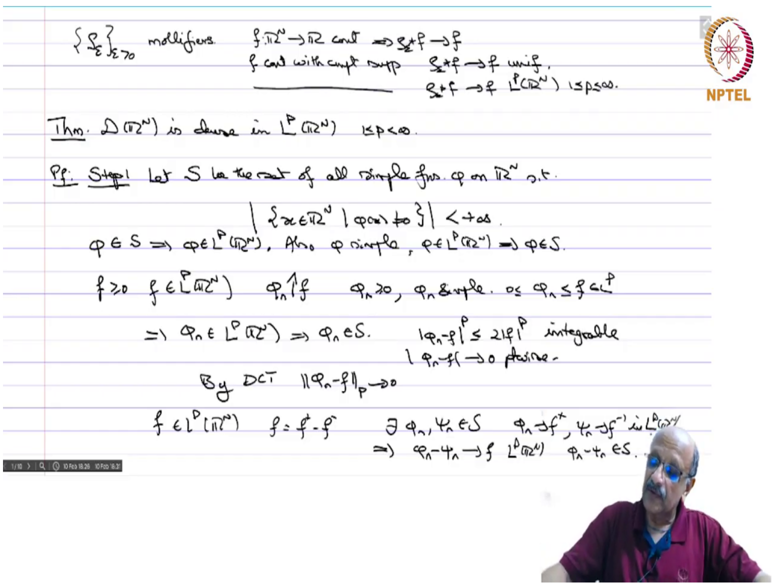
Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Lecture 13
Convolution of functions - Part 3

(Refer Slide Time: 00:17)



So, we have proved up to now, that if $\{\rho_\epsilon\}_{\epsilon>0}$ is the family of mollifiers and $f: \mathbb{R}^N \to \mathbb{R}$ continuous then $\rho_\epsilon * f \to f$ pointwise. And if f is continuous with compact support, then $\rho_\epsilon * f \to f$ uniformly. Finally, we also saw that f is continuous with compact support, then $\rho_\epsilon * f \to f$ in $L^p(\mathbb{R}^N)$, $1 \le p \le \infty$.

We will now use this to prove some density theorems.

Theorem: $D(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, $1 \le p < \infty$.

proof. So, first step.

So, let S be the set of all simple functions ϕ on $\mathbb{R}^N$ such that

$$|\{x \in \mathbb{R}^N : \phi(x) \neq 0\}| < \infty.$$

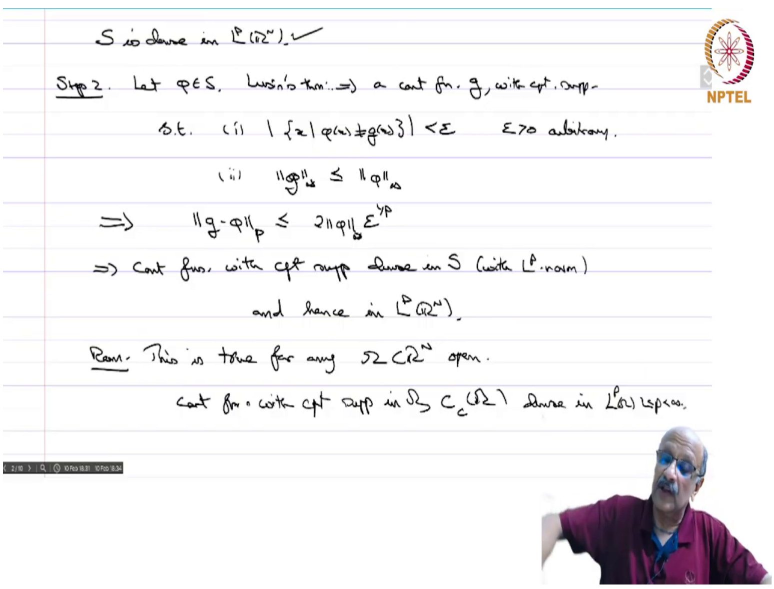
So, the function vanishes outside a set of finite measures. So, then ϕ in S if and only if, so ϕ in S implies automatically ϕ belongs $L^p(\mathbb{R}^N)$ to because after all what is the P integral, integral of mod phi power P, it is nothing but sigma mod alpha i measure of a i where alpha a i is the set where the value of alpha is taken. So, if alpha is not 0 all these sets of finite measures, so they add up to this.

