Sobolev Space and Partial Differential Equations

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Exercise - Part 2

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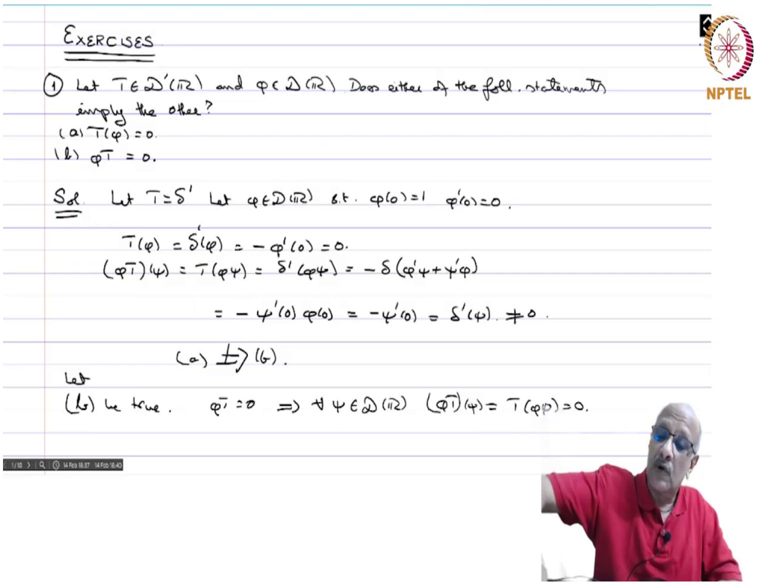

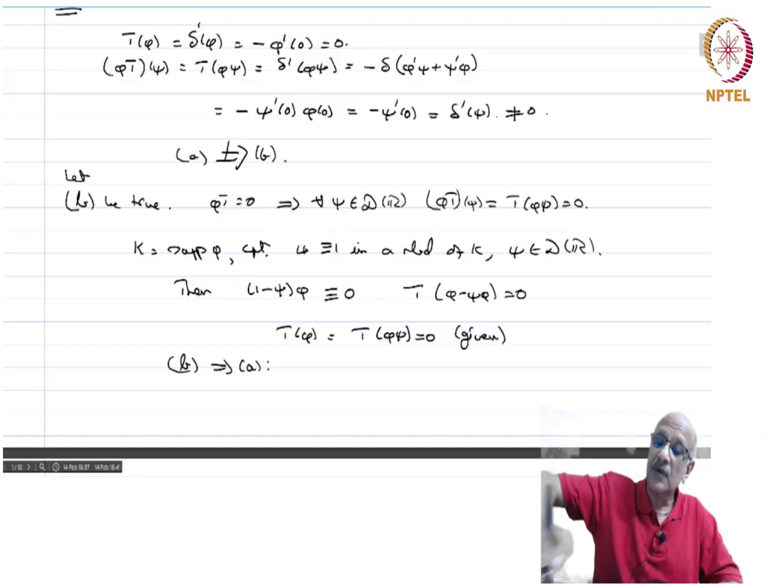

Before we continue with further topics on distributions, we will do some exercises. So, first 1, let $T \in D'(\mathbb{R})$ and $\varphi \in D(\mathbb{R})$. Does either of the following statements imply the other

(a) $T(\varphi) = 0,$

(b) $\varphi T = 0$.

So, we have 2 statements and we want to know if (a) implies (b) or (b) implies (a). Now, what do we guess from this, first one is a very weak statement, it just says $T(\varphi) = 0$.

You cannot expect really that it should have widespread repercussions whereas, phi of T equals 0 is a statement which refers to all C infinity functions because $\varphi T = 0$, means $\varphi T(\varphi_1) = 0$ for every $\varphi_1 \in D(\mathbb{R})$. So, you can expect (B) to be a strong statement and therefore, we do not really expect a to imply (b) but (b) might imply (a), we have to check that. So, let us first see the following. So, we will give an example. So, let

$$T = \delta'$$

and let $\varphi \in D(\mathbb{R})$ such that $\varphi(0) = 1$ and $\varphi'(0) = 1$.

