Sobolev Space and Partial Differential Equations

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Convolution of Distribution – part 2

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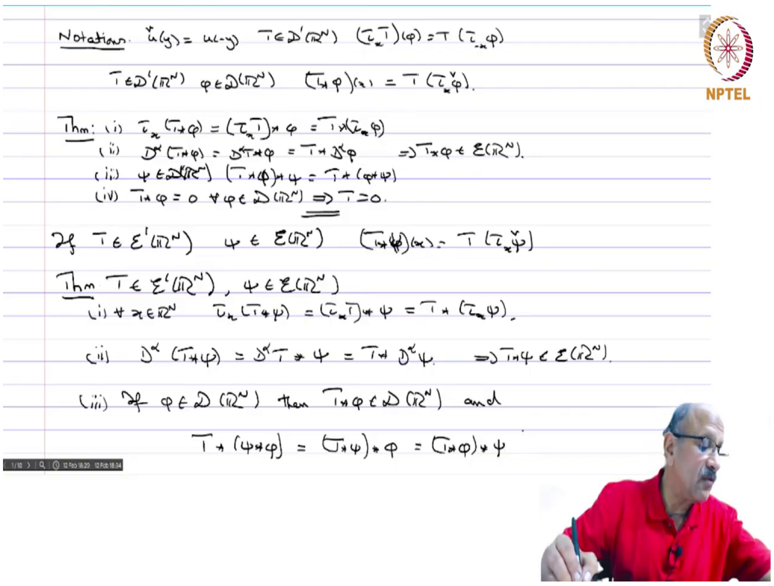

So, we were looking at convolution of distributions, so the notations we were using,

$u^{\vee}(y) = u(-y)$, $T \in D'(\mathbb{R}^N)$, $(\tau_x T)(\phi) = T(\tau_x \phi)$, $T * \phi(x) = T(\tau_x \phi^{\vee})$.

Now, the properties of this function are we proved, so we prove the following theorem:

Theorem: $T \in D'(\mathbb{R}^N)$, $\phi \in D(\mathbb{R}^N)$.

(i) for any $x \in \mathbb{R}^N$, $\tau_x(T * \phi) = \tau_x T * \phi = T * \tau_x \phi$.

(ii) for all multi-index α, $D^{\alpha}(T * \phi) = D^{\alpha}T * \phi = T * D^{\alpha}\phi$. In particular $T * \phi \in C^{\infty}(\mathbb{R}^N)$.

(iii) if $\psi \in D(\mathbb{R}^N), T * (\phi * \psi) = (T * \phi) * \psi$.

(iv) if $T * \phi = 0, \forall\phi \in D(\mathbb{R}^N), \ then \ T = 0$.

So, now it is clear that if $T \in E'(\mathbb{R}^N)$ and $\psi \in E(\mathbb{R}^N)$, then again we can define

$$T * \psi(x) = T(\tau_x \psi^{\vee}).$$

Now, we have the following theorem which is analogue of the previous theorem:

Theorem: $T \in E'(\mathbb{R}^N), \ \phi \in E(\mathbb{R}^N)$.

(i) for any $x \in \mathbb{R}^N$, $\tau_x(T * \phi) = \tau_x T * \phi = T * \tau_x \phi$.

(ii) for all multi-index α, $D^{\alpha}(T * \phi) = D^{\alpha}T * \phi = T * D^{\alpha}\phi$. In particular $T * \phi \in E(\mathbb{R}^N)$.

(iii) if $\psi \in D(\mathbb{R}^N), \ then \ T * \phi \in D(\mathbb{R}^N), \ and$

$$T * (\phi * \psi) = (T * \phi) * \psi = (T * \psi) * \phi.$$

So these are the three theorems.

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Pf: (i) & (ii) Exactly as in preceding thm.

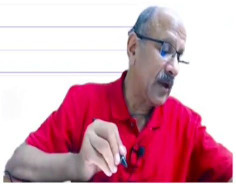
(iii). $K = \operatorname{supp}(T)$ $H = \operatorname{supp}(\varphi)$ both cpt.

