Sobolev Spaces and Partial Differential Equations
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Lecture 14
Convolution of Distributions – Part 1

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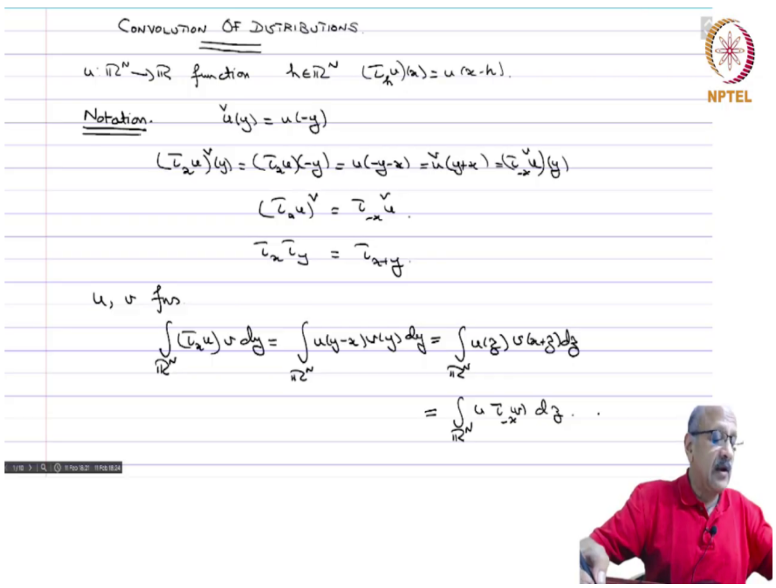

We will now extend the notion of convolution to certain types of distributions. So, convolution of distributions. So, obviously we will be trying to copy what we did for functions and therefore, we need to establish some notation which we used for functions, for instance, if you have

$u: \mathbb{R}^N \to \mathbb{R}$ is a function and $h \in \mathbb{R}^N$. Then we are introduced.

$$\tau_h u(x) = u(x - h).$$

So, from this, you can easily follow the following relations. So, and now we define another notation:

Notation: $\check{u}(y) = u(-y)$.

$$(\tau_x u)^{\vee}(y) = (\tau_x u)\ (-y) = u(-y-x) = u^{\vee}(x+y) = \tau_{-x} u^{\vee}(y).$$

$$(\tau_x u)^{\vee} = \tau_{-x} u^{\vee}$$

$$\tau_x \tau_y = \tau_{x+y}.$$

So, we have these notations. So, if u and v are two functions locally integrable for instance. So, let us take integral

$$\int_{\mathbb{R}^N} (\tau_x u) v dy = \int_{\mathbb{R}^N} u(x-y) v(y) dy = \int_{\mathbb{R}^N} u(z) v(x+z) dz = \int_{\mathbb{R}^N} u \tau_x v\, dz.$$

So, you copy this, and therefore, we define the translation of a distribution.

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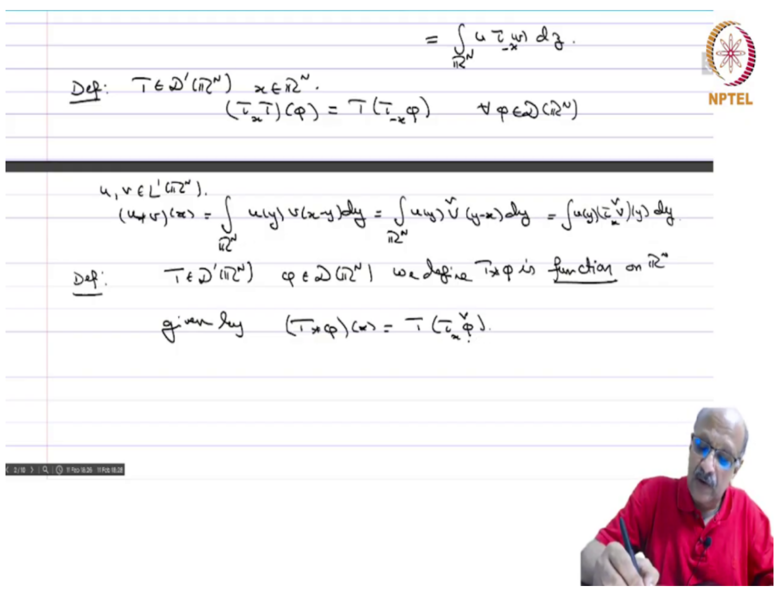

So, definition. So, we have tau x. So, T belongs to $D'(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. So, then we define the translation tau x of T acting on any ϕ. So, this is for every $\phi \in D(\mathbb{R}^N)$. So, if you look at what we have done here imagine instead of you having T. So, this is nothing but T of tau minus x of phi.

