

Sobolev Space and Partial Differential Equations

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Convolution of Distribution – part 2

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Notation. $u^v(y) = u(-y)$, $T \in \mathcal{D}'(\mathbb{R}^N)$, $(\tau_x T)(\phi) = T(\tau_x \phi)$
 $T \in \mathcal{D}'(\mathbb{R}^N)$, $\phi \in \mathcal{D}(\mathbb{R}^N)$, $(\tau_x \phi)(y) = \tau_x(\tau_{-x} \phi)$.

Thm: (i) $\tau_x(\tau_y \phi) = (\tau_{x+y})\phi = \tau_{x+y}(\tau_y \phi)$
(ii) $\mathcal{D}'(\tau_x \phi) = \tau_x \mathcal{D}'\phi = \tau_x \mathcal{D}'\phi \Rightarrow \tau_x \mathcal{D}'\phi \in \mathcal{D}'(\mathbb{R}^N)$.
(iii) $\phi \in \mathcal{D}(\mathbb{R}^N)$, $(\tau_x \phi) * \psi = \tau_x(\phi * \psi)$
(iv) $\tau_x \phi = 0 \forall \phi \in \mathcal{D}(\mathbb{R}^N) \Rightarrow \tau_x = 0$.

Def $T \in \mathcal{D}'(\mathbb{R}^N)$, $\phi \in \mathcal{D}(\mathbb{R}^N)$, $(\tau_x \phi)(y) = \tau_x(\tau_{-x} \phi)$

Thm $T \in \mathcal{D}'(\mathbb{R}^N)$, $\phi \in \mathcal{D}(\mathbb{R}^N)$
(i) $x \in \mathbb{R}^N$, $\tau_x(\tau_x \phi) = (\tau_x \tau_x) * \phi = \tau_x(\tau_x \phi)$.
(ii) $\mathcal{D}'(\tau_x \phi) = \tau_x \mathcal{D}'\phi = \tau_x \mathcal{D}'\phi \Rightarrow \tau_x \mathcal{D}'\phi \in \mathcal{D}'(\mathbb{R}^N)$.
(iii) $\mathcal{D}'\phi \in \mathcal{D}'(\mathbb{R}^N)$ then $\tau_x \mathcal{D}'\phi \in \mathcal{D}'(\mathbb{R}^N)$ and
 $\tau_x(\phi * \psi) = (\tau_x \phi) * \psi = (\tau_x \phi) * \psi$.

So, we were looking at convolution of distributions, so the notations we were using,

$$u^v(y) = u(-y), \quad T \in \mathcal{D}'(\mathbb{R}^N), \quad (\tau_x T)(\phi) = T(\tau_x \phi), \quad T * \phi(x) = T(\tau_x \phi^v).$$

Now, the properties of this function are we proved, so we prove the following theorem:

Theorem: $T \in \mathcal{D}'(\mathbb{R}^N)$, $\phi \in \mathcal{D}(\mathbb{R}^N)$.

$$(i) \text{ for any } x \in \mathbb{R}^N, \quad \tau_x(T * \phi) = \tau_x T * \phi = T * \tau_x \phi.$$

(ii) for all multi-index α , $D^\alpha(T * \phi) = D^\alpha T * \phi = T * D^\alpha \phi$. In particular $T * \phi \in C^\infty(\mathbb{R}^N)$.

(iii) if $\psi \in D(\mathbb{R}^N)$, $T * (\phi * \psi) = (T * \phi) * \psi$.

(iv) if $T * \phi = 0$, $\forall \phi \in D(\mathbb{R}^N)$, then $T = 0$.

So, now it is clear that if $T \in E'(\mathbb{R}^N)$ and $\psi \in E(\mathbb{R}^N)$, then again we can define

$$T * \psi(x) = T(\tau_x \psi^\vee).$$

Now, we have the following theorem which is analogue of the previous theorem:

Theorem: $T \in E'(\mathbb{R}^N)$, $\phi \in E(\mathbb{R}^N)$.