$(T*\varphi)(x) = T(\tau_x \check{\varphi})$ $\quad \operatorname{supp}(\tau_x \check{\varphi}) \subset x - H$.

Thus $(T*\varphi)(x)$ will vanish if $(x-H)\cap K = \phi$.

$\Rightarrow \operatorname{supp}(T*\varphi) \subset \operatorname{supp} T + \operatorname{supp} \varphi = K+H$ cpt.

$\Rightarrow T*\varphi \in \mathcal{D}(\mathbb{R}^N)$.



(ii) $D^\alpha(T*\psi) = D^\alpha T * \psi = T * D^\alpha \psi$. $\Rightarrow T*\psi \in \mathcal{E}(\mathbb{R}^N)$.

(iii) If $\varphi \in \mathcal{D}(\mathbb{R}^N)$ then $T*\varphi \in \mathcal{D}(\mathbb{R}^N)$ and

$T*(\psi*\varphi) = (T*\psi)*\varphi = (T*\varphi)*\psi$ $\quad \psi$

Pf: (i) & (ii) Exactly as in preceding thm.

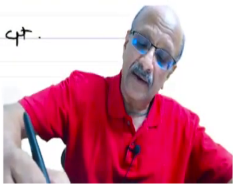
(iii). $K = \operatorname{supp}(T)$ $H = \operatorname{supp}(\varphi)$ both cpt.

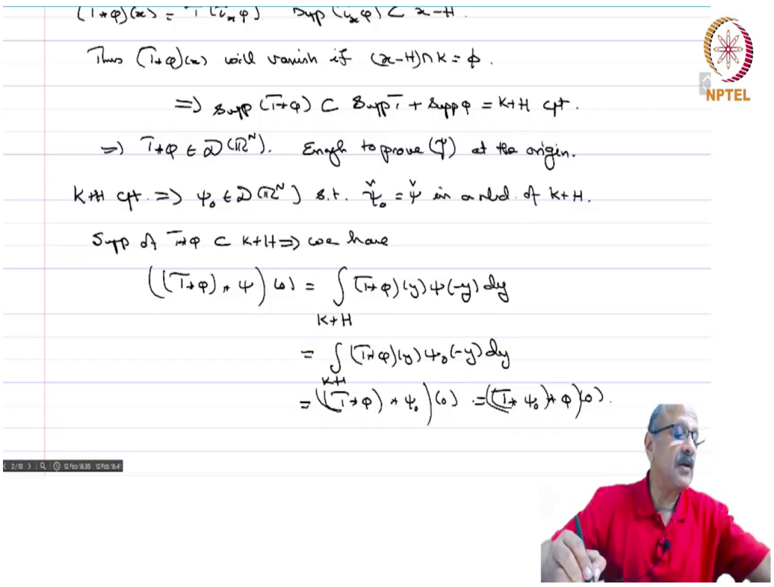
$(T*\varphi)(x) = T(\tau_x \check{\varphi})$ $\quad \operatorname{supp}(\tau_x \check{\varphi}) \subset x - H$.

Thus $(T*\varphi)(x)$ will vanish if $(x-H)\cap K = \phi$.

$\Rightarrow \operatorname{supp}(T*\varphi) \subset \operatorname{supp} T + \operatorname{supp} \varphi = K+H$ cpt.

$T*\varphi \in \mathcal{D}(\mathbb{R}^N)$





proof: (i) and (ii) are exactly as before as in preceding theorem, so we do not have to spend time calling them, so now we only have to prove, so we are proving three now.

(iii) So, let us take $K = supp(T)$, $H = supp(\phi) -$ both compact.

So $T * \phi(x) = T(\tau_x \check{\phi})$, $supp(\tau_x \check{\phi}) \subset x - H$.