So, this defines a new distribution. So, tau x T is a distribution which is defined this way. Now, if u and v are two functions for which if they are L1 functions for instance, so, u v say in L1 of $\mathbb{R}^N$ then. What is u star v of x? This equal to integral over $\mathbb{R}^N$ of uy, v of x minus y dy, and this equal to integral over $\mathbb{R}^N$ of uy, v chesh of y minus x dy, and that is equal to integral uy tau x of v chesh at y dy. So, if we copy this given T in so definition again T in $D'(\mathbb{R}^N)$ and phi in $D(\mathbb{R}^N)$ then we define T star phi is a function not it is a function on $\mathbb{R}^N$ given by T star phi any point x is equal to T acting on tau x of phi chesh. So, again, if phi and is a sequence which converges to 0, then tau x phi chesh is also a sequence converging to 0 and $D(\mathbb{R}^N)$ and so, this defines, therefore, T of that will go to 0 and therefore, this defines a well-defined function.

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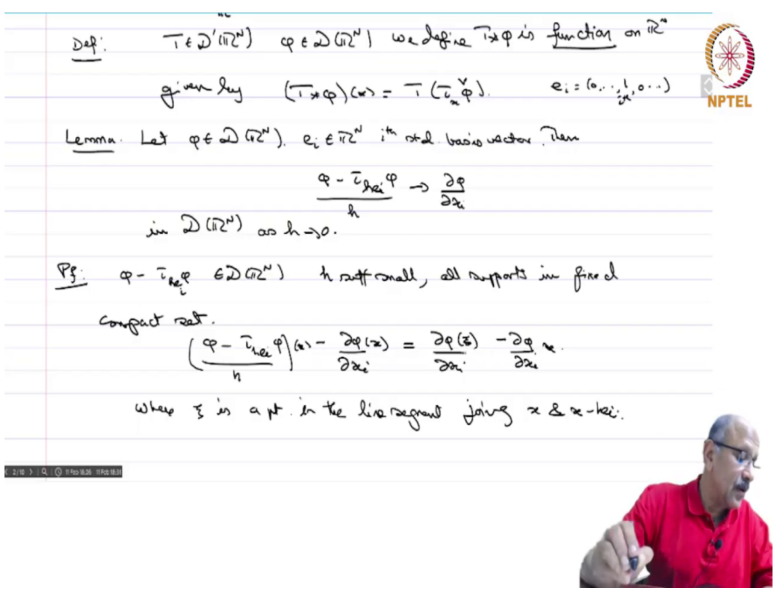

So, now, before we go to examine the properties of the convolution of a distribution and (())(07:01) infinity function with compact support, we need the following technical lemma.

Lemma : Let $\phi \in D(\mathbb{R}^N)$, $e_i \in \mathbb{R}^N$. Then

$$\frac{\phi - \tau_{he_i}\phi}{h} \to \frac{\partial\phi}{\partial x_i} \text{ in } D(\mathbb{R}^N) \text{ as } h \to 0.$$

proof: $\phi - \tau_{he_i}\phi \in D(\mathbb{R}^N)$.

$\frac{\phi - \tau_{he_i}\phi}{h} - \frac{\partial\phi}{\partial x_i} = \frac{\partial\phi}{\partial x_i}(\xi) - \frac{\partial\phi}{\partial x_i}(x)$, where ξ is a point in the line segment joining x and $x - he_i$.

So, there is just a mean value theorem.

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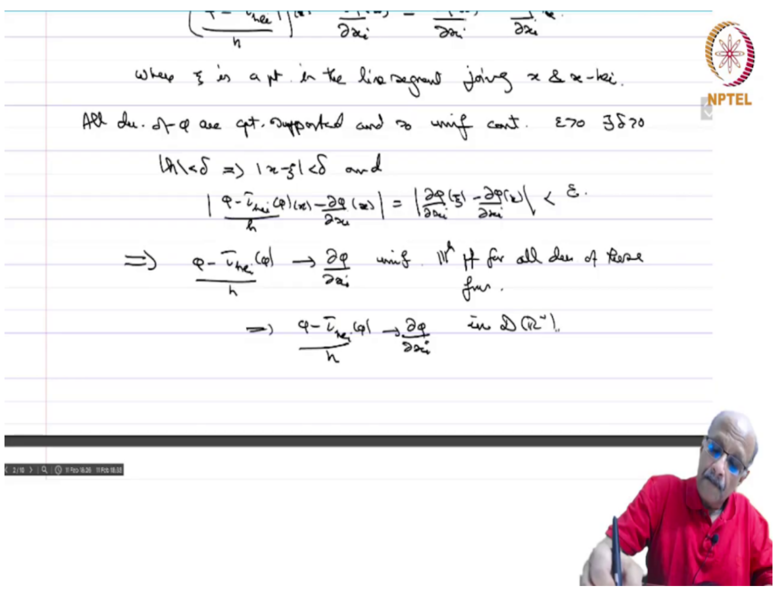