(i) for any $x \in \mathbb{R}^N$, $\tau_x(T * \phi) = \tau_x T * \phi = T * \tau_x \phi$.

(ii) for all multi-index α , $D^\alpha(T * \phi) = D^\alpha T * \phi = T * D^\alpha \phi$. In particular $T * \phi \in E(\mathbb{R}^N)$.

(iii) if $\psi \in D(\mathbb{R}^N)$, then $T * \phi \in D(\mathbb{R}^N)$, and

$$T * (\phi * \psi) = (T * \phi) * \psi = (T * \psi) * \phi.$$

So these are the three theorems.

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Pf: (i) & (ii) Exactly as in preceding thm.

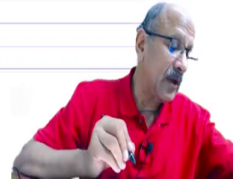
(iii), $K = \text{supp } T$, $H = \text{supp } \varphi$ with $\varphi \neq 0$.

$$(T * \varphi)(x) = T(\check{\varphi}_x) \quad \text{supp } (\check{\varphi}_x) \subset x - H.$$

Thus $(T * \varphi)(x)$ will vanish if $(x - H) \cap K = \emptyset$.

$$\Rightarrow \text{supp } (T * \varphi) \subset \text{supp } T + \text{supp } \varphi = K + H \text{ cpt.}$$

$$\Rightarrow T * \varphi \in \mathcal{D}'(\mathbb{R}^n).$$



$$(ii) \quad \partial^\alpha (T * \varphi) = \partial^\alpha T * \varphi = T * \partial^\alpha \varphi \Rightarrow T * \varphi \in \mathcal{E}'(\mathbb{R}^n).$$

(iii) If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then $T * \varphi \in \mathcal{D}'(\mathbb{R}^n)$ and

$$T * (\varphi * \psi) = (T * \varphi) * \psi = (T * \varphi) * \psi \quad \psi \in \mathcal{D}'$$



Pf: (i) & (ii) Exactly as in preceding thm.

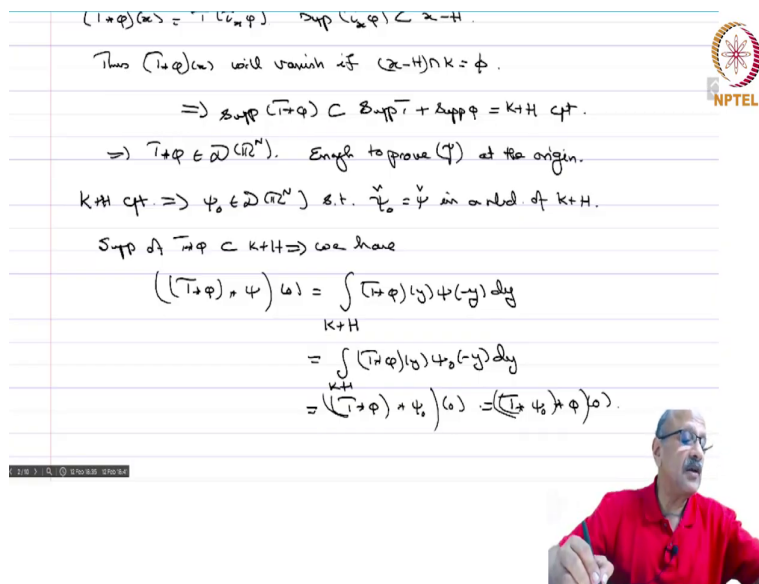
(iii), $K = \text{supp } T$, $H = \text{supp } \varphi$ with $\varphi \neq 0$.

$$(T * \varphi)(x) = T(\check{\varphi}_x) \quad \text{supp } (\check{\varphi}_x) \subset x - H.$$

Thus $(T * \varphi)(x)$ will vanish if $(x - H) \cap K = \emptyset$.