Thus $T * \phi(x)$ will vanish if $(x - H) \cap K = \phi$.

$$\Rightarrow supp(T * \phi) \subset supp(T) + supp(\phi) = K + H \text{ compact.}$$

$$\Rightarrow T * \phi \in D(\mathbb{R}^N).$$

So now we have to prove the other relation.

So, let us call that relation something, let me call it dagger. So, enough to prove dagger at the origin for the x equals 0, after that you apply tau x operation and use the first part of the theorem and therefore, you can push the tau anywhere and consequently if you can throw it at the origin, you can prove it for any other point, so we want to just prove it at the origin.

Now, k plus H is compact and therefore, you can find psi naught in D of Rn, such that psi naught chesh is equal to psi chesh in the neighbourhood of k plus H all you have to do is to take a function in D of Rn which is one in the neighbourhood of k plus H and multiply psi chesh with that function, so that will be equal to psi naught chesh call that psi naught chest and therefore, psi naught chesh will be equal to psi chesh in a neighbourhood of k plus H, so this is just multiplying by a cut off function and therefore, you can do it.

Now, supp($T * \phi$)$\subset K + H$. So we have

$$T * (\phi * \psi)(0) = \int_{K+H} T * \phi(y)\psi(-y)dy.$$

$$= \int_{K+H} T * \phi(y)\psi_0(-y)dy.$$

$$= ((T * \phi) * \psi_0)(0) = ((T * \psi_0) * \phi)(0).$$

You play commutativity so you get psi naught star phi and then again you can push the psi naught inside because both functions are in C infinity with compact support, so this is equal to T star psi not star phi evaluated at 0. So, just think about it I have used commutativity and associativity as per the previous theorem, then commutativity of the convolution in the previous for C infinity functions and then I have again used the associativity of the previous theorem, because both these functions are in C infinity with compact support.

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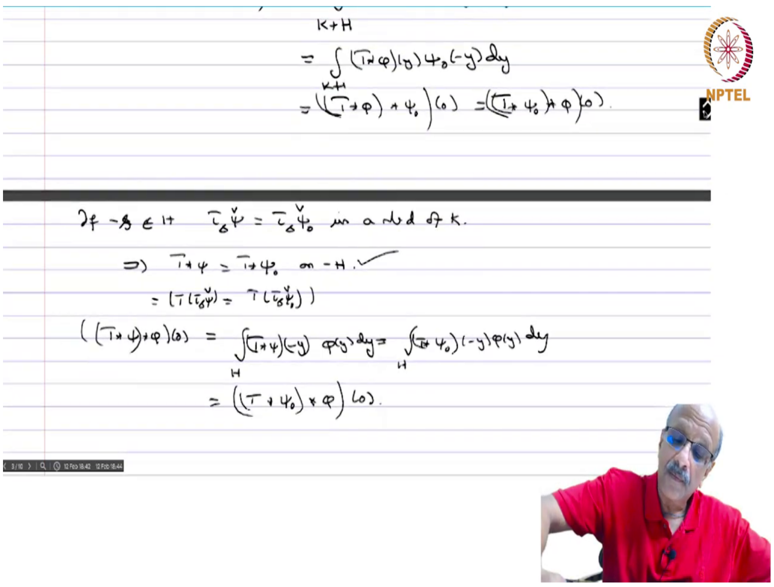

So, now if $-s \in H$, $\tau_s \overset{\vee}{\psi} = \tau_s \overset{\vee}{\psi_0}$ in a nbd of K.