Now, all derivatives of phi are compactly supported and so uniformly continuous. Therefore, given any $\epsilon > 0$ there exists a $\delta > 0$ such that $|h| < \delta \Rightarrow |x - \xi| < \delta$ and you have

$$\left|\frac{\phi - \tau_{he_i}\phi}{h} - \frac{\partial\phi}{\partial x_i}\right| = \left|\frac{\partial\phi}{\partial x_i}(\xi) - \frac{\partial\phi}{\partial x_i}(x)\right| < \epsilon$$

$$\Rightarrow \frac{\phi - \tau_{he_i}\phi}{h} \to \frac{\partial\phi}{\partial x_i} \text{ uniformly.}$$

$$\Rightarrow \frac{\phi - \tau_{he_i}\phi}{h} \to \frac{\partial \phi}{\partial x_i} \text{ in } D(\mathbb{R}^N).$$

So, this completes the proof of this lemma.

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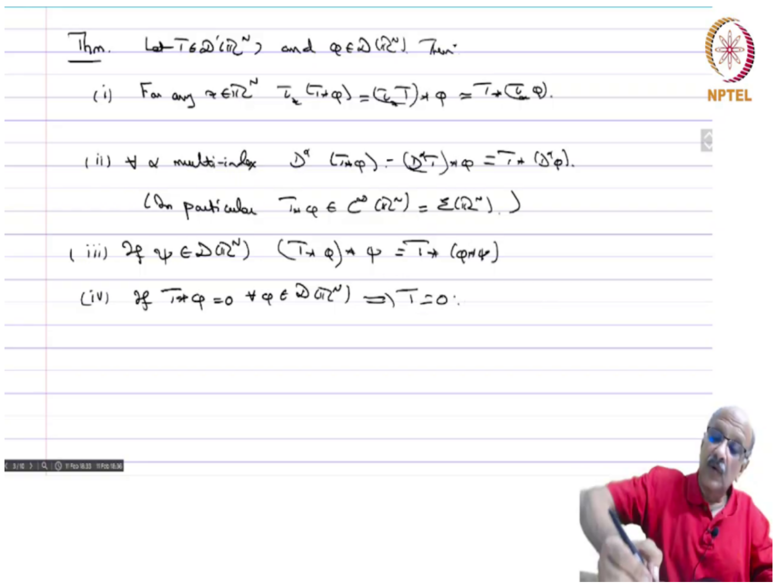

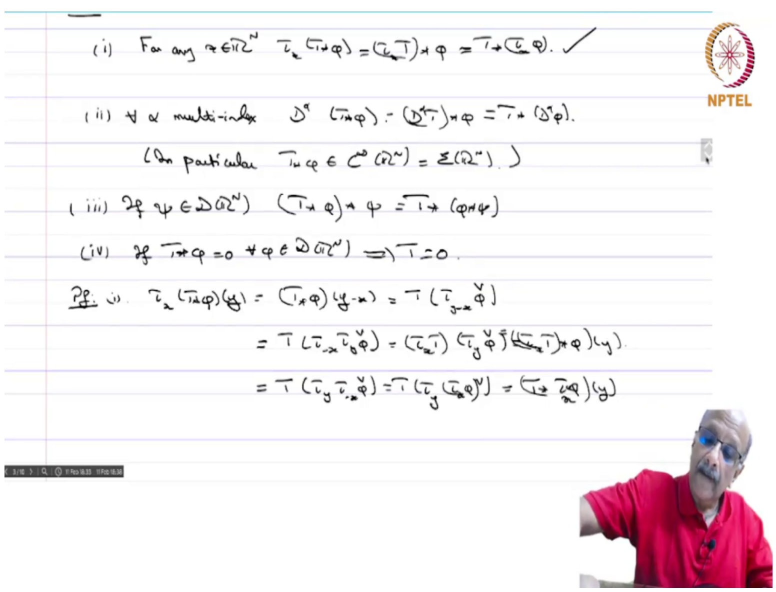

So, now we have the following theorem.

Theorem: $T \in D'(\mathbb{R}^N)$, $\phi \in D(\mathbb{R}^N)$.