Conversely, also ϕ simple, ϕ in $L^p(\mathbb{R}^N)$ implies ϕ is in S, because it cannot be nonzero on the set of infinite measures being a simple function because then the P integral will blow up. So, we have this. Now, let f be greater than equal to 0, f in $L^p(\mathbb{R}^N)$. Now, any non-negative measurable function you can approximate by an increasing sequence of simple functions.

So, $\phi_n \geq 0$, ϕ_n simple. So, ϕ_n is less than or equal to f, there is no need for modulus, everything is non negative and f belongs to L^p. So, this implies phi n is in LP of RN implies phi n belongs to S, S we have just observed. Further, mod ϕ_n minus f power p mod phi n minus f is less than equal to 2-time CF, so that is equal to 2 mod f power p and this is integrable and mod phi n minus f goes to 0 point wise.

Therefore, by the dominated convergence theorem ϕ_n minus f in $L^p(\mathbb{R}^N)$ goes to 0. Therefore, this phi n which increases to f also approximates f in LP and they are all in S. Now, if f belongs to $L^p(\mathbb{R}^N)$ arbitrary function, then you write f equals f plus minus f minus then there exists ϕ_n in S. So, ϕ_n goes to f plus psi goes to f minus in $L^p(\mathbb{R}^N)$ and therefore, this implies that ϕ_n minus ψ_n goes to f in $L^p(\mathbb{R}^N)$. And of course, fi n minus ψ_n belongs to S.

(Refer Slide Time: 05:43)



So, the conclusion is S dense in $L^p(\mathbb{R}^N)$.

Now, step 2.

So, let ϕ belong to S, then we quote a theorem Lusin's theorem from measure theory. So, what does this theorem imply, there exists a continuous function g with compact support such that 1, measure of set of all x, such that $\phi(x)$ is not equal to $g(x)$, can be made less than epsilon. So, ϵ greater than 0 is arbitrary.

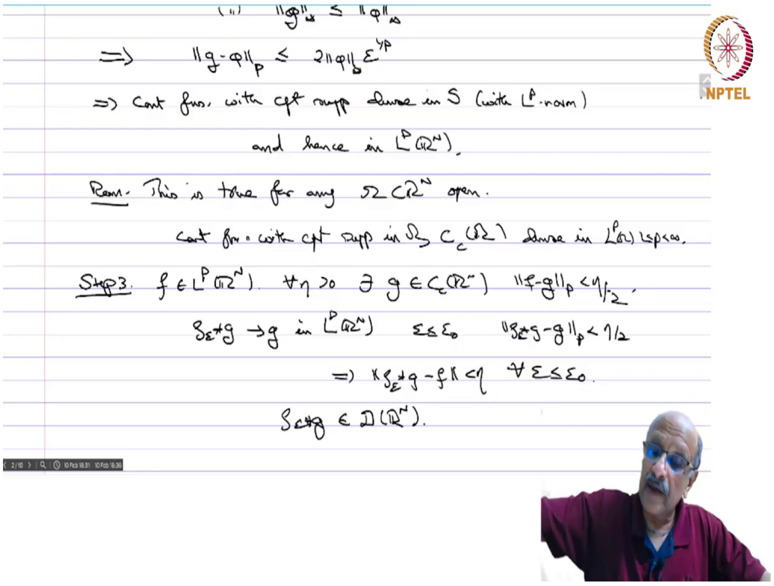
So, for every epsilon we can find such a g, and secondly norm of ϕ g in L infinity is less than equal to normal ϕ in L^∞. So, once you have this it automatically means that norm of g minus phi in L^p, what is this less than or equal to after all you have to take the integral only where these functions are different, everywhere else this difference is 0 and that is of measure less than epsilon and the function itself is less than 2 times norm of ϕ.