$$T * \psi = T * \psi_0 \ on\ H.$$

$$((T * \psi) * \phi)(0) = \int_{K+H} T * \psi(-y)\phi(y)dy = \int_{K+H} T * \psi_0(-y)\phi(y)dy$$

$$= ((T * \psi_0) * \phi)(0)$$

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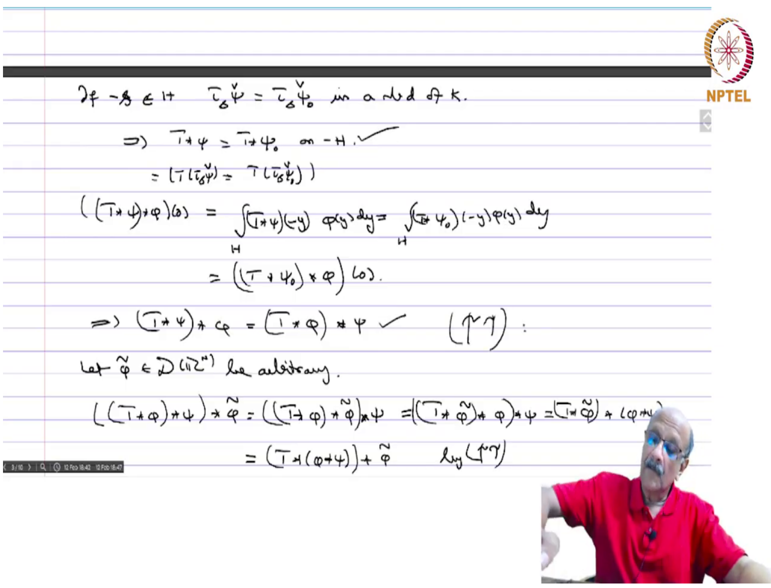

So, let $\phi \in D(\mathbb{R}^N)$ be arbitrary.

$$((T * \phi) * \psi) * \tilde{\phi} = ((T * \phi) * \tilde{\phi}) * \psi = ((T * \tilde{\phi}) * \phi)) * \psi.$$

$$= (T * (\phi * \psi)) * \tilde{\phi}.$$

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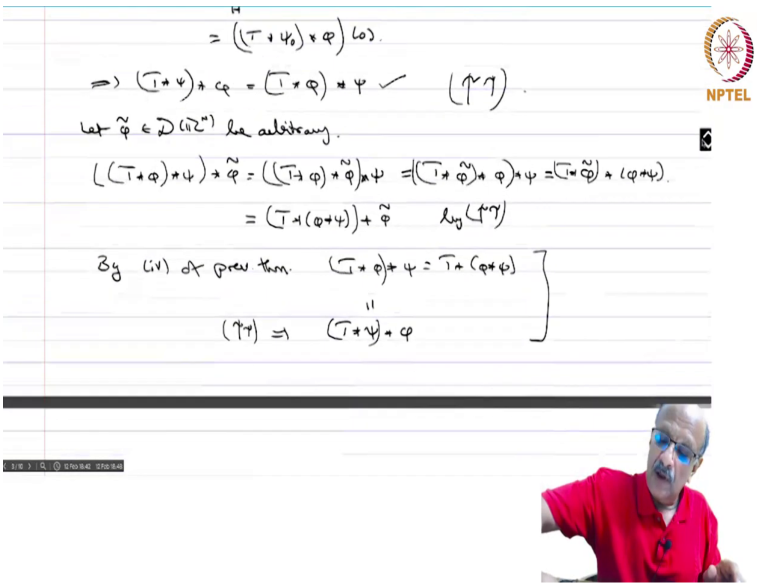

So, now for all phi tilde these two are equal, so by (iv) of previous theorem, we get