(i) for any $x \in \mathbb{R}^N$, $\tau_x(T * \phi) = \tau_x T * \phi = T * \tau_x \phi$.

(ii) for all multi-index α, $D^\alpha(T * \phi) = D^\alpha T * \phi = T * D^\alpha \phi$. In particular $T * \phi \in C^\infty(\mathbb{R}^N)$.

(iii) if $\psi \in D(\mathbb{R}^N)$, $T * (\phi * \psi) = (T * \phi) * \psi$.

(iv) if $T * \phi = 0,\ \forall \phi \in D(\mathbb{R}^N),\ then\ T = 0$.

Therefore, the convolution is also C infinity functions with compact support, and therefore, they can convolve it with a distribution that is well defined by what we have seen. So, and this shows some kind of associative law. So, 4 this will be useful to set later on. If T star phi equal to 0 for all phi in $D(\mathbb{R}^N)$ This implies then T equal to 0. So, it is only the 0 distribution. So, proof.

So, first. So, you take tau x of T star phi evaluated at any y equal to T star phi at y minus x. And how is this defined? This is T of tau y minus x phi chesh that is the definition of the thing. Now, tau y minus x. I can write a tau y tau minus x and that is committed. So, I can compute it in two ways. So, this is tau T of tau minus x tau y of phi chesh which is tau x of T acting on tau y of phi chesh, and by definition, this is tau x of T star phi evaluated at y.

But this can also be witnessed as T of tau y tau minus x phi chesh. Now, tau minus x phi chesh is nothing but T of tau y we have already seen this tau x of phi chesh, and therefore, this is nothing by definition is nothing but T star tau x of phi evaluated at y. So, these two are equal. So, that proves the first statement.

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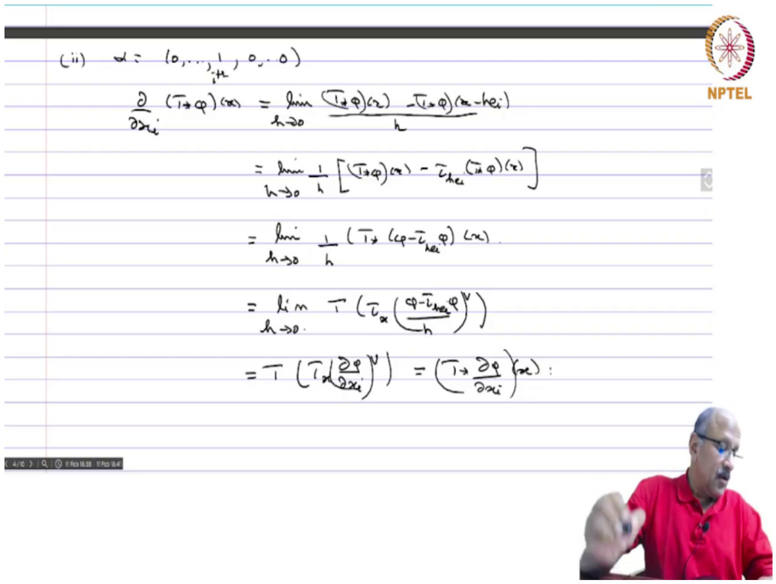

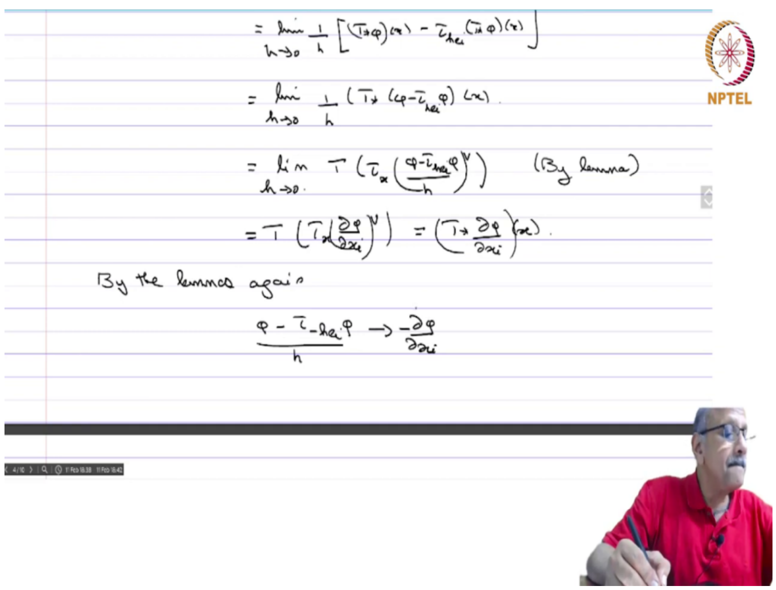

Second one. So, we will prove it let alpha equal to 0,1 in the ith place 0 elsewhere. We will prove it for one derivative namely d by dxi and then we will iterate it to get for any other derivative. So, let us take d by dxi of T star phi of x. So, this is the limit as h goes to 0. If this limit exists, of course, T star phi of x minus T star phi of x minus hei divided by h this is the usual definition of a derivative. This equals the limit h going to 0,1 by h of T star phi of x minus tau of hei T star phi of x.