So, then what is $T(\varphi)$?

$$T(\varphi) = \delta'(\varphi) =- \varphi'(0) = 0$$

Whereas, what is φT?

$$\varphi T(\psi) = T(\varphi\psi) = \delta'(\varphi\psi) =- \delta(\varphi'\psi + \psi'\varphi)$$

$$=- \psi'(0)\varphi(0) = \psi'(0) = \delta'(\psi) \neq 0.$$

therefore you have that (a) does not imply (b), as it is to be expected. Now, let us assume (b) true, therefore you have $\varphi T = 0$, what does it mean, this means that for

$\psi \in D(\mathbb{R})$ you have $(\varphi T)(\psi) = T(\varphi\psi) = 0$,

So, we want to know in particular this $T\varphi = 0$. So, let us take

$K = supp(\varphi)$ compact and let us take $\psi = 1$ in a neighborhood of K and $\psi \in D(\mathbb{R})$.

So, we know that such cut off functions exist. And therefore, you have then, what do you have $(1 - \psi)\varphi = 0$ Now, ϕ, ψ is identically 1 in the neighbourhood of the support φ, so this is 0 in the neighbourhood of support of φ, this is 0 outside support of ψ. So, this is identically 0. So,

$$T(\varphi - \psi\varphi) = 0$$

$$T(\varphi) = T(\varphi\psi) = 0$$

because that is given and therefore,

$$T(\varphi) = 0,$$

so (b) implies (a). So, that is the exercise.

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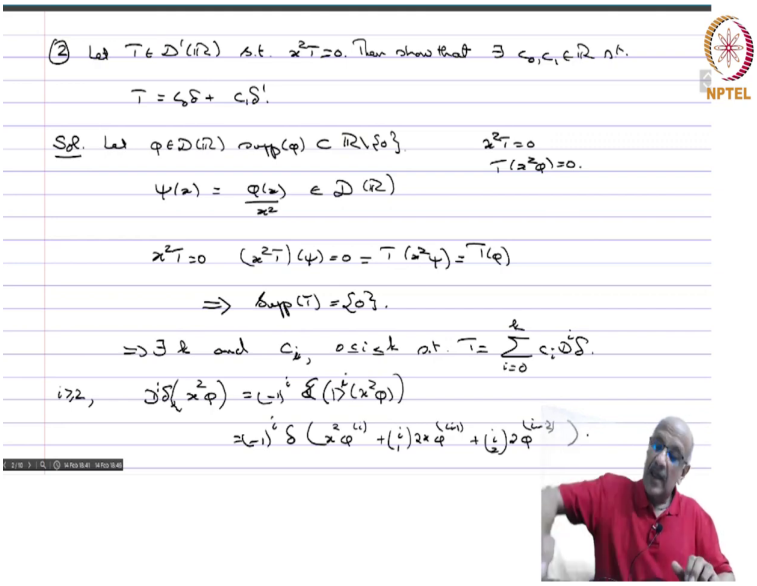

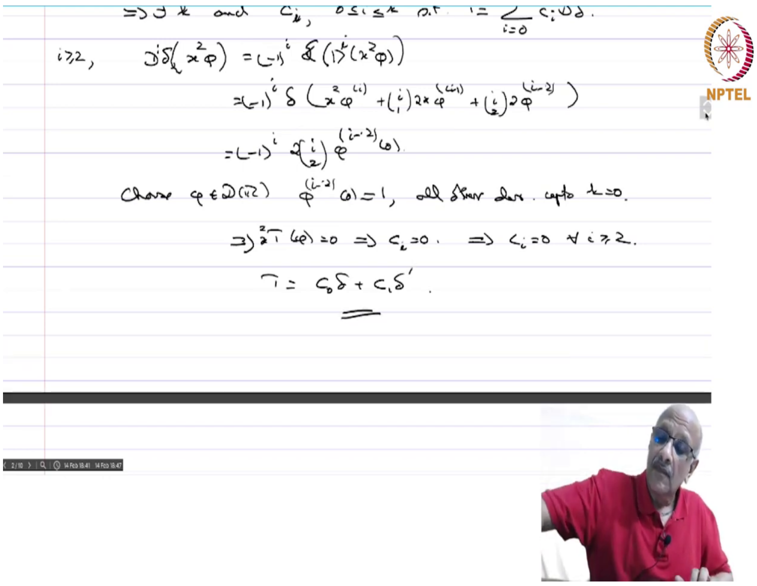

Second one.

