Sobolev Space and Partial Differential Equations

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Convolution of Distribution - Part 3

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So, we had $T, S \in D'(\mathbb{R}^N)$ at least one of them with compact support. Then we define $T * S \in D'(\mathbb{R}^N)$ which is characterized by the following conditions :

$$(T * S)(\varphi) = (T * (S * \check{\varphi}))(0)$$

$$(T * S) * \varphi = T * (S * \varphi)$$

and both these for all $\varphi \in D(\mathbb{R}^N)$. So, either of these we will verify, we will characterize T star it is a unique one.

So, now, we want to show that this convolution has all the properties with functions and therefore, we have the following theorem. So, T_1 and T_2 distributions on $\mathbb{R}^N$ at least one with compact support then

(i) $T_1 * T_2 = T_2 * T_1$

(ii) T_1 and T_2 as the above. Then

$$supp(T_1 * T_2) \subset supp(T_1) + supp(T_2).$$

(iii) $T_1, T_2, T_3 \in D'(\mathbb{R}^N)$ at least 2 of them with compact support. Then

$$T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3$$

(iv) $T_1, T_2 \in D'(\mathbb{R}^N)$ as in (i) α any multi-index. Then

$$D^{\alpha}(T_1 * T_2) = D^{\alpha}T_1 * T_2 = T_1 * D^{\alpha}T_2.$$

So, you can push the derivative wherever you like.

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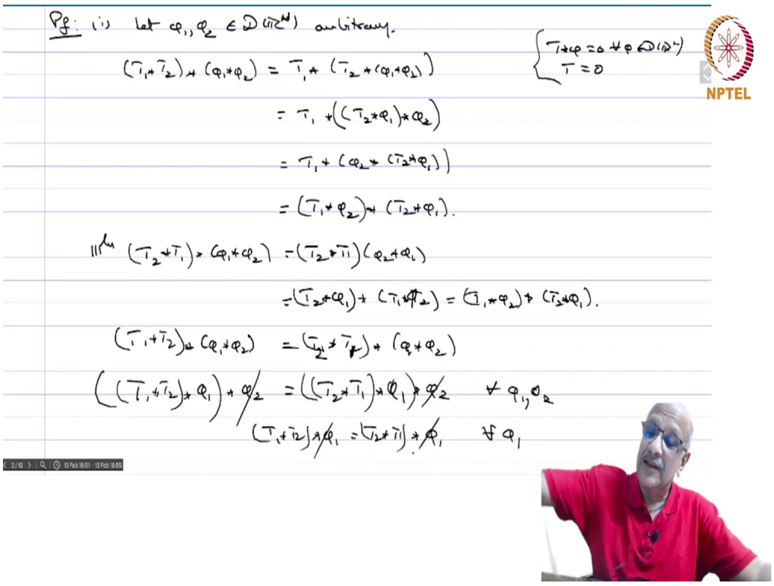

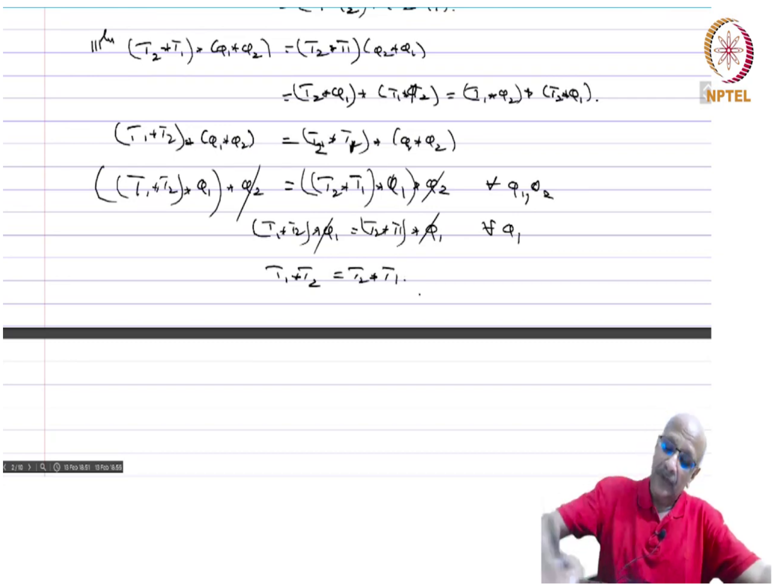

So, now let us give a proof of these properties which mirror the properties of convolution of functions. So, first one, so let $\varphi_1, \varphi_2 \in D(\mathbb{R}^N)$ arbitrary. So, let us take $(T_1 * T_2) * (\varphi_1 * \varphi_2)$. So, this again an infinity functions with compact support. So, therefore, we know you have 2 functions 2 distributions 1 of them compact support and c infinity functions with compact support. So, we have proved this theorem

$$= T_1 * (T_2 * (\varphi_1 * \varphi_2))$$

$$= T_1 * ((T_2 * \varphi_1) * \varphi_2)$$

Again, we have shown this when you have 2 with compact support and 1 is a distribution So, you have this, so you have distribution into functions compact support, so once again I am using the theorem. Now, this is equal to

$$= T_1 * (\varphi_2 * (T_2 * \varphi_1))$$

I am just using the commutativity of the convolution of functions.