Therefore, you get 2 times norm ϕ in L infinity epsilon power 1 by P and therefore, this goes to 0. So, this implies that continuous functions with compact support are dense in S with L^p norm. And hence, since S itself by step 1 is dense in $L^p(\mathbb{R}^N)$, you have and hence in $L^p(\mathbb{R}^N)$.

Remark. This is true for any $\Omega \subset \mathbb{R}^N$ open.

So, in any, so continuous functions with compact support in omega, so denoted $C_c(\Omega)$ is dense in $L^p(\Omega)$ 1 less than equal to p less than ∞ because we are taking integrals, we are not taking L^∞ norm.

(Refer Slide Time: 08:51)

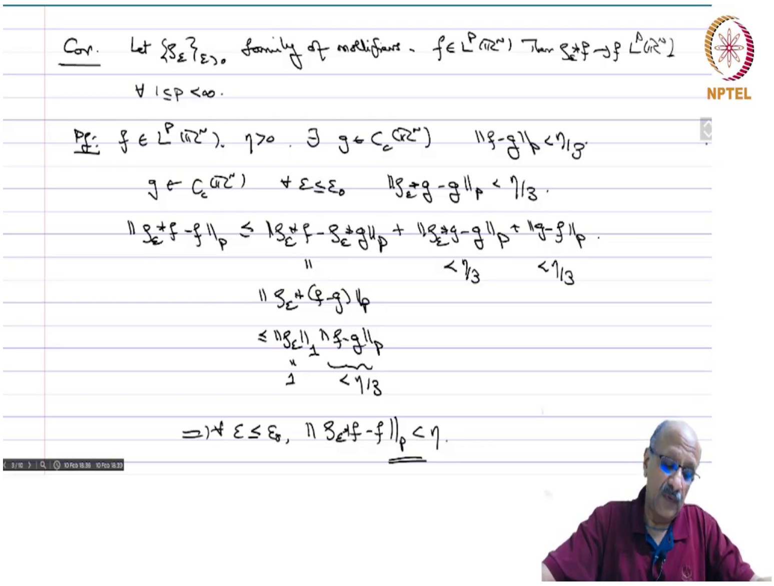


So, then step 3.

So, f is can be approximated, so, f in $L^p(\mathbb{R}^N)$ then for every eta positive there exists g in $C_c(\mathbb{R}^N)$, norm f minus g in $L^p(\mathbb{R}^N)$ is less than eta, then we know that $\rho_\epsilon * g$ converges to g in $L^p(\mathbb{R}^N)$, we have already proved that. And therefore, that exists for ϵ less than equal to some ϵ_0, norm of $\rho_\epsilon * g$ minus g in $L^p(\mathbb{R}^N)$ is less than η, let us say η by 2 both of these and this implies that norm of $\rho_\epsilon * g$ minus f is less than η for all ϵ less than equal to ϵ_0 and $\rho_\epsilon * g$.

Remember, it is a convolution of C infinity function and with compact support, with a continuous function with compact support and therefore, the function is C infinity and has compact support as well. And therefore, this proves the theorem completely.

(Refer Slide Time: 10:15)



So, $D(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$. Corollary, so let ρ_ϵ, ϵ greater than equal to 0 family of mollifiers. And let f belong to $L^p(\mathbb{R}^N)$, then $\rho_\epsilon * f$ converges to f in $L^p(\mathbb{R}^N)$. So, every continuous function can be approximated in the $L^p(\mathbb{R}^N)$ norm by a C infinity function now, because f is locally integrable being a $L^p(\mathbb{R}^N)$ function, rhi epsilon is C infinity with compact support that does not matter.

So, the rho epsilon star f is C infinity and it approximates f in the L^p norm and this is true for all 1 less than equal to P less than infinity. So, proof. So, you take f in $L^p(\mathbb{R}^N)$, and let eta be greater than 0 fixed quantity. So, then there exists a g in Cc of RN continuous function with compact support, such that norm f minus g in $L^p(\mathbb{R}^N)$ is less than eta by 2 because that is the first two steps of the theorem.