Let $T \in D'(\mathbb{R})$ for such that $x^2T = 0$. Then show that there exists $c_0, c_1 \in \mathbb{R}$ such that

$$T = c_0\delta + c_1\delta'.$$

So, this means that if you have support $x^2T = 0$ then T must be a combination of Dirac and its first derivative only.

So solution, so, let $\varphi \in D(\mathbb{R})$ and $supp(\varphi) \subset \mathbb{R} - \{0\}$. So, we have a, what do we expect at 0 x is 0, outside 0 x is not 0 and we expect it to be 0, roughly these were functions that is what would happen, and that is what we are trying to show also for the distribution. So, we want to show that the support of T is at the origin, in this case T will be some linear combination of Dirac and its derivatives.

So, we are first going to show that the support of φ is that. So, for that we look atφ, support of φcontained in $\mathbb{R} - \{0\}$. Then you take

$$\psi(x) = \frac{\varphi(x)}{x^2} \in D(\mathbb{R})$$

because this has support away from the origin, therefore when you divide by x square it is perfectly well defined at the origin, near the origin the function is 0 and therefore, division 0 does not matter. So, this is a function which is very defined and it is in $D(\mathbb{R})$.

Therefore, you can have, but

$$x^2T = 0, \text{ so } x^2T(\psi) = 0 = T(x^2\psi) = T(\varphi)$$

So, $T\varphi = 0$ for all φ such that support of φ is away from this. So, what does this imply, T vanishes on$\mathbb{R} - \{0\}$, so that is the largest possible open set you can have and therefore, $supp(T) = \{0\}$.

So, this implies that there exists a k and c_i, $0 \leq k \leq i$ such that we have proved this

$$T = \sum_{i=0}^{k} c_i D^i\delta$$

So, this is what we have. And then, but we are, we also have that x square $x^2T = 0$. So, let us take

$$D^i\delta(x^2\varphi) = (-1)^i\delta(D^i(x^2\varphi))$$

$$=(-1)^i\delta(x^2\varphi^{(i)} + \binom{i}{1} 2x\varphi^{(i+1)} + \binom{i}{2}2\varphi^{(i-2)}$$

$$= (-1)^i 2\binom{i}{2}\varphi^{(i-2)}(0)$$

so, I have used some different notation, so let just say delta acting on $D^{(i)}x^2\varphi$. Now you apply like this formula.

And thereafter there would not be any more derivatives because x square will not have any more derivatives after that. So, if i is bigger than 2 this is what we have.

so choose $\varphi \in D(\mathbb{R})$ $\varphi^{(i-2)}(0) = 1$, all other derivatives up to k equal to 0.

And this will imply if you, then if you apply it to

$$\Rightarrow x^2 T(\varphi) = 0 \Rightarrow c_i = 0$$

. Therefore, so this implies that

$$c_i = 0, \forall i \geq 2$$

$$\Rightarrow T = c_0\delta + c_1\delta^{(1)}.$$

So, you just plug it in and you check the fact.

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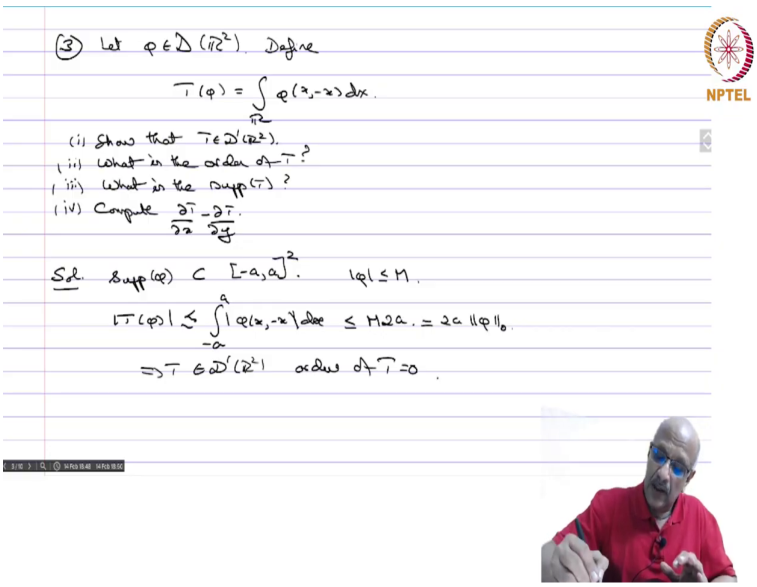