$$(T * \phi) * \psi = T * (\phi * \psi).$$

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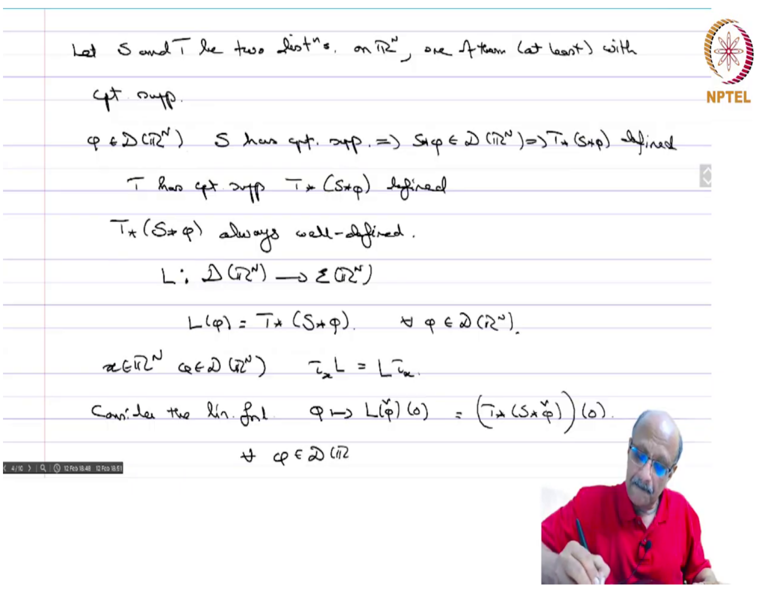

So, now let S and T be two distributions on $\mathbb{R}^N$ one of them at least with compact support, so we can define the convolution of two functions one C infinity, one C infinity with compact support or continuous with compact support. Similarly, two distributions we are going to find the convolution when at least one of them has compact support.

So, let us take $\phi \in D(\mathbb{R}^N)$ so S has compact support let us assume then, this means that $S * \phi \in D(\mathbb{R}^N)$ by the proceeding theorem and so we can define $T * (S * \phi)$.

Therefore, $T * (S * \phi)$ is always well defined, so let us take

$$L: D(\mathbb{R}^N) \to E(\mathbb{R}^N)$$

$$L(\phi) = T * (S * \phi), \forall \phi \in D(\mathbb{R}^N).$$

$$x \in \mathbb{R}^N, \phi \in D(\mathbb{R}^N), \tau_x L = L\tau_x.$$

Now, consider the linear functional $\phi \to L(\tilde{\phi})(0) = T * (S * \phi^{\vee})(0)$.

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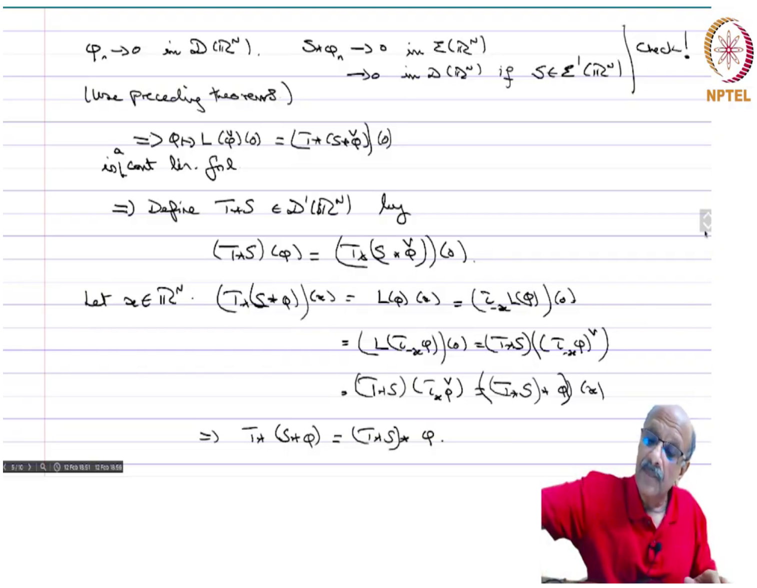

So, let $\phi_n \to 0 \; in \; D(\mathbb{R}^N)$. Then $S * \phi_n \to 0 \; in \; E(\mathbb{R}^N)$ and $S * \phi_n \to 0 \; in \; D(\mathbb{R}^N)$ if $\in E'(\mathbb{R}^N)$.

$$\Rightarrow \phi \to L(\tilde{\phi})(0) \text{ is continuous linear functional.}$$

$$\Rightarrow \; Define, \; T * S \in D'(\mathbb{R}^N) \text{ by}$$

$$T * S(\phi) = (T * (S * \tilde{\phi}))(0).$$

So, now let $x \in \mathbb{R}^N$.