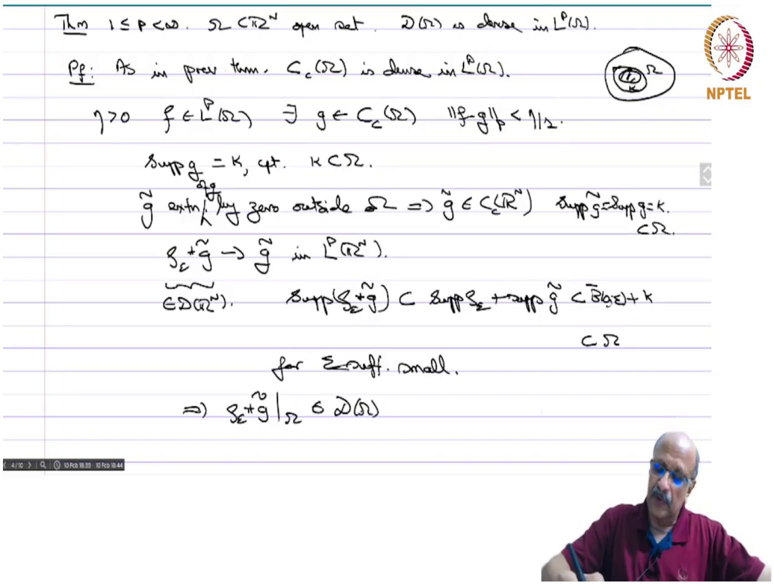
Namely, we saw that continuous functions with compact support were dense and then of course, we can conclude it for C infinity functions with compact support. Now, since g is in $C_c(\mathbb{R}^N)$, so for all ϵ, let us say some ϵ_0 you will have norm of $\rho_\epsilon * g$ minus g in $L^p(\mathbb{R}^N)$ is less than say η.

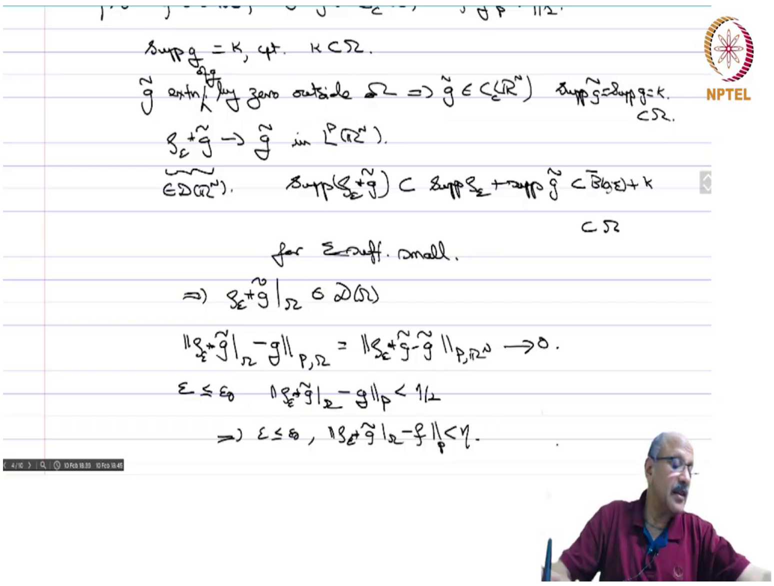
So, let me take it with a little foresight, let me write η by 3. So, this is also η by 3. So, norm of rho epsilon star f minus f in $L^p(\mathbb{R}^N)$ is less than or equal to norm of rho epsilon star f minus rho epsilon star g in $L^p(\mathbb{R}^N)$ by triangle inequality, this is rho epsilon star g minus g in

$L^p(\mathbb{R}^N)$ and then plus norm of g minus f in $L^p(\mathbb{R}^N)$. So, now, this is less than eta by 3, this is also less than eta by 3 and this is equal to norm of rho epsilon star f minus g in $L^p(\mathbb{R}^N)$.