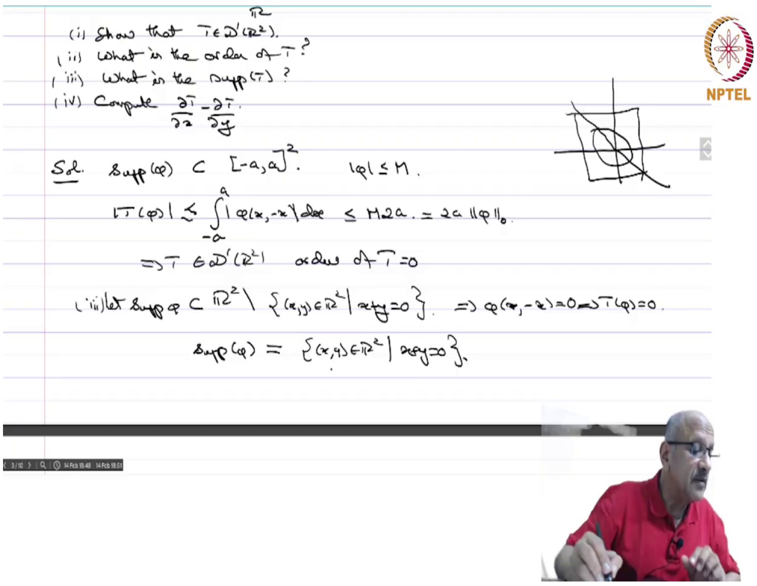

(3), let $\varphi \in D(\mathbb{R}^2)$. Define

$$T(\varphi) = \int_{\mathbb{R}} \varphi(x, -x)dx$$

(i) show that $T \in D'(\mathbb{R}^2)$ that it is a distribution.

(ii) What is the order of T?

(iii) What is the support of T? and

(iv) Compute $\frac{T}{x} - \frac{T}{y}$

So, solution – so let us take $supp(\varphi) \subset [-a, a]^2$ and let us say $|\varphi| \leq M$

$$|T(\varphi)| \leq \int_{-a}^{a} |\varphi(x, -x)|dx \leq M2a = 2a||\varphi||_0$$

$$\Rightarrow T \in D'(\mathbb{R}^2)$$

and order of $T = 0$ because you have such an inequality. Remember we proved the theorem for every phi if you can prove such a thing with a constant depending on the support of φ , which here it is 2a.

And therefore, you and here have an independent of the support this for all seen for the D functions with compact support and therefore, the order of T is 0. So, that solves (i) and (ii), ⇒ what about the support of T. So (iii) so let us take

$$supp(\varphi) \subset \mathbb{R}^2 - \{(x, y) \in \mathbb{R}^2 | x + y = 0 \} \Rightarrow \varphi(x, -x) = 0 \Rightarrow T(\varphi) = 0.$$

$$\Rightarrow supp(\varphi) \subset \{(x, y) \in \mathbb{R}^2 | x + y = 0 \}$$

But if you take line y=x, you can always find a function which is supported in any part of that line. So, the integral will not be equal to 0. So, in fact, you can say that

$$\Rightarrow supp(\varphi) = \{(x, y) \in \mathbb{R}^2 | x + y = 0 \}$$

So, now, let us compute the derivative.

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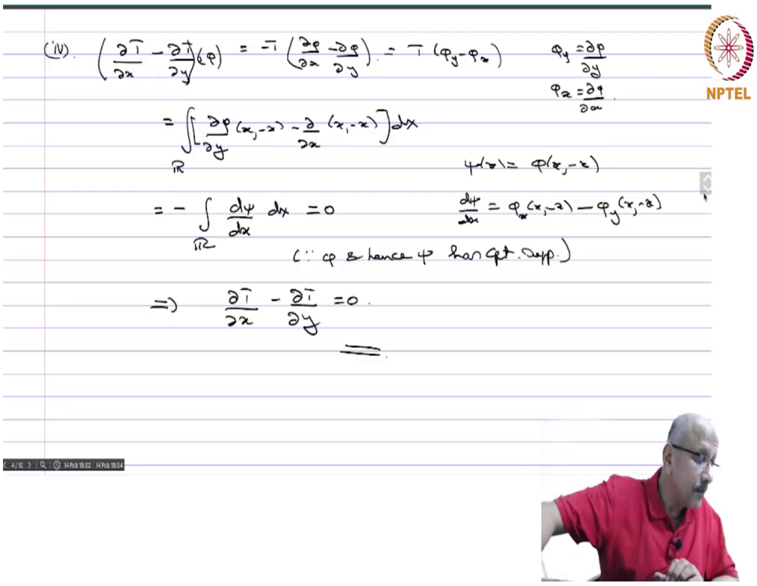

