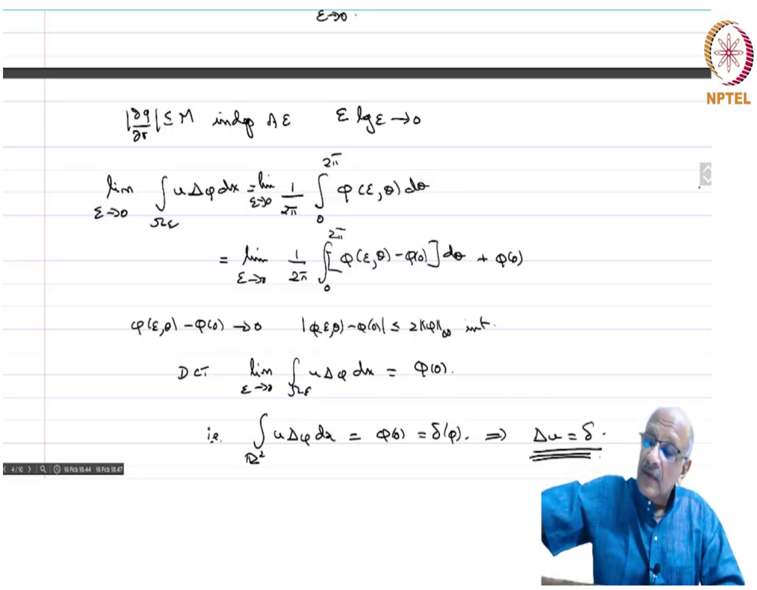

So,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} u\Delta\varphi\, dx = \frac{1}{2\pi}\int_{0}^{2\pi} \varphi(\varepsilon, \theta)d\theta$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} [\varphi(\varepsilon, \theta) - \varphi(0)] d\theta + \varphi(0)$$

so we just have to find this limit here now what do you know about this limit? Now

$$\varphi(\varepsilon, \theta) - \varphi(0) \to 0$$

in fact it goes to 0 uniformly but anyway it does not matter, it goes to 0 and

$$|\varphi(\varepsilon, \theta) - \varphi(0)| \leq 2||\varphi||_\infty$$

and that is integrable, because it is a constant and you have a finite interval.

So by the dominated convergence theorem, you have that

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} u(x)\Delta\varphi(x)dx = \varphi(0)$$

and this proves the theorem, because so that this integral lower

$$\text{i.e., } \int_{\mathbb{R}^2} u(x)\Delta\varphi(x)dx = \varphi(0) = \delta(\varphi) \Rightarrow \Delta u = \delta$$

so that is a fundamental solution to the equations.

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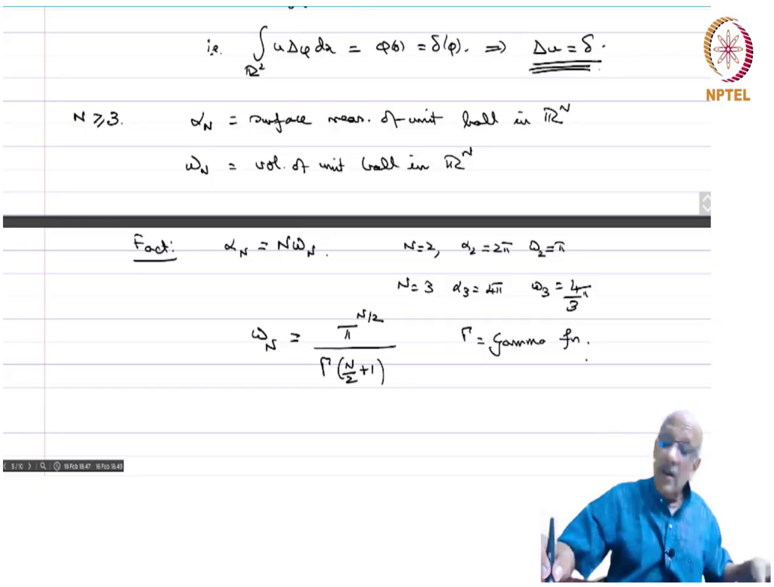

So, now what about if $N \geq 3$, so if $N \geq 3$,

then you have let $\alpha_N = surface\ measure\ of\ unit\ ball\ in\ \mathbb{R}^N$

and let $\omega_N = volume\ of\ the\ unit\ ball\ in\ \mathbb{R}^N$

you know, that means n dimensional lebesgue measure of the unit ball, then we can show, so fact is a very nice application of the Gauss divergence theorem,

$$\alpha_N = N\omega_N$$

So, let us see when $N = 2$, you have $\alpha_2 = 2\pi$ and $\omega_2 = \pi$

$N = 3$, you have $\alpha_3 = 4\pi$ and $\omega_3 = \frac{4}{3}\pi$

In fact, it is true for all this thing and what about

$$\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)} \quad \Gamma = Gamma\ function$$

so we omit the details of this you can find it is a very interesting calculation, maybe we will see later on.