Now

$$(T * (S * \phi))(x) = L(\phi)(x) = (\tau_{-x} L(\phi))(0) = (L(\tau_{-x}\phi))(0) = (T * S)((\tau_{-x}\phi)^{\vee})$$

$$= (T * S)(\tau_x \phi)^{\vee} = ((T * S) * \phi)(x).$$

$$\Rightarrow T * (S * \phi) = (T * S) * \phi.$$

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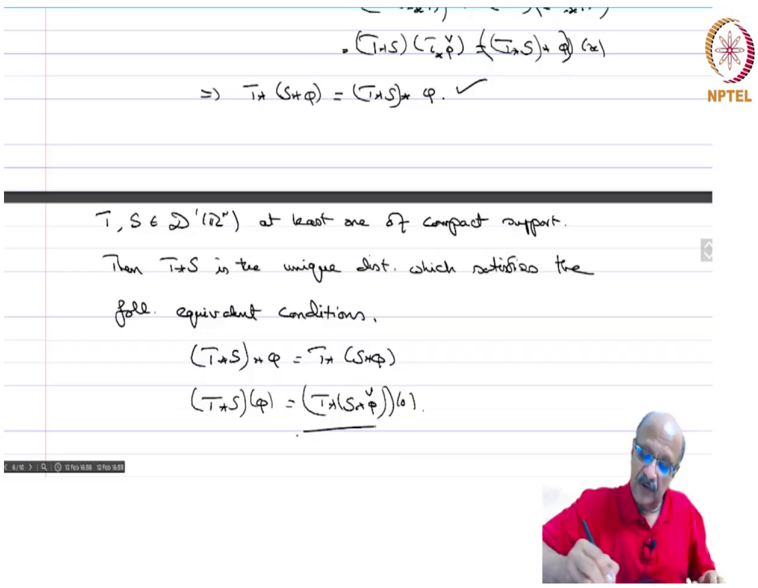

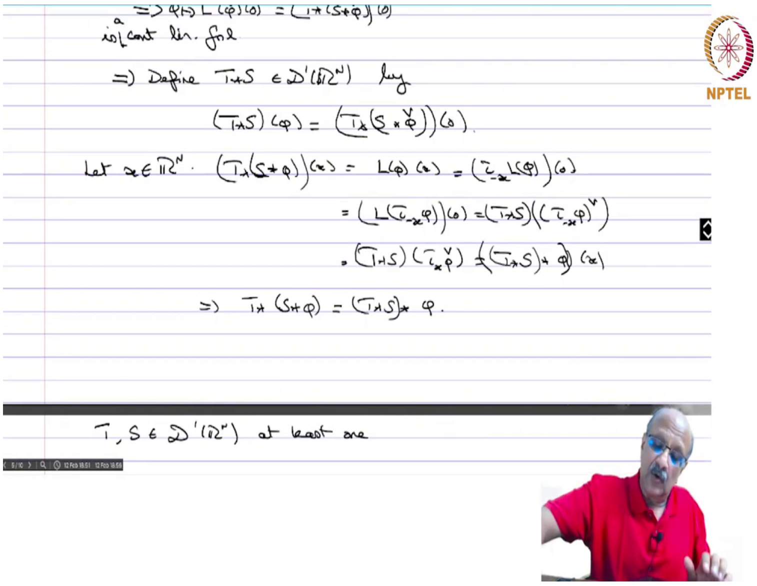

So, let $S, T \in D'(\mathbb{R}^N)$, at least one of them has cpt support, then $T * S$ is the unique distribution which satisfies the following equivalent conditions.

$$T * (S * \phi) = (T * S) * \phi.$$

$$T * S(\phi) = (T * (S * \tilde{\phi}))(0).$$

So, it just types from those calculations here so, this defines.