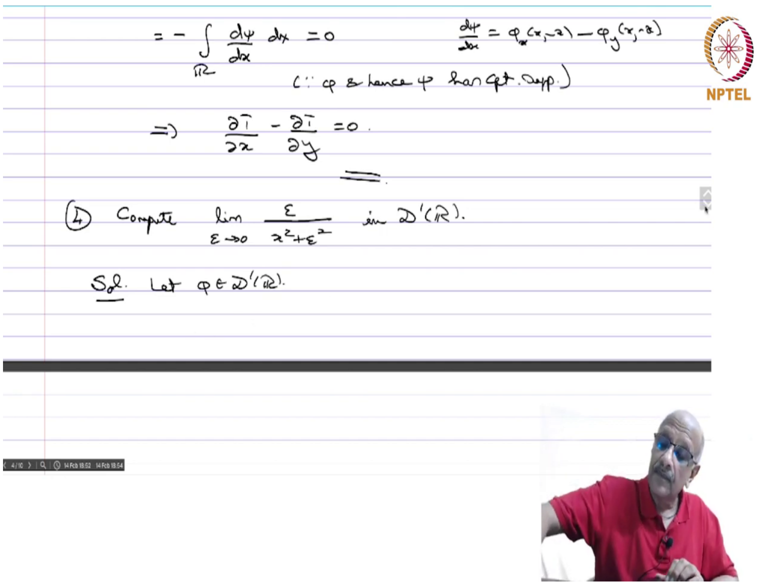

So, what is

$$\frac{T}{x} - \frac{T}{y}(\varphi) = -T(\frac{\varphi}{x} - \frac{\varphi}{y}) = T(\varphi_y - \varphi_x) \qquad \varphi_y = \frac{\varphi}{y}$$

$$\varphi_x = \frac{\varphi}{x}$$

I am just using that notation. So, and what is that equal to,

$$= \int_{\mathbb{R}} [\frac{\varphi}{y}(x, -x) - \frac{\varphi}{x}(x, -x)]dx \qquad \psi(x) = \varphi(x, -x)$$

$$= \int_{\mathbb{R}} \frac{d\psi}{dx} dx = 0 \qquad \Rightarrow \frac{d\psi}{dx} = \varphi_x(x, -x) - \varphi_y(x, -x)$$

but φ has compact support, since φ and hence ψhas compact support. So, this implies, so this is true for every φ, so

$$\frac{T}{x} - \frac{T}{y} = 0.$$

(iv), compute

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon^2} \quad \text{in} \quad D'(\mathbb{R}\ )$$

Solution, so let $\varphi \in D'(\mathbb{R}\ )$.

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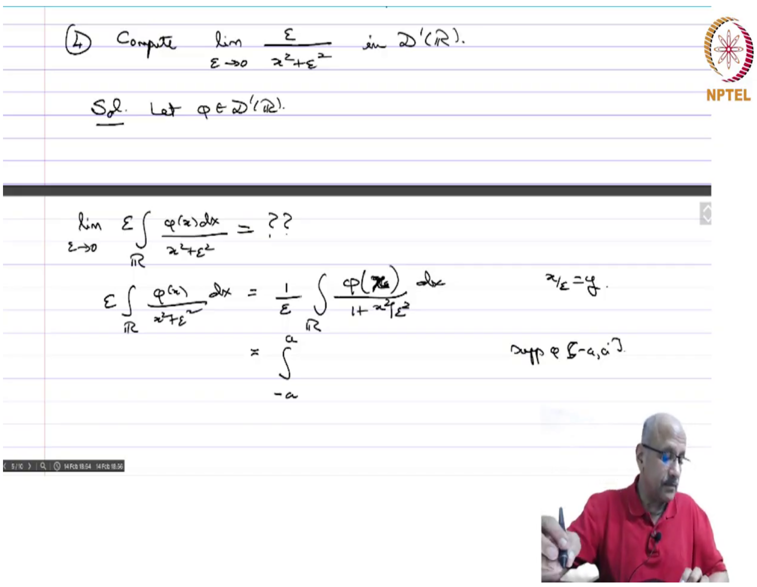