So, now, of course, we have the choice of pushing the T of tau star h wherever we like. So, we have limit h tending to 0,1 by h T star phi minus tau hei of phi evaluated at x and then we saw this goes to. What is this definition? So, limit as h goes to 0 of T of tau x of phi minus tau hei phi over h chesh. That is the definition of the convolution of a function, but we saw that this function here goes to d by dxi as in $D(\mathbb{R}^N)$. So, tau x also will do that, and therefore, this is nothing but T of tau x d phi by dxi chesh and that is nothing but T star d phi by dxi evaluated at x.

So, we have shown that the limit exists and in fact, the derivative is equal to this. Now, we can take the tau in another way also. So, we can also have a d by dxi of T star phi at x before they do that. So, by the same lemma, so, this is the lemma. So, by the lemma again we also have phi minus tau or minus hei phi by h. Now, this will go to minus d phi by dxi because we are taking the difference quotient in the opposite direction: tau of minus hei will be phi of x plus hei. So, phi of x minus phi of x plus hei by h will go to minus d phi by dxi by the same lemma.

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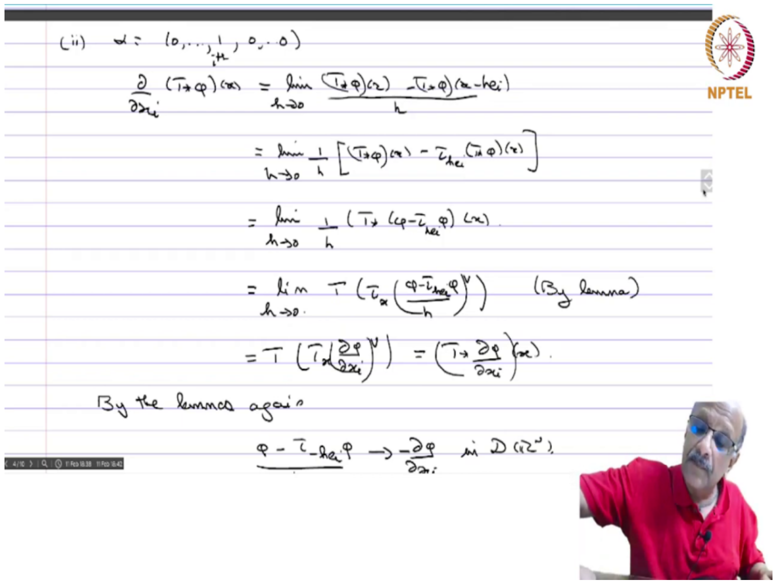