And by Young's inequality this less than equal norm rho epsilon 1 norm f minus g in $L^p(\mathbb{R}^N)$. Now, rho epsilon 1 is the integral of rho epsilon is always unity and this is less than epsilon by 3 and therefore, we have that for all epsilon S equal to epsilon naught rho epsilon star f minus f in $L^p(\mathbb{R}^N)$ is less than eta, where eta is any arbitrary positive constant and therefore, arbitrarily small and therefore, we approved this Corollary.

(Refer Slide Time: 13:42)





Theorem. $1 \leq p < \infty$. $\Omega \subset \mathbb{R}^N$ *open set*. Then $D(\Omega)$ is dense in $L^p(\Omega)$.

And therefore, use which he said we will prove later. So, we are redeeming our promise now, so we are proving this theorem in this case. So, proof. So, as in the previous theorem $C_c(\Omega)$ continuous functions with compact support is dense in $L^p(\Omega)$. So, I have already made that remark. So, we already have this result.

So, given eta positive arbitrarily small positive quantity in f in $L^p(\Omega)$ there exists a g in Cc omega. Such that norm of f minus g in $L^p(\Omega)$is less than epsilon, eta by 2. Now, g is continuous with compact support contained in omega, support of g equal to K compact and K is contained in omega. So, you have omega here and you have a complex set K which is contained in it which is compact.

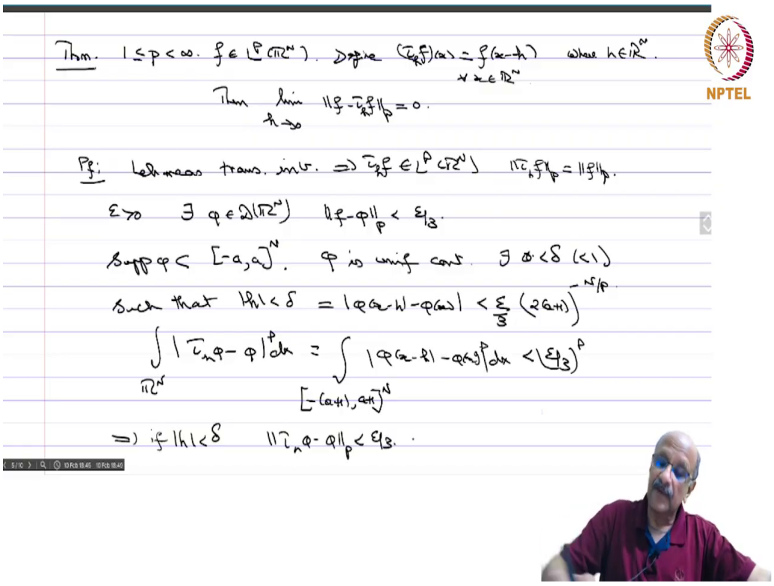
So, if you take g tilde extension by 0 outside omega extension of g by 0 outside omega then you have the g tilde belongs to $C_c(\mathbb{R}^N)$. And what is g tilde, support of g tilde is the same as support of g which is K which is contained in omega. Now, ρ_ϵ sub g tilde converges to g tilde in $L^p(\mathbb{R}^N)$ and this belongs to $D(\mathbb{R}^N)$. Because it is a C infinity function and its support is contained in, support for rho epsilon.

Now, support of rho epsilon star g tilde is contained in support of rho epsilon plus support of g tilde which is contained in B0 epsilon, the ball center the origin radius epsilon, close to ball plus K and if epsilon and that we still contained in omega because you are going to take epsilon distance from K. So, that will be still contained in Ω for epsilon sufficiently small.

So, you have that rho epsilon sub g tilde restricted to omega belongs to $D(\Omega)$, because the sub is a C infinity function and the support is contained in omega. And since rho epsilon g tilde converges to g tilde in $L^p(\mathbb{R}^N)$. So, norm of rho epsilon star g tilde restricted omega minus g in $L^p(\Omega)$. This is the same as norm of rho epsilon star g tilde minus g tilde in $L^p(\mathbb{R}^N)$ and we know that this goes to 0. Because everything is 0 outside and so this goes to 0.