Now, so now, I want suppose T_2 has compact support then this has $T_2 * \varphi_1$ will also have compact support we have seen that and this has compact support, so we can use the associativity law. If T_1has compact support then you have again φ_2 has compact support and this is the c infinity function once again we have used we had the second theorem which we proved that can be used and therefore, this can be written as $T_1 * ((T_2 * \varphi_1) * \varphi_2)$ using the 2 preceding theorems.

Similarly,

$$(T_2 * T_1) * (\varphi_1 * \varphi_2) = (T_1 * T_2) * (\varphi_1 * \varphi_2$$

. And now going by what went, you have first with the second and second with the first. So, this is

$$= (T_2 * \varphi_1) * (T_1 * \varphi_2)$$

$$= (T_1 * \varphi_2) * (T_2 * \varphi_1)$$

So, these 2 are equal, so we have

$$(T_1 * T_2) * (\varphi_1 * \varphi_2) = (T_2 * T_1) * (\varphi_1 * \varphi_2)$$

And now, this is

$$((T_1 * T_2) * \varphi_1) * \varphi_2 = ((T_2 * T_1) * \varphi_1) * \varphi_2$$

But then by the fourth part of the first one that is if

$T * \varphi = 0$ for all $\varphi \in D(\mathbb{R}^N)$ then $T = 0$.

So, this is a very important thing. So, I can cancel this, and so you get

$$(T_1 * T_2) * \varphi_1 = (T_2 * T_1) * \varphi_1$$

this is for all φ_1, φ_2 and this is for all φ_1.

Again, this is true for all φ_1 and therefore, I can once more cancel it and so, I will get

$$T_1 * T_2 = T_2 * T_1$$

So, this proves the first part

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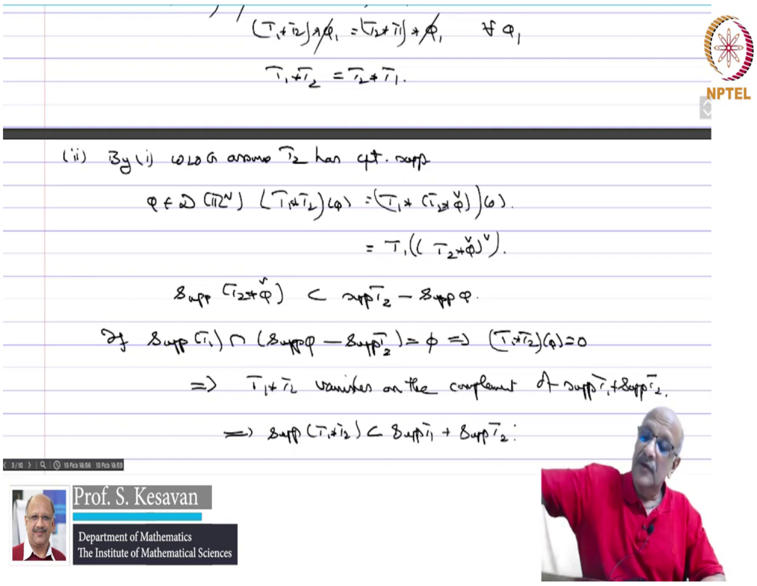