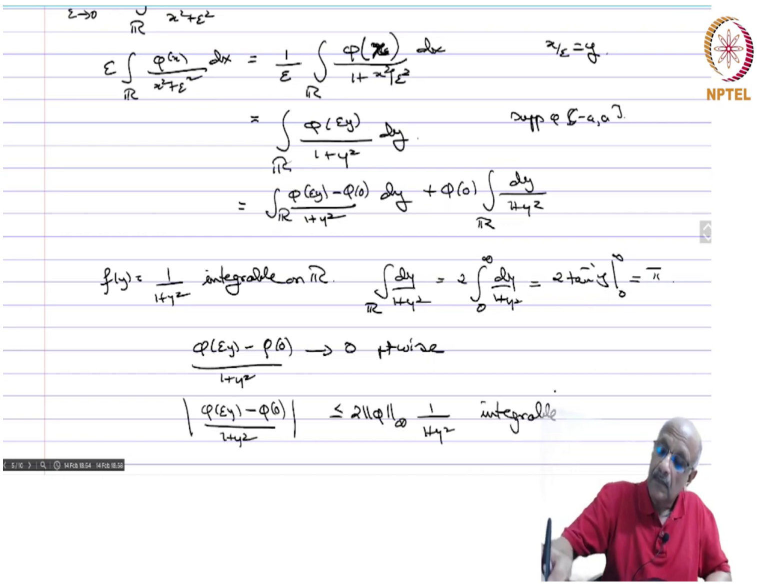

So, we have to look at

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\mathbb{R}} \frac{\varphi(x)dx}{x^2+\varepsilon^2} = ??$$

$$\varepsilon \int_{\mathbb{R}} \frac{\varphi(x)dx}{x^2+\varepsilon^2} = \int_{\mathbb{R}} \frac{\varphi(\varepsilon y)dy}{1+y^2} \quad \text{put} \quad \frac{x}{\varepsilon} = y$$

So, that will just give you that.

So, I am going to write this as integral over $\mathbb{R}$.

$$=\int_{\mathbb{R}} \frac{\varphi(\varepsilon y)-\varphi(0)dy}{1+y^2} + \varphi(0)\int_{\mathbb{R}} \frac{dy}{1+y^2}$$

so, $f(y) = \frac{1}{1+y^2}$ is integrable over $\mathbb{R}$.

So, you can check that. So, away from the origin, this is near the origin this is a nice continuous function it does not vanish and in the neighbourhood of the origin and away from the origin it is less than equal to $\frac{1}{y^2}$and which is a integrable function. And therefore, you have that $f(y)$is integral.

So then, and also you can compute the integral,

$$\int_{\mathbb{R}} \frac{1}{1+y^2} = 2\int_0^{\infty} \frac{1}{1+y^2} = 2\, tan^{-y}|_0^{\infty} = \pi$$

So now,

$\frac{\varphi(\varepsilon y)-\varphi(0)}{1+y^2}$ converges to 0 pointwise and

$|\frac{\varphi(\varepsilon y)-\varphi(0)dy}{1+y^2}| \leq 2||\varphi||_{\infty} \frac{1}{1+y^2}$ is integrable