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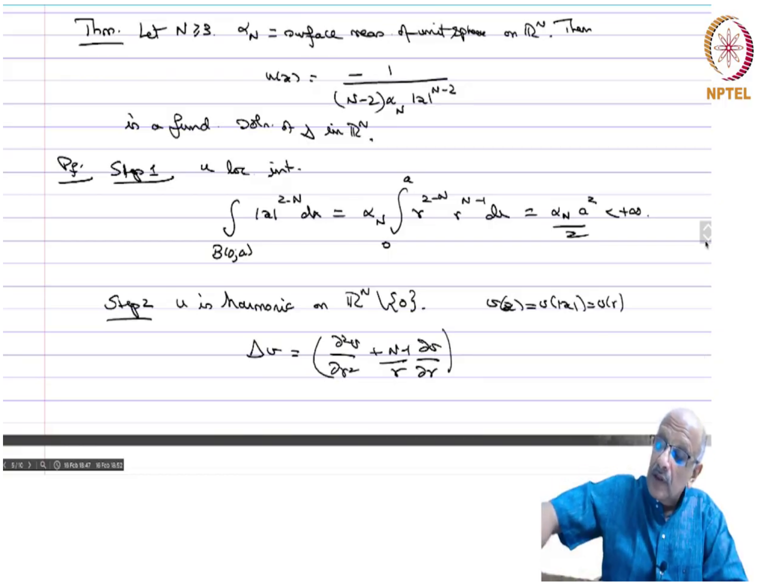

So, now we state the theorem, we will very rapidly prove it because we have more or less done most of the work in the previous theorem,

Theorem: Let $N \geq 3$, $\alpha_N = surface\ measure\ of\ unit\ ball\ in\ \mathbb{R}^N$ that means, the areas surface masseur of the unit sphere units, so let me call it unit sphere here also let me write, then

$$u(x) = \frac{-1}{(N-2)\alpha_N |x|^{(N-2)}} \text{ is fundamental solution of } \Delta \text{ in } \mathbb{R}^N$$

so we will proof.

So, step 1:

u is locally integrable why is it, so you have if you again you only need to check in a neighbourhood of origin, so

$$\int_{B(0,a)} |x|^{2-N} dx = \alpha_N \int_0^a r^{2-N} r^{N-1} dr = \alpha_N \frac{a^2}{2} < +\infty$$

, so that which is finite.

So, this is just polar coordinates in n dimensions and we are using the further no in the previous 1 we had $rd\theta$ we had a 2π the 2π came because that is the $\alpha_2 = \pi$, because the integral again was the radial function we integrated out and you got the2π. Now, when the $drd\theta$ we wrote now, we are not writing the other part we have integrated it out and produced it comes out as the surface measure of the unit ball.

Step 2:

u is harmonic on $\mathbb{R}^N - \{0\}$, so all you have to do is if $v\, is\, radial\; v(x) = v(|x|) = v(r)$, so, then we have

$$\Delta u = (\frac{^2 v}{r^2} + \frac{N-1}{r}\frac{v}{r})$$

so you just substitute and you calculate, so this will give you the delta of this function is 0 in Rn minus origin.

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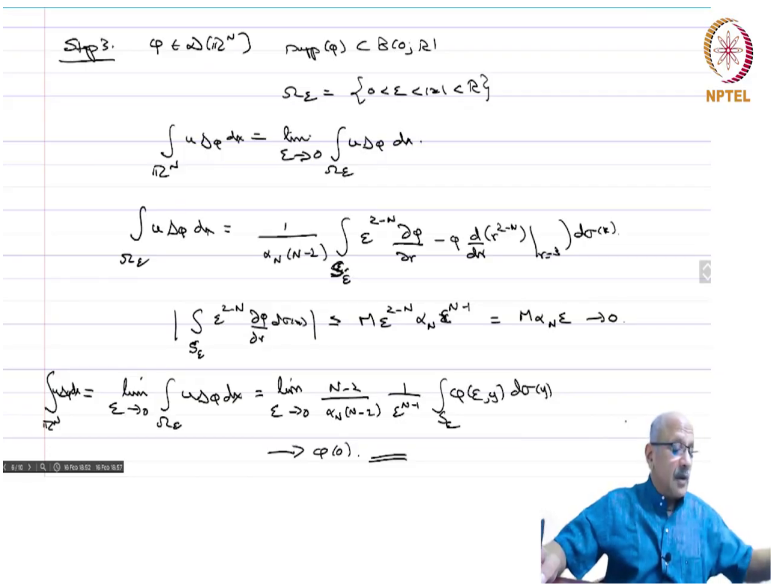