$$\Rightarrow \text{supp } (T * \varphi) \subset \text{supp } T + \text{supp } \varphi = K + H \text{ cpt.}$$

$$T * \varphi \in \mathcal{D}'(\mathbb{R}^n)$$



$(T * \phi)(x) = T(\tau_x \phi)$ $\text{supp}(\tau_x \phi) \subset x - H$.
 Thus $(T * \phi)(x)$ will vanish if $(x - H) \cap K = \emptyset$.
 $\Rightarrow \text{supp}(T * \phi) \subset \text{supp}(T) + \text{supp}(\phi) = K + H$ cft.
 $\Rightarrow T * \phi \in \mathcal{D}'(\mathbb{R}^N)$. Enough to prove (*) at the origin.
 $K + H$ cft. $\Rightarrow \psi_0 \in \mathcal{D}'(\mathbb{R}^N)$ s.t. $\check{\tau}_0 = \check{\psi}$ in a nbhd of $K + H$.
 Supp of $\tau * \phi \subset K + H \Rightarrow$ we have

$$\begin{aligned} ((T * \phi) * \psi)(0) &= \int_{K+H} (T * \phi)(y) \psi(-y) dy \\ &= \int_{K+H} (T * \phi)(0) \psi_0(-y) dy \\ &= ((T * \phi) * \psi_0)(0) = ((T * \psi_0) * \phi)(0). \end{aligned}$$

proof: (i) and (ii) are exactly as before as in preceding theorem, so we do not have to spend time calling them, so now we only have to prove, so we are proving three now.

(iii) So, let us take $K = \text{supp}(T)$, $H = \text{supp}(\phi)$ — both compact.

So $T * \phi(x) = T(\tau_x \phi^\vee)$, $\text{supp}(\tau_x \phi^\vee) \subset x - H$.

Thus $T * \phi(x)$ will vanish if $(x - H) \cap K = \emptyset$.

$$\Rightarrow \text{supp}(T * \phi) \subset \text{supp}(T) + \text{supp}(\phi) = K + H \text{ compact.}$$

$$\Rightarrow T * \phi \in \mathcal{D}'(\mathbb{R}^N).$$

So now we have to prove the other relation.

So, let us call that relation something, let me call it dagger. So, enough to prove dagger at the origin for the x equals 0, after that you apply τ_x operation and use the first part of the theorem and therefore, you can push the τ anywhere and consequently if you can throw it at the origin, you can prove it for any other point, so we want to just prove it at the origin.

Now, $K + H$ is compact and therefore, you can find ψ_0 in \mathcal{D} of \mathbb{R}^N , such that ψ_0 is equal to 1 in the neighbourhood of $K + H$ all you have to do is to take a function in \mathcal{D} of \mathbb{R}^N which is one in the neighbourhood of $K + H$ and multiply ψ

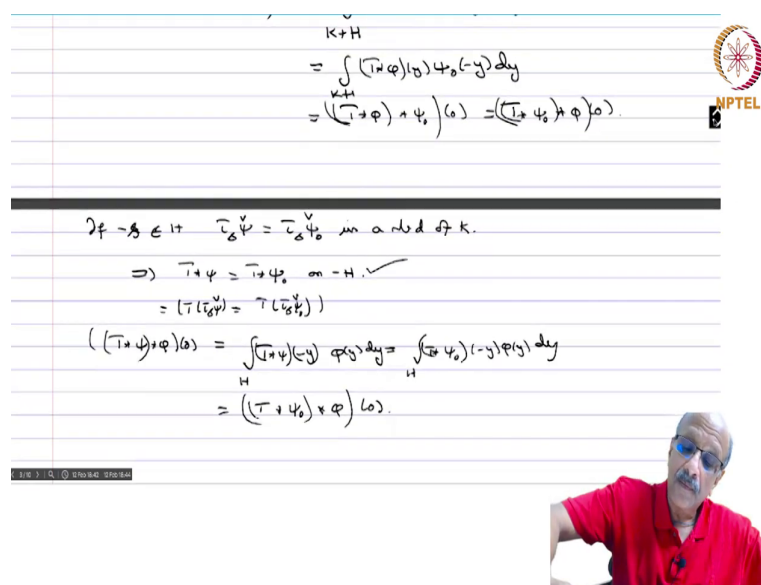
chesh with that function, so that will be equal to psi naught chesh call that psi naught chesh and therefore, psi naught chesh will be equal to psi chesh in a neighbourhood of k plus H, so this is just multiplying by a cut off function and therefore, you can do it.