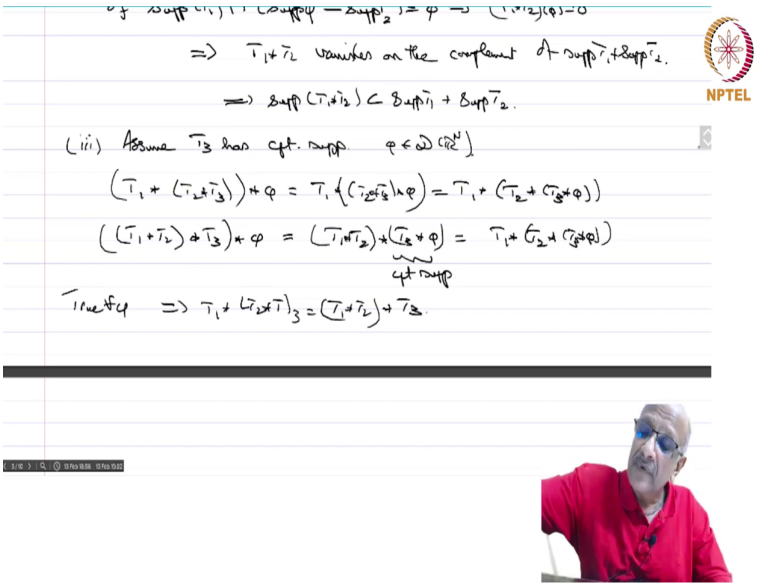

So, the second part, so by 1 without loss of generality assume T_2 has compact support. So, we have T_1 and T_2. So, if 1 of them has compact support, if you, if T_1 has compact support then $T_1 * T_2 = T_2 * T_1$. So, the second 1 you can say always has compact support. Now, if you take any $\varphi \in D(\mathbb{R}^N)$, then you have the

$$(T_1 * T_2)(\varphi) = (T_1 * (T_2 * \check{\varphi}))(0)$$

So, you have, this is c infinity function with compact support, this distribution and you are evaluating at 0. So,

$$= T_1((T_2 * \check{\varphi})^{\vee})$$

That is I am just using the definition of the function, the convolution of this distribution and the c infinity function with compact support. But what is support of T_2star phi chesh,

$$supp(T_2 * \check{\varphi}) \subset supp(T_2) - supp(\varphi)$$

So, if

$$supp(T_1) \cap (supp(T_2) - supp(\varphi)) = \phi \quad \Rightarrow (T_1 * T_2)(\varphi) = 0$$

.So, that means, this will be 0 because this has a support which is the minus of this 1. So, we have to take the minus of that, because we have a check support φto minus support of T_2 because that is the minus of this and therefore, which is the support of T_2star φ check check and therefore, the these 2 are disjoint then this will be 0.

So, this implies that 2 should not be disjoint that means, implies support of T_1 and T_2. So, the $T_1 * T_2$ will vanish on the complement of, so this implies T_1 start T_2 vanishes on the complement of support T_1plus support $T_2$from this it follows. And therefore, this means

$$supp(T_1 * T_2) \subset supp(T_1) + supp(T_2)$$

So, that is the second part. So, now, let us take the third one.

So, if T_1, T_2, T_3 two of them have compact support, I already checked that you can define any of those triple operations. So, let us assume, that assume T_3 has compact support, at least 2 of them will have. So, let us assume $T_3$is compact support. Let $\varphi \in D(\mathbb{R}^N)$. So,

$$(T_1 * (T_2 * T_3)) * \varphi = T_1 * ((T_2 * T_3) * \varphi)$$

Again, I am using the fact that you have 2 distributions and the star, so it is characterised the convolution is characterized this way. And that equal to

$$= T_1 * (T_2 * (T_3 * \varphi))$$

$$((T_1 * T_2) * T_3) * \varphi = (T_1 * T_2) * (T_3 * \varphi) = T_1 * (T_2 * (T_3 * \varphi))$$

And then $((T_1 * T_2) * T_3) * \varphi$ now the last 1 is possible because T_3 start phi is a infinity function with compact support because phi and T_3 are both having compact support.

So, this has compact support. And therefore, I can use the characterization and so these 2 are equal. So, they are equal for all phi, so this implies true for all phi. So, T_1 star by the cancellation theorem, you have

$$T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3$$

So, what happens if T_3 does not have compact support?

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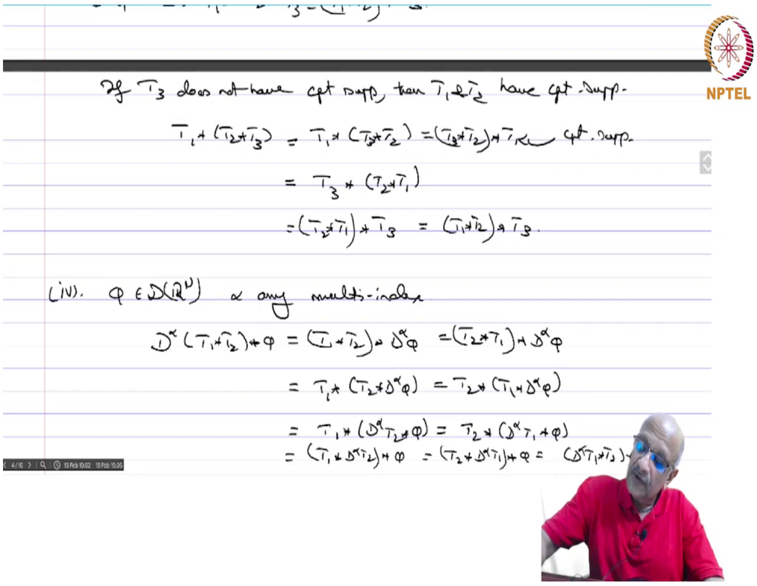