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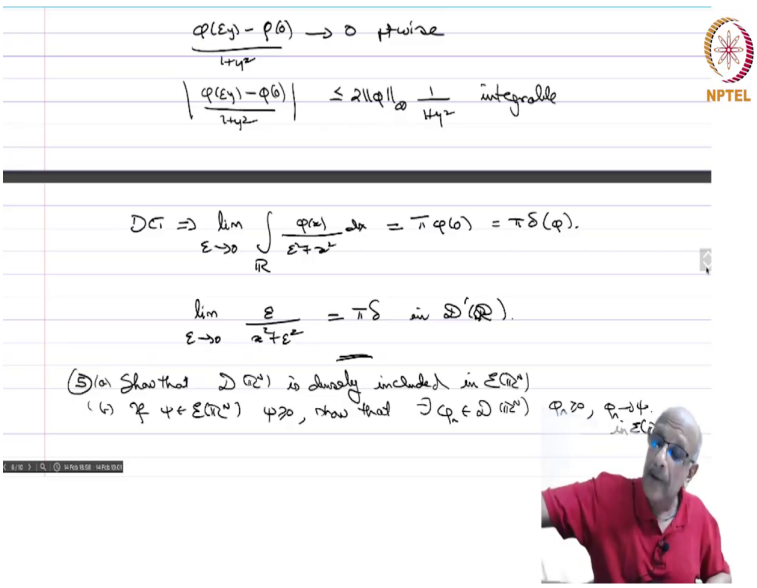


$$\varepsilon\int_{\mathbb{R}} \frac{\varphi(x)}{x^2+\varepsilon^2}\,dx = \frac{1}{\varepsilon}\int_{\mathbb{R}}\frac{\varphi(x)}{1+x^2/\varepsilon^2}\,dx \qquad x/\varepsilon = y.$$

$$= \int_{\mathbb{R}} \frac{\varphi(\varepsilon y)}{1+y^2}\,dy. \qquad \text{supp}\,\varphi \subset [-a,a]$$

$$= \underbrace{\int_{\mathbb{R}} \frac{\varphi(\varepsilon y)-\varphi(0)}{1+y^2}\,dy}_{\downarrow\, 0} + \varphi(0)\int_{\mathbb{R}}\frac{dy}{1+y^2}$$

$f(y)=\frac{1}{1+y^2}$ integrable on $\mathbb{R}$. $\int_{\mathbb{R}}\frac{dy}{1+y^2} = 2\int_0^\infty \frac{dy}{1+y^2} = 2\tan^{-1}y\Big|_0^\infty = \pi.$

$\frac{\varphi(\varepsilon y)-\varphi(0)}{1+y^2} \to 0$ ptwise

$\left|\frac{\varphi(\varepsilon y)-\varphi(0)}{1+y^2}\right| \le 2\|\varphi\|_\infty \frac{1}{1+y^2}$ integrable

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{\varepsilon\,\varphi(x)}{\varepsilon^2+x^2}\,dx = \pi\varphi(0) = \pi\delta(\varphi).$$

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{x^2+\varepsilon^2} = \pi\delta \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

(5)(a) Show that $\mathcal{D}(\mathbb{R}^N)$ is densely included in $\mathcal{E}(\mathbb{R}^N)$

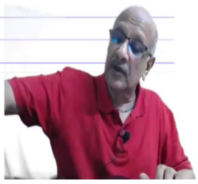
(b) If $\psi\in\mathcal{E}(\mathbb{R}^N)$, $\psi\ge 0$, show that $\exists\, \varphi_n\in\mathcal{D}(\mathbb{R}^N)$, $\varphi_n\ge 0$, $\varphi_n\to\psi$ in $\mathcal{E}(\mathbb{R}^N)$.

Sol. (a) Let $\varphi_n\in\mathcal{D}(\mathbb{R}^N)$ (i) $0\le\varphi_n\le 1$.

(ii) $\varphi_n\equiv 1$ in a nbd. of $\overline{B}(0;n)$.

(iii) supp $\varphi_n\subset\overline{B}(0;n+1)$

$\psi\in\mathcal{E}(\mathbb{R}^N)$ Consider $\varphi_n\psi\in\mathcal{D}(\mathbb{R}^N)$ supp$(\varphi_n\psi)\subset\overline{B}(0;n)$



(5)(a) Show that $\mathcal{D}(\mathbb{R}^N)$ is densely included in $\mathcal{E}(\mathbb{R}^N)$

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$\psi\in\mathcal{E}(\mathbb{R}^N)$ Consider $\varphi_n\psi\in\mathcal{D}(\mathbb{R}^N)$ supp$(\varphi_n\psi)\subset\overline{B}(0;n)$ cpt.

K any cpt set in $\mathbb{R}^N$.