Now, $\text{supp}(T * \phi) \subset K + H$. So we have

$$\begin{aligned} T * (\phi * \psi)(0) &= \int_{K+H} T * \phi(y) \psi(-y) dy. \\ &= \int_{K+H} T * \phi(y) \psi_0(-y) dy. \\ &= ((T * \phi) * \psi_0)(0) = ((T * \psi_0) * \phi)(0). \end{aligned}$$

You play commutativity so you get psi naught star phi and then again you can push the psi naught inside because both functions are in C infinity with compact support, so this is equal to T star psi not star phi evaluated at 0. So, just think about it I have used commutativity and associativity as per the previous theorem, then commutativity of the convolution in the previous for C infinity functions and then I have again used the associativity of the previous theorem, because both these functions are in C infinity with compact support.

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The slide shows a handwritten derivation of the commutativity of convolution for distributions. The equations are as follows:

$$\begin{aligned} &= \int_{K+H} (T * \phi)(y) \psi_0(-y) dy \\ &= ((T * \phi) * \psi_0)(0) = ((T * \psi_0) * \phi)(0). \end{aligned}$$

Below this, there is a horizontal line and then the following text and equations:

$\partial_f \rightarrow \partial_g \in 1 + \tau_{\partial} \psi = \tau_{\partial} \psi_0$ on a neighbourhood of k .

$\Rightarrow T * \psi = T * \psi_0$ on $-H$. ✓

$= (T * \psi_0) * \phi = (T * \psi_0) * \phi$

$((T * \psi) * \phi)(0) = \int_H (T * \psi)(-y) \phi(y) dy = \int_H (T * \psi_0)(-y) \phi(y) dy$

$= ((T * \psi_0) * \phi)(0).$

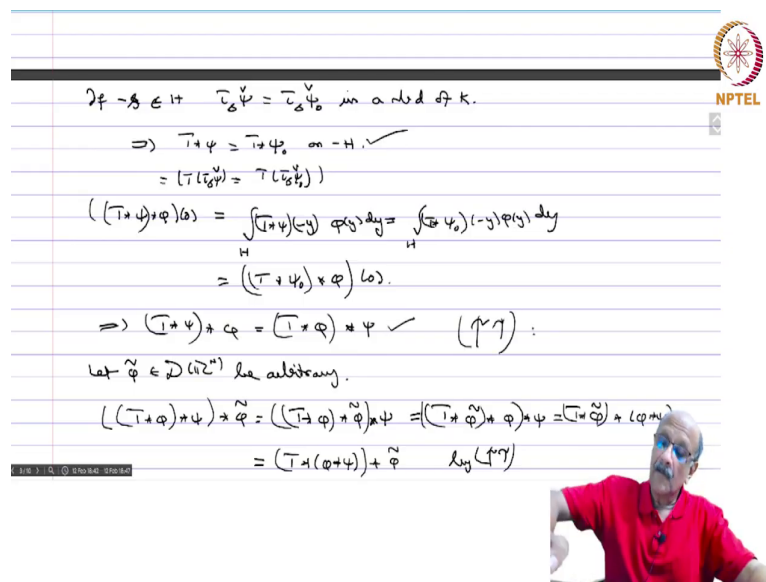
In the bottom right corner, there is a video inset of a man with a mustache, wearing a red shirt and glasses, looking towards the camera.