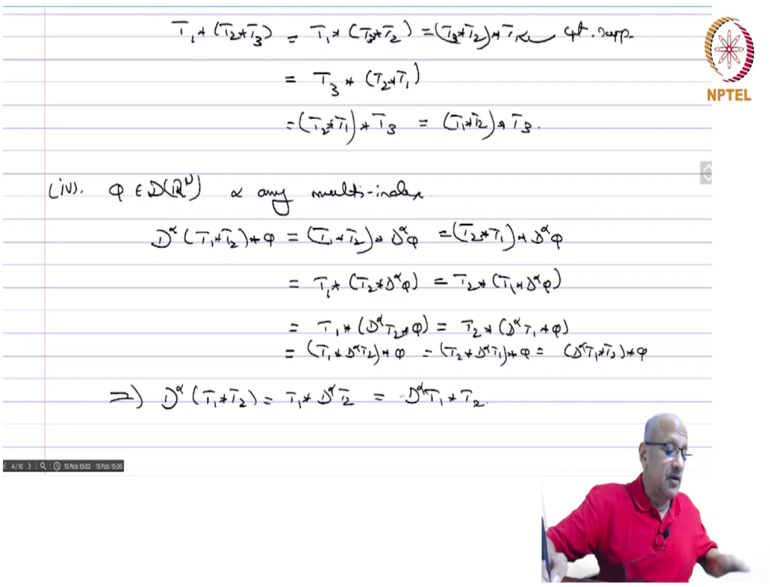

Now, we use the commutativity T if T_3 does not have compact support then T_1 and T_2 have compact support. So, we get

$$T_1 * (T_2 * T_3) = T_1 * (T_3 * T_2) = (T_3 * T_2) * T_1$$

by the commutativity this equal to T_3 start T_1 . Now, T_1 has compact support and therefore, by the first part of this, of this section, so you have this and that proves the whole thing.

So, now 4, so $\varphi \in D(\mathbb{R}^N)$ and α any multi-index, so then again we will use the cancellation

$$D^{\alpha}(T_1 * T_2) * \varphi = (T_1 * T_2) * D^{\alpha}\varphi = (T_2 * T_1) * D^{\alpha}\varphi$$

$$= T_1 * (T_2 * D^{\alpha}\varphi) = T_2 * (T_1 * D^{\alpha}\varphi)$$

$$= T_1 * (D^{\alpha}T_2 * \varphi) = T_2 * (D^{\alpha}T_1 * \varphi)$$

$$= (T_1 * D^{\alpha}T_2) * \varphi = (T_2 * D^{\alpha}T_1) * \varphi = (D^{\alpha}T_1 * T_2) * \varphi$$

$$\Rightarrow D^{\alpha}(T_1 * T_2) = T_1 * D^{\alpha}T_2 = D^{\alpha}T_1 * T_2$$

So, this proves the (())(18:51).

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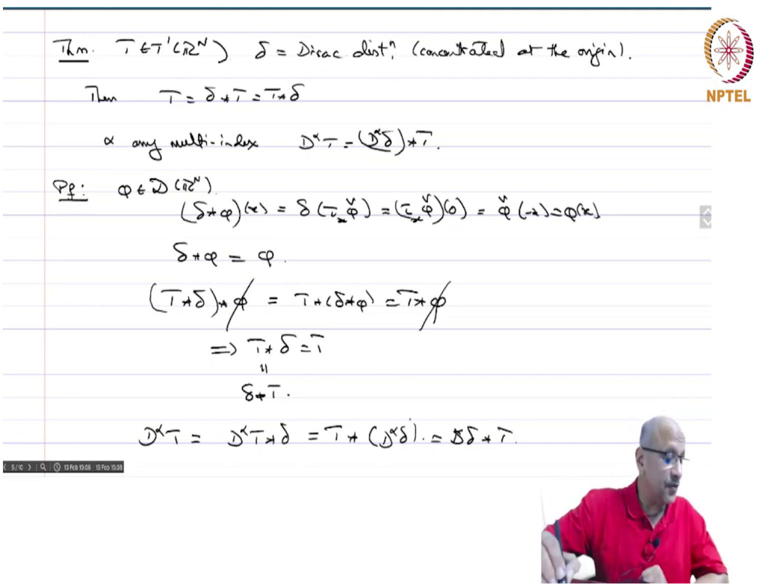

So, now, as an example, and also it is important the Dirac distribution plays a particular important state, has a special status with respect to convolution.