Step 3:

is the calculation which are going to go through rapidly, so $\varphi \in D(\mathbb{R}^N)$ $supp(\varphi) \subset B(0, R)$ ball centre origin radius R,

$$\Omega_\varepsilon = \{x:\ 0 < \varepsilon < |x| < R\}$$

so you have the same thing and therefore,

$$\int_{\mathbb{R}^N} u\Delta\varphi\, dx = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} u\Delta\varphi\, dx$$

Again I am going to use the green side entity and once more if I get that part $\Delta u\varphi$will go away to 0, because of the harmonic nature of u and therefore, you will have that is equal

$$\int_{\Omega_\varepsilon} u\Delta\varphi\, dx = \frac{1}{\alpha_N(N-2)}\int_{S_\varepsilon}\left(\varepsilon^{2-N}\frac{\varphi}{r} - \varphi\frac{d(r^{2-N})}{dr}\Big|_{r=\varepsilon}\right) d\sigma(x)$$

Again, this part will go to 0, why, because you have

$$|\int_{S_\varepsilon}\left(\varepsilon^{2-N}\frac{\varphi}{r}\right) d\sigma(x)\,| \le M\varepsilon^{N-1}\varepsilon^{2-N}\alpha_N = M\varepsilon\ \alpha_N \to 0$$

Therefore, you have that

$$\int_{\mathbb{R}^N} u\Delta\varphi\, dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} u\Delta\varphi\, dx = \lim_{\varepsilon\to 0}\frac{N-2}{\alpha_N(N-2)}\frac{1}{\varepsilon^{N-1}}\int_{S_\varepsilon}\varphi(\varepsilon, y)\, d\sigma(y) \ \to \varphi(0)$$

And again this $d\sigma(y)$ will be nothing but $\varepsilon^{N-1}\alpha_N$ times integral over the unit ball, so when you convert this to integral over the unit ball, you will get this ε^{N-1} will go and you can easily show that this goes to $\varphi(0)$ the usually add and subtract $\varphi(0)$ and you will get it.

So, this completes the proof, so you will try to do this calculation yourself, so this thing you add and subtract 0 and then you will show that the whole thing can go to 0, so this is about the fundamental solution of the Laplacian.

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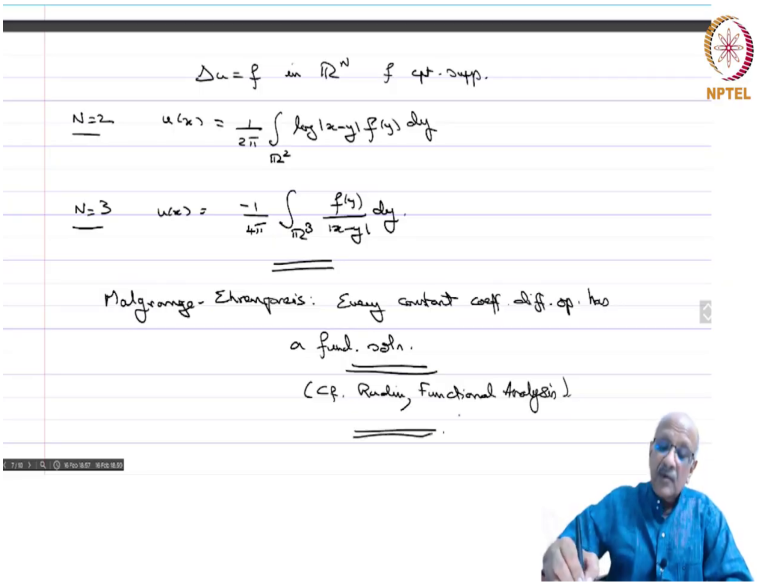

So, now if you have, suppose you have want to solve

$$\Delta u = f, \text{ f have compact support}$$

then we already saw fundamental solution star f is a solution, so in

$$N = 2, u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} log|x - y|\, f(y)\, dy$$

$$N = 3, u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy$$

You might have seen such formulae earlier when studying partial differential equations, in particular the Laplace operator, so this comes from this thing. So, finally this it is not necessary that f should have compact support; it is enough f has sufficiently good D-K properties, so that this integral makes sense then also one can show that this is the solution.

Now, as an application of the Han Banach theorem, Malgrange and Ehrenpreis have shown that every constant coefficient differential operator has a fundamental solution. So, this is Malgrange and Ehrenpreis theorem for instance, you can find a proof en route in functional analysis, so with that we will stop this discussion.