So, now if $-s \in H$, $\tau_s \psi^V = \tau_s \psi_0^V$ in a nbd of K .

$$T * \psi = T * \psi_0 \text{ on } H.$$

$$\begin{aligned} ((T * \psi) * \phi)(0) &= \int_{K+H} T * \psi(-y) \phi(y) dy = \int_{K+H} T * \psi_0(-y) \phi(y) dy \\ &= ((T * \psi_0) * \phi)(0) \end{aligned}$$

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$\text{If } -s \in H, \tau_s \psi^V = \tau_s \psi_0^V \text{ in a nbd of } K.$
 $\Rightarrow T * \psi = T * \psi_0 \text{ on } -H. \checkmark$
 $= (\tau(\tau_s \psi) = \tau(\tau_s \psi_0))$
 $((T * \psi) * \phi)(0) = \int_H (T * \psi)(-y) \phi(y) dy = \int_H (T * \psi_0)(-y) \phi(y) dy$
 $= ((T * \psi_0) * \phi)(0).$
 $\Rightarrow (T * \psi) * \phi = (T * \psi_0) * \phi \checkmark \quad (17.7)$
 Let $\tilde{\phi} \in \mathcal{D}(\mathbb{R}^N)$ be arbitrary.
 $((T * \phi) * \psi) * \tilde{\phi} = ((T * \phi) * \tilde{\phi}) * \psi = ((T * \tilde{\phi}) * \phi) * \psi$
 $= (T * (\phi * \psi)) * \tilde{\phi} \quad \text{by (17.7)}$

So, let $\phi \in D(\mathbb{R}^N)$ be arbitrary.

$$\begin{aligned} ((T * \phi) * \psi) * \tilde{\phi} &= ((T * \phi) * \tilde{\phi}) * \psi = ((T * \tilde{\phi}) * \phi) * \psi \\ &= (T * (\phi * \psi)) * \tilde{\phi}. \end{aligned}$$

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$$= ((T + \psi) * \phi)(0).$$


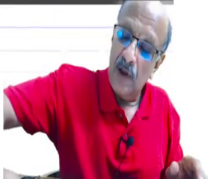
$$\Rightarrow (T + \psi) * \phi = (T * \phi) + \psi \quad (*)$$

Let $\tilde{\phi} \in \mathcal{D}(\mathbb{R}^n)$ be arbitrary.

$$((T + \phi) * \psi) * \tilde{\phi} = ((T * \phi) + \psi) * \tilde{\phi} = (T * \tilde{\phi}) * \phi + \psi * \tilde{\phi} = (T * \tilde{\phi}) * (\phi + \psi).$$

$$= (T * (\phi + \psi)) * \tilde{\phi} \quad \text{by } (*)$$

By (iv) of previous theorem, $(T * \phi) * \psi = T * (\phi * \psi)$

$$(*) \Rightarrow (T * \psi) * \phi = T * (\psi * \phi)$$



So, now for all ϕ these two are equal, so by (iv) of previous theorem, we get

$$(T * \phi) * \psi = T * (\phi * \psi).$$

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Let S and T be two distributions on \mathbb{R}^n , one of them (at least) with compact support.

$\phi \in \mathcal{D}(\mathbb{R}^n)$ S has cpt. supp. $\Rightarrow S * \phi \in \mathcal{D}(\mathbb{R}^n) \Rightarrow T * (S * \phi)$ defined

T has cpt. supp. $T * (S * \phi)$ defined

$T * (S * \phi)$ always well-defined.



$L: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$

$$L(\phi) = T * (S * \phi) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

$x \in \mathbb{R}^n \quad \phi \in \mathcal{D}(\mathbb{R}^n) \quad \tau_x L = L \tau_x$

Consider the lin. fnl $\phi \mapsto L(\tilde{\phi})(0) = (T * (S * \tilde{\phi}))(0)$

$\forall \phi \in \mathcal{D}(\mathbb{R}^n)$

So, now let S and T be two distributions on \mathbb{R}^n one of them at least with compact support, so we can define the convolution of two functions one C^∞ , one C^∞ with compact support or continuous with compact support. Similarly, two distributions we are going to find the convolution when at least one of them has compact support.

So, let us take $\phi \in D(\mathbb{R}^N)$ so S has compact support let us assume then, this means that $S * \phi \in D(\mathbb{R}^N)$ by the proceeding theorem and so we can define $T * (S * \phi)$.