So, theorem $T \in D'(\mathbb{R}^N)$, $\delta=$ Dirac distribution concentrated at 0. Then

$$T = \delta * T = T * \delta$$

So, if you, it is liking acting like the identity element in this binary operation. Delta is like the binary.

If α any multi-index then

$$D^{\alpha}T = (D^{\alpha}\delta) * T$$

Remember, delta has support singleton and the origin so do all its derivatives. And therefore, it is a distribution with compact support therefore, you can convolve it with any other distribution there is no problem. So, proof let $\varphi \in D(\mathbb{R}^N)$ then

$$(\delta * \varphi)(x) = \delta(\tau_x \check{\varphi}) = (\tau_x \check{\varphi})(0) = \check{\varphi}(-x) = \varphi(x)$$

$$\Rightarrow \quad \delta * \varphi = \varphi$$

So, if you take

$$(T * \delta) * \varphi = T * (\delta * \varphi) = T * \varphi$$

and so, by the cancellation like and again cancel this and therefore, you get

$$T * \delta = T = T * \delta.$$

Now, if you take

$$D^{\alpha}T = D^{\alpha}T * \delta = T * (D^{\alpha}\delta) = D^{\alpha}\delta * T$$

So, this proves this theorem. So, if you have a remark.

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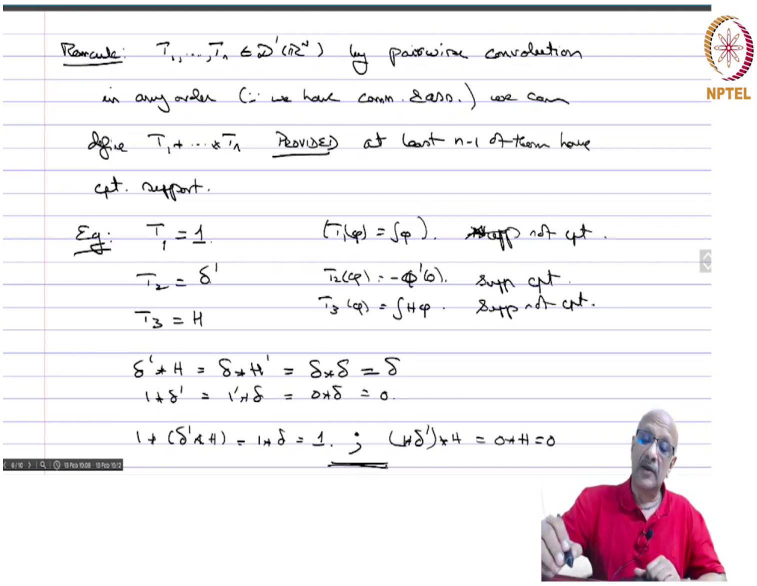

If you have T_1 to $T_n \in D'(\mathbb{R}^N)$ then by pairwise convolution in any order since we have commutativity and associativity, we can define $T_1 * \ldots * T_n$ provided this is really important, at least n minus 1 of them have compact support. So, this, without this compact support, you cannot have the associative law and therefore, you cannot in an unambiguous way define the convolution in all this. So, let us give an example.

So, let us take 3 distributions, So,

$$T_1 = 1, \qquad T_1(\varphi) = \int \varphi$$

$$T_2 = \delta', \qquad T_2(\varphi) = -\varphi'(0)$$

$$T_3 = H, \qquad T_3(\varphi) = \int H\varphi$$

So, this is 3 distributions we have. So let us take

$$\delta' * H = \delta * H' = \delta * \delta = \delta,$$

Now,

$$1 * \delta' = 1' * \delta = 0 * \delta = 0$$

So, let us take

$$1 * (\delta' * H) = 1 * \delta = 1;$$

On the other hand you have

$$(1 * \delta') * H = 0 * H = 0;$$

so these 2 are not equal. So, you do not have the associative law. So, therefore, it is important and why is it so, this is not of compact, not compact support so support not compact, in fact, is entire real line. This is support compact, and this support not compact, it is the negative, positive real axis.

So, therefore, you have to have them with non-compact supports and therefore, you cannot do this, so much about the convolution of distributions. Now, we will look at some applications.