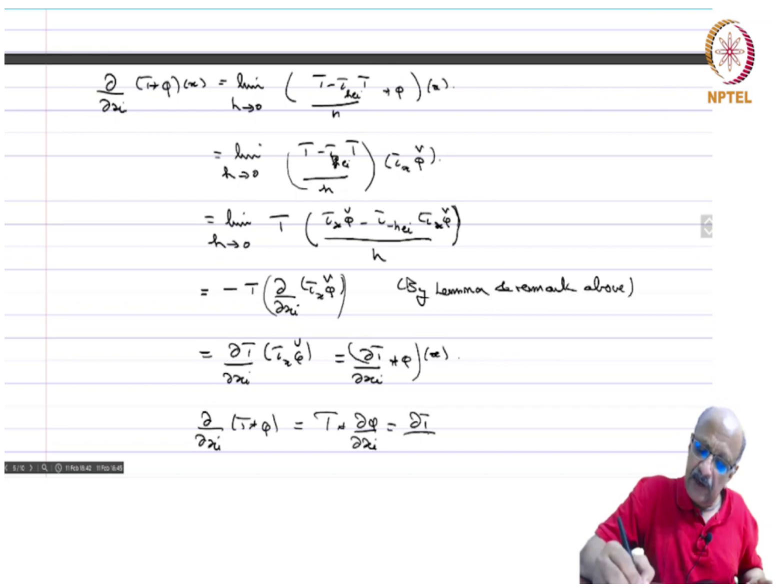

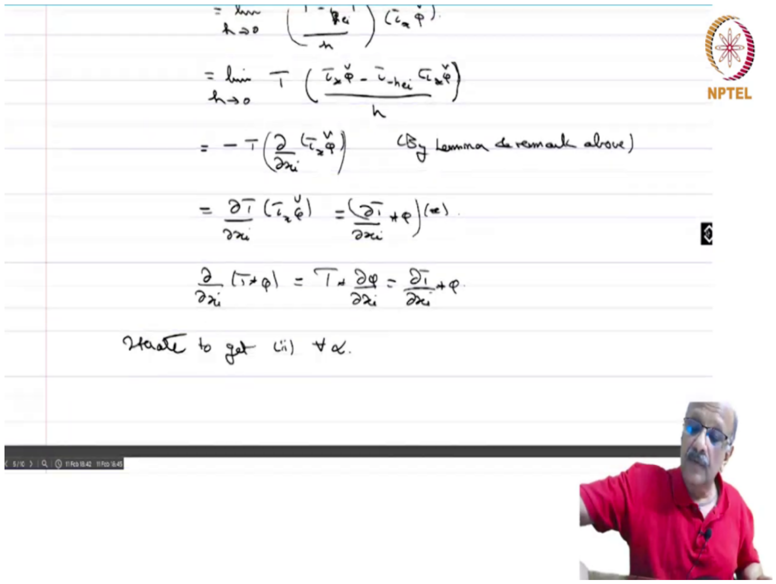

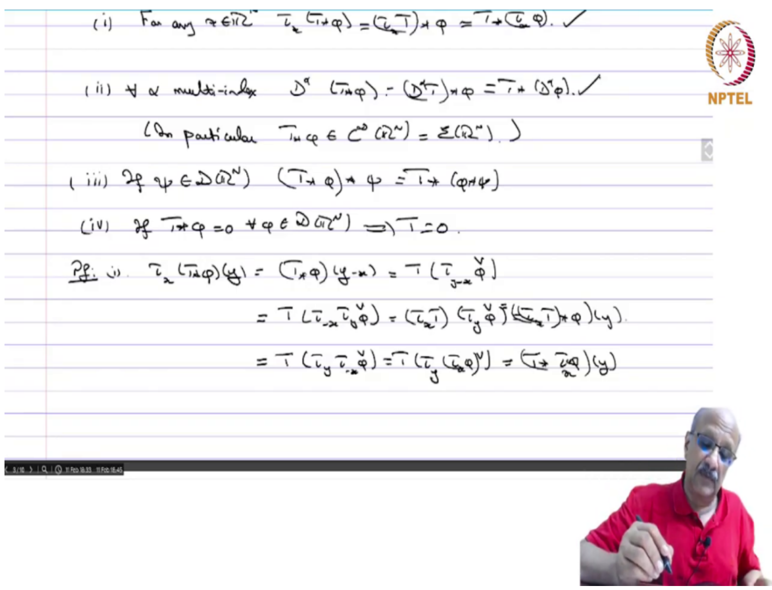

So, therefore in $D(\mathbb{R}^N)$. So, d by dxi of T star phi at any x is limit h going to 0. Now, I will take the tau to a different place. So, this tau can be pushed. We pushed it to the phi earlier. Now, we will push it to the T. So, this will be equal to T minus tau hei of T by h star phi that is equal to limit h going to 0 of T minus tau hei of T by h acting on tau x of phi chesh.

Which is equal to limiting h tending to 0 of T acting on tau x of phi chesh minus. So, we are taking the translation of T this is minus tau of minus hei of tau x of phi chesh by h and just by remark we just saw that this is giving you minus T d by dxi tau x of phi chesh by lemma and remarkable. But what is this minus T of d by dxi is nothing but dT by dxi acting on tau x phi chesh and this is nothing but dT by dxi star phi evaluated at x.

Therefore, you have d by dxi of T star phi equals T star d phi by dxi equals dT by dxi star phi and now you iterate to get to for all multi this is alpha. So, you see you can push the derivative wherever you like and in, therefore, that makes it the C infinity function because you have just shown that.