$\exists\, n_0$ s.t. for $n\ge n_0$, $K\subset\overline{B}(0;n)$.

Therefore, by the dominated convergence theorem, we get the limit we know what the limit is, so we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \frac{\varphi(x)dx}{x^2+\varepsilon^2} = \pi\varphi(0) = \pi\delta(0)$$

So, the first term goes to 0 by the dominated convergence theorem and the second one we just know computed that is equal

Therefore, $\lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon^2} = \pi\delta$ in $D'(\mathbb{R})$

(v), have already used this and I said it is very easy, so let us do this when we discussed compact, distributions with compact support. So, let

(a) show that $D(\mathbb{R}^N)$ is densely included in $\mathcal{E}(\mathbb{R}^N)$. And

(b) , if $\psi \in E(\mathbb{R}^N)$, $\psi \geq 0$, show that there exists $\varphi_n \in D(\mathbb{R}^N)$,$\varphi_n \geq 0$,$\varphi_n \to \psi$ in $\mathcal{E}(\mathbb{R}^N)$

So, you can approximate any non negative c infinity function by a non-negative c infinity function with compact support.

So, this non negative D can be the, so solution. So, again very, very easy. So a, so let us takeφ_n

let$\varphi_n \in D(\mathbb{R}^N)$, such that

(i) $0 \leq \varphi_n \leq 1,$

(ii) $\varphi_n \equiv 1$ in the neighbourhood of $\overline{B}(0; n)$

(iii) $supp\ \varphi_n \subset \overline{B}(0; n+1)$

you can put anything bigger than n here it does not matter what you are going to put here.

So this, is given. So now let take

$\psi \in \mathcal{E}(\mathbb{R}^N)$ and consider $\varphi_n \psi$. This is my product of a distribution with compact support and seen, I mean seen infinity function with compact support and see infinity function, so then this

$\varphi_n \psi \in D(\mathbb{R}^N)$In fact $supp(\varphi\psi) \subset \overline{B}(0; n)$. So that is compact.

So, now let K be any compact set. Then for the n, so $\exists n_0$such that for $\exists n \geq n_0$ we have, $K \subset \overline{B}(0; n)$ after some time you can absorb it in any big ball. So, since it is a compact set. So, compact sets are bonded and therefore, you can put them inside any big ball after sometime.

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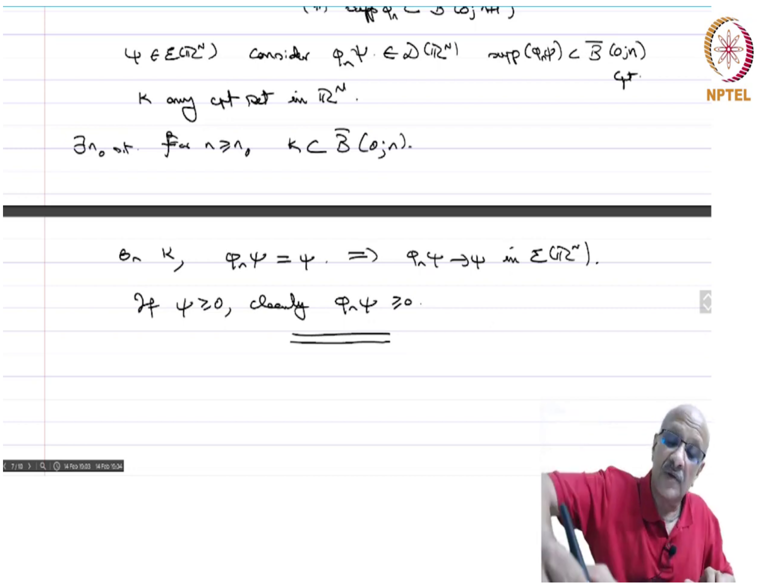

So, on K we have that $\varphi_n \equiv 1$, so $\varphi_n\psi = \psi$. So, trivially these functions agree on terminally on any compact set, so this implies that

$$\varphi_n\psi \rightarrow \psi \text{ in } \mathcal{E}(\mathbb{R}^N).$$

And therefore, you have approximated everything. Now if

$$\psi \geq 0 \text{, clearly} \qquad \varphi_n\psi \geq 0.$$

So, that completes. So, we will stop with this and now we will continue with some further topics in the theory of distributions next time.