Therefore, $T * (S * \phi)$ is always well defined, so let us take

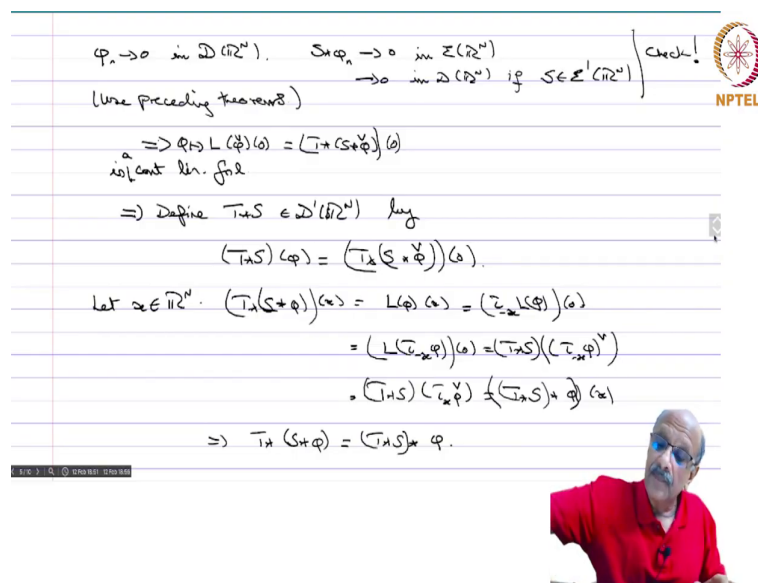
$$L: D(\mathbb{R}^N) \rightarrow E(\mathbb{R}^N)$$

$$L(\phi) = T * (S * \phi), \forall \phi \in D(\mathbb{R}^N).$$

$$x \in \mathbb{R}^N, \phi \in D(\mathbb{R}^N), \tau_x L = L\tau_x.$$

Now, consider the linear functional $\phi \rightarrow L(\tilde{\phi})(0) = T * (S * \phi^V)(0)$.

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Handwritten notes on a slide:

$$\begin{aligned} &\phi_n \rightarrow 0 \text{ in } D(\mathbb{R}^N), \quad S * \phi_n \rightarrow 0 \text{ in } E(\mathbb{R}^N) \\ &\quad \rightarrow 0 \text{ in } D(\mathbb{R}^N) \text{ if } S \in E'(\mathbb{R}^N) \quad \left. \begin{array}{l} \text{Check!} \\ \text{NPTEL} \end{array} \right\} \\ &\text{(Use preceding theorem)} \\ &\Rightarrow \phi \mapsto L(\tilde{\phi})(0) = (T * (S * \phi^V))(0) \\ &\quad \text{is a cont. lin. fcn.} \\ &\Rightarrow \text{Define } T * S \in D'(\mathbb{R}^N) \text{ by} \\ &\quad (T * S)(\phi) = (T * (S * \phi^V))(0). \\ &\text{Let } x \in \mathbb{R}^N, (T * (S * \phi))(\tau_x) = L(\tilde{\phi})(x) = (T * L(\tilde{\phi}))(\tau_x) \\ &\quad = (T * L(\tau_x \tilde{\phi}))(\tau_x) = (T * S)((\tau_x \tilde{\phi})^V) \\ &\quad = (T * S)(\tau_x \tilde{\phi}) = (T * S) * \tilde{\phi}(\tau_x) \\ &\Rightarrow T * (S * \phi) = (T * S) * \phi. \end{aligned}$$

Video inset shows a professor in a red shirt.

So, let $\phi_n \rightarrow 0$ in $D(\mathbb{R}^N)$. Then $S * \phi_n \rightarrow 0$ in $E(\mathbb{R}^N)$ and $S * \phi_n \rightarrow 0$ in $D(\mathbb{R}^N)$ if $S \in E'(\mathbb{R}^N)$.

$\Rightarrow \phi \rightarrow L(\tilde{\phi})(0)$ is continuous linear functional.