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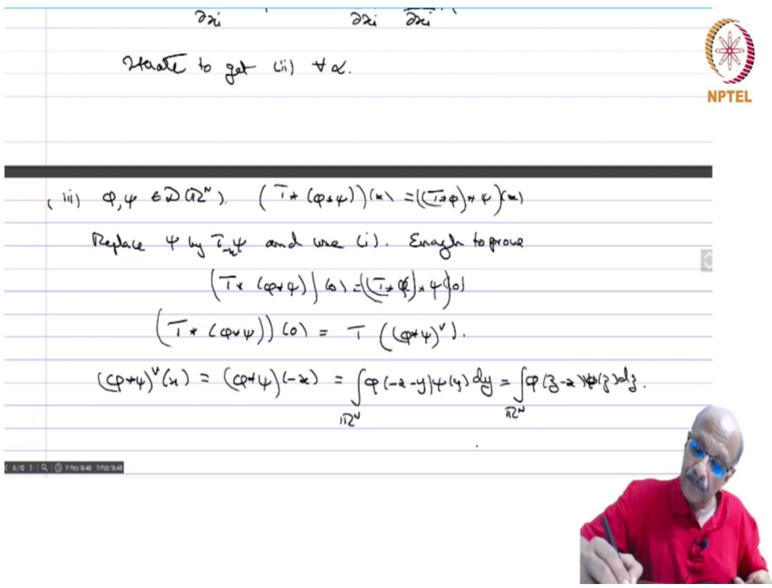

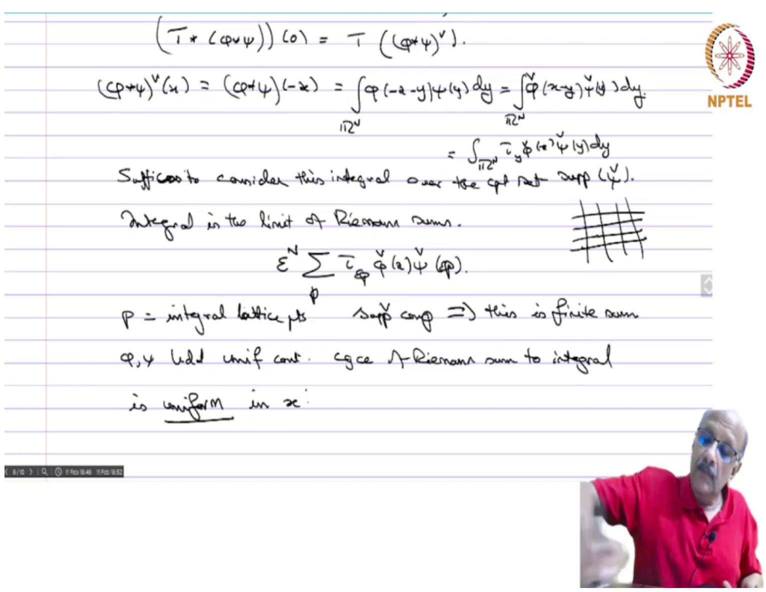

So, now we have to prove the third one is a little sticky. So, let us carefully do it. So, phi psi in $D(\mathbb{R}^N)$. So, we want to show T star, phi star psi at any x is T star phi star psi at any x. Now, we can replace tau x psi by tau minus xi psi. Because it is also C infinity functions with compact support. So, enough to prove and use 1.

So, enough to prove T stat phi star psi at 0 equals T star phi star psi at 0. Because once you prove at 0 the value at any x is tau of minus x or the value at 0 and the tau minus x can be pushed anywhere. So, in particular, you can push it to psi and use it by the previous theorem. It is so this is enough to prove so just check that.

So, let us take it. So, T star phi star psi evaluated at 0. What is this? This is T acting on tau of 0 which is the identity of phi star psi chesh. So, let us see what is phi star psi evaluated at any x this is nothing but phi star psi evaluated at minus x and that is equal to the integral $\mathbb{R}^N$ of phi of minus x minus y psi y dy. So, now I will change y to minus y.

So, this is equal to the integral $\mathbb{R}^N$ minus phi of z minus x psi z dz. So, this is equal to sorry let me write it is clear. So, this is minus x plus y. And, I can write this as a phi chesh of x minus y dy. So, this is equal to the integral lower $\mathbb{R}^N$ tau y of phi chesh at x and psi chesh of y dy. So, they changed y to minus y. So, then I wrote it in a slightly different form.

So, now we are going to this integral to consider the integral over the compact set. Which is support of psi chesh. Then the integral is the limit of the Riemann sums. So, you have I am going to take the integral lattice points with epsilon as the mesh size. So, this will be each cube has volume epsilon power N into sigma over all lattice integral lattice points p tau of epsilon phi chesh of x psi chesh of epsilon.

So, this is I am just replacing it as a Riemann sum evaluating at all these points which are the integral. Now, p equals integral lattice point but support psi chesh compact implies. This is finite because it is a compact set; only a finite number of integral lattice points will be there; all the others this will be 0 and therefore. I can just ignore them and therefore, this is essentially a finite sum and then both phi and psi are bounded; they are of compact support. So, they are bounded and uniformly continuous, and therefore, convergence of Riemann sum to integral is uniform in x.