And therefore, you have shown that therefore, for an epsilon sufficiently small you have a norm of rho epsilon star g tilde (())(19:09) to omega minus g in L^p is less than eta by 2. And therefore, this implies for epsilon, this equals to epsilon naught norm of rho epsilon star g tilde is secret to omega minus f is less than eta in L^p. And that proves the theorem.

(Refer Slide Time: 19:30)



Then one more very useful theorem.

Theorem: $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^N)$. Define $\tau_h f(x) = f(x - h)$, $where\ h \in \mathbb{R}^N$.

Then $\lim_{h \to \infty} ||f - \tau_h f||_p = 0$.

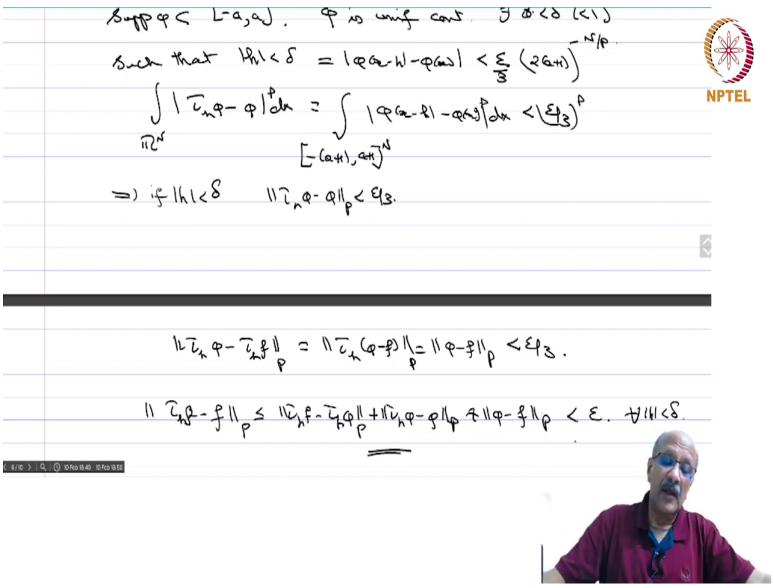
proof. So, lebesgue measure is translation in variant. So, this implies that $\tau_h f$ is also in $L^p(\mathbb{R}^N)$. And in fact, it says the same norm, norm of $\tau_h f$ in $L^p(\mathbb{R}^N)$ is the same as norm of f in $L^p(\mathbb{R}^N)$, that is just the translation invariance of the lebesgue measure. And now, let epsilon be great than or equal to 0, small positive quantity, then there exists ϕ in $D(\Omega)$ such that norm of f minus phi in $L^p(\mathbb{R}^N)$ is less than epsilon by 3.

Now, let us support a ϕ being contained in some minus a, a power N. So, it is a big box in which we are putting it. Then phi is uniformly continuous. And therefore, there exists a 0 less than delta which we can assume to be less than 1 because you can take it as small as you like such that mod h less than delta implies mod phi of x minus h minus phi x is as small as we place and I am going to put it as less than epsilon by 3 times 2 into a plus 1 power minus N by P.

So, then if you compute tau h phi minus phi power P integral over RN dx, this is equal to we just have to take the integral because h is less than 1 and therefore, this will be over the integral minus a plus 1 power n that will contain all the supports phi of x minus h minus phi of x power P dx and that you will get less than epsilon by 3 power P. Because of the choice of this when you have mod h is less than delta. So, this implies

if $|h| < \delta, \Rightarrow ||\tau_h\phi - \phi||_p < \frac{\epsilon}{3}.$

(Refer Slide Time: 23:28)



Now,

$$||\tau_h\phi - \tau_h f||_p = ||\tau_h(\phi - f)||_p = ||\phi - f||_p < \frac{\epsilon}{3}.$$

$$||\tau_h f - f||_p \leq ||\tau_h\phi - \tau_h f||_p + ||\tau_h f - f||_p + ||\phi - f||_p < \epsilon, \forall\, |h| < \delta.$$

So, we will, our next thing is to extend the notion of convolution to that of distributions which we will do next.