\Rightarrow Define, $T * S \in D'(\mathbb{R}^N)$ by

$$T * S(\phi) = (T * (S * \tilde{\phi}))(0).$$

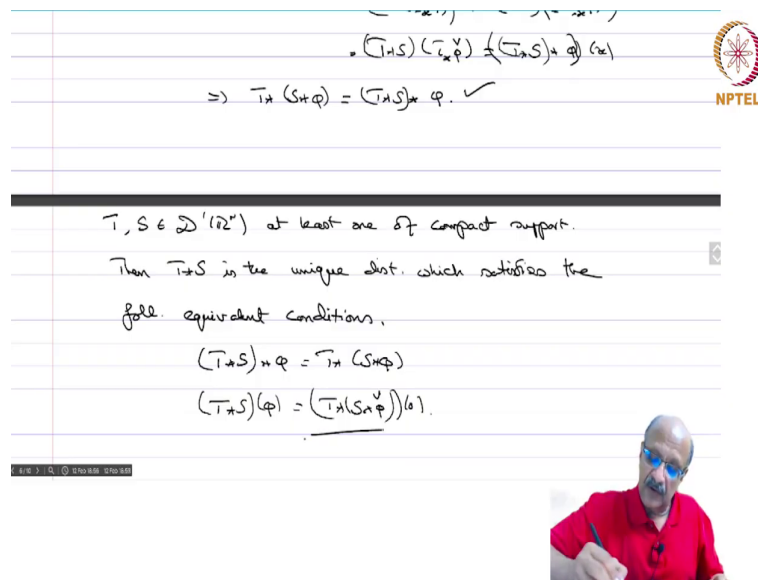
So, now let $x \in \mathbb{R}^N$.

Now

$$\begin{aligned} (T * (S * \phi))(x) &= L(\phi)(x) = (\tau_{-x} L(\phi))(0) = (L(\tau_x \phi))(0) = (T * S)((\tau_x \phi)^V) \\ &= (T * S)(\tau_x \phi)^V = ((T * S) * \phi)(x). \end{aligned}$$

$$\Rightarrow T * (S * \phi) = (T * S) * \phi.$$

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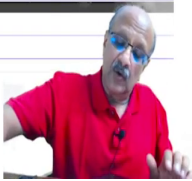
$(T * S)(\tau_x \phi)^V = ((T * S) * \phi)(x)$
 $\Rightarrow T * (S * \phi) = (T * S) * \phi. \checkmark$

NPTEL

$T, S \in \mathcal{D}'(\mathbb{R}^n)$ at least one of compact support.
 Then $T * S$ is the unique dist. which satisfies the
 foll. equivalent conditions,
 $(T * S) * \phi = T * (S * \phi)$
 $(T * S)(\phi) = (T * (S * \phi))(0).$

$\Rightarrow \varphi \mapsto L(\varphi)(0) = (T * (S * \varphi))(0)$
 is const lin. fcn
 \Rightarrow Define $T * S \in \mathcal{D}'(\mathbb{R}^N)$ by
 $(T * S)(\varphi) = (T * (S * \tilde{\varphi}))(0)$
 Let $x \in \mathbb{R}^N$. $(T * (S * \varphi))(x) = L(\varphi)(x) = (\tilde{\varphi} * L)(x)$
 $= (L(\tilde{\varphi} * \varphi))(x) = (T * S)(\tilde{\varphi} * \varphi)$
 $= (T * S)(\tilde{\varphi})(\tilde{\varphi} * \varphi)(x)$
 $\Rightarrow T * (S * \varphi) = (T * S) * \varphi$

$T, S \in \mathcal{D}'(\mathbb{R}^N)$ at least one

So, let $S, T \in \mathcal{D}'(\mathbb{R}^N)$, at least one of them has cpt support, then $T * S$ is the unique distribution which satisfies the following equivalent conditions.

$$T * (S * \varphi) = (T * S) * \varphi.$$

$$T * S(\varphi) = (T * (S * \tilde{\varphi}))(0).$$

So, it just types from those calculations here so, this defines.