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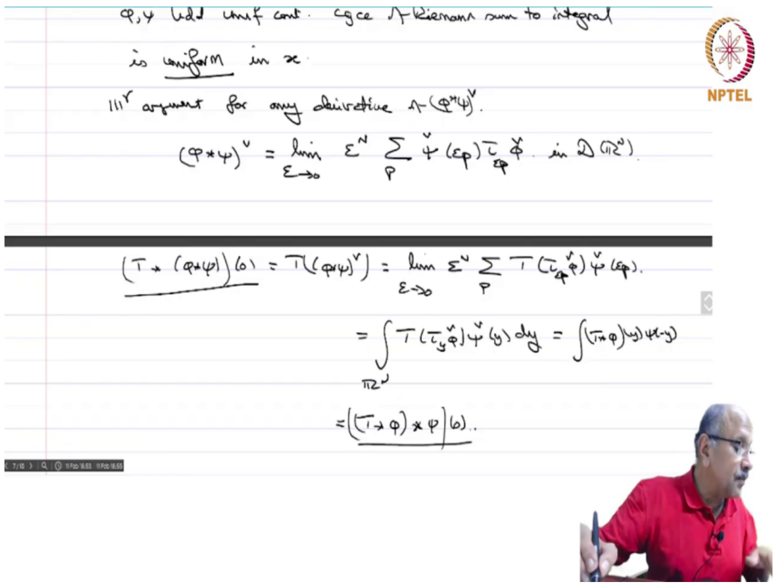

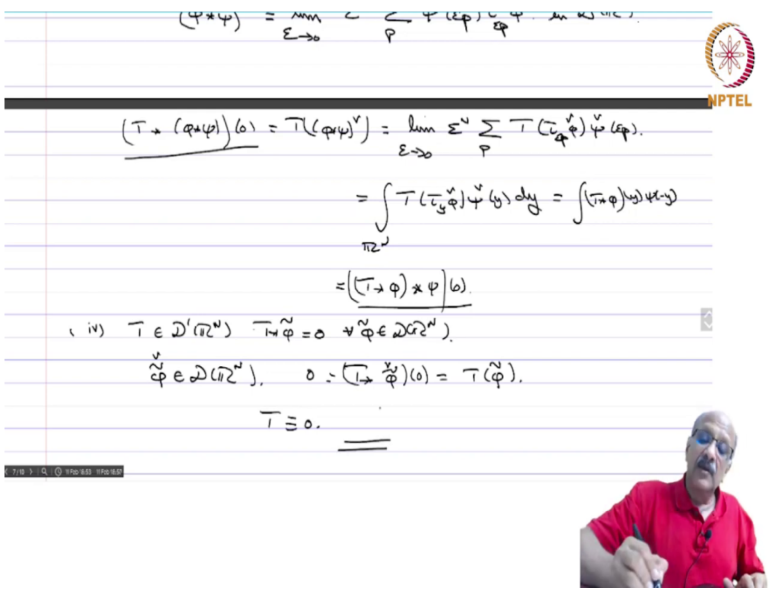

Similar argument for any derivative of $(\phi * \psi)^{\vee}$.

$$(\phi * \psi)^{\vee} = \lim_{\epsilon \to 0} \epsilon^{N} \sum_{P} \psi^{\vee}(\epsilon P)\tau_{\epsilon P}(\phi^{\vee}) \text{ in } D(\mathbb{R}^{N}).$$

So, once you have this convergence. Then, I have to take T of that therefore,

$$T * (\phi * \psi)(0) = T((\phi * \psi)^{\vee}) = \lim_{\epsilon \to 0} \epsilon^N \sum_P T(\tau_{\epsilon P}\phi^{\vee})\psi^{\vee}(\epsilon P).$$

$$= \int_{\mathbb{R}^N} T(\tau_y \phi^{\vee})\psi^{\vee}(y)dy = \int_{\mathbb{R}^N} (T * \phi)(y)\psi(-y).$$

$$= ((T * \phi) * \psi)(0).$$

(iv) $T \in D'(\mathbb{R}^N)$, $T * \tilde{\phi} = 0$, $\forall \tilde{\phi} \in D(\mathbb{R}^N)$.

$$\tilde{\phi}^{\vee} \in D(\mathbb{R}^N), \quad 0 = (T * \tilde{\phi}^{\vee})(0) = T(\tilde{\phi}).$$

$$T \equiv 0.$$