Sobolev Spaces and Partial Differential Equations
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Lecture 12
Convolution of Functions Part - 2

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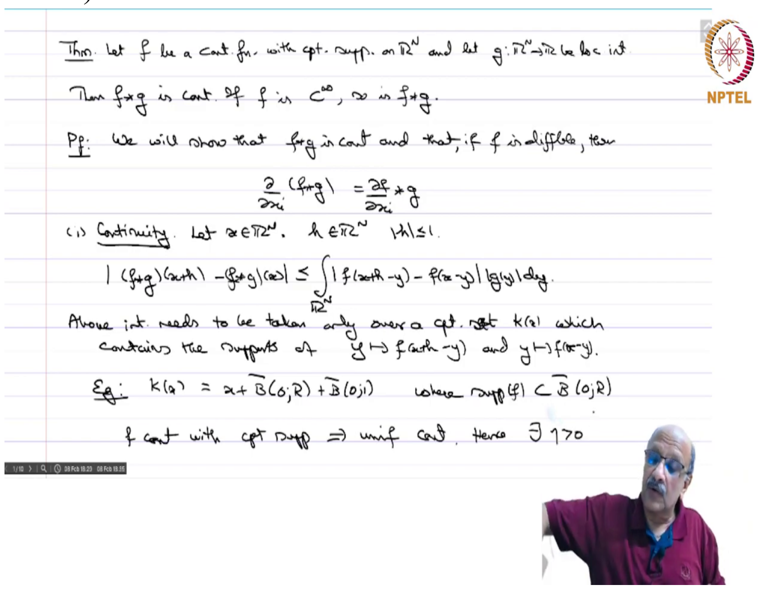

So, we continue with the property convolution. So, now we prove an important theorem, which is really the reason why convolution is a very powerful tool in analysis.

Theorem: Let f be a continuous function with compact support on $\mathbb{R}^N$ and let $g:\mathbb{R}^N \rightarrow \mathbb{R}$ be locally integrable. Then, $f * g$ is continuous. If f is C^{∞} so is $f * g$.

So, we saw that convolution can be defined for a continuous function with compact support and a locally integral function. So, there is no difficulty in making sense of that integral.

So, here is the very beautiful property: you take any continuous locally integrable function, convolve it with a continuous function with compact support, and you produce a continuous function. And if you convolve it with a C^{∞} function, then you produce a C^{∞} function. And if you in fact, we will see in the proof that if you convolve it with a ck function the resultant function will also be ck. So, you take any rough function and by convolution you make it smooth, a very smooth function.

So, this is called a smoothing operation. So, let us prove.

proof: So, we will show that $f * g$ is continuous. And that if f is differentiable, then

$$\frac{\partial}{\partial x_i}(f * g) = \frac{\partial f}{\partial x_i} * g.$$

So, you see the derivative of this is, the derivative of f is again a continuous function with compact support. And then you can convolve it with g. So, every time you want to differentiate, you put the derivative on f. So, if f is C^{∞} you can put any derivative on f.

continuity: $x \in \mathbb{R}^N$, $h \in \mathbb{R}^N$ s.t. $|h| \leq 1$.

$$|(f * g)(x + h) - (f * g)(x)| \leq \int_{\mathbb{R}^N} |f(x + h - y) - f(x - y)||g(y)|dy$$

Now, the above integral needs to be taken only over a compact set $K(x)$ which contains the supports of $y \to f(x + h - y)$ and $y \to f(x - y)$.

So, you can take for instance, so example.

example: $K(x) = x + \overline{B(0, R)} + \overline{B(0, 1)}$, where supp(f)$\subset\overline{B(0, R)}$.

Now f has compact support, implies uniformly continuous. Hence, there exists an $\eta > 0$.

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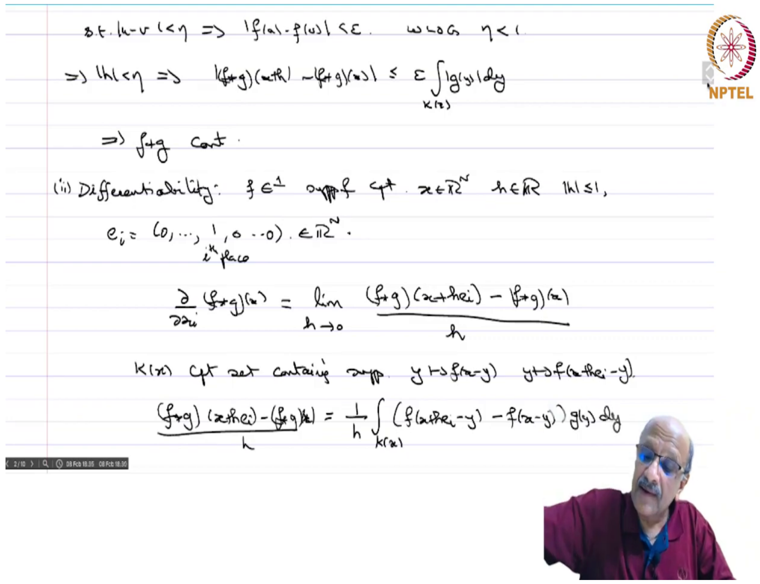

Such that $|u - v| < \eta \Rightarrow |f(x) - f(y)| < \epsilon$. WLOG $\eta < 1$.

$$|h| < \eta \Rightarrow |(f * g)(x + h) - (f * g)(x)| \leq \epsilon \int_{K(x)} |g(y)|dy$$

$\Rightarrow f * g$ is continuous.

differentiability : f *is* C^1 , supp(f) is compact, $x \in \mathbb{R}^N$, $h \in \mathbb{R}^N$ s.t. $|h| \leq 1$

Let us take $e_i = (0, 0, \ldots\ldots, 1, \ldots., 0) \in \mathbb{R}^N$ (1 in k-th position).

Then

$$\frac{\partial}{\partial x_i}(f * g)(x) = \lim_{h \to 0} \frac{(f*g)(x+he_i)-(f*g)(x)}{h}$$

So, now again if you take K(x) is a compact set containing the support of $y \to f(x - y)$ and $y \to f(x + he_i - y)$, then you have

$$\frac{(f*g)(x+he_i)-(f*g)(x)}{h} = \frac{1}{h} \int_{K(x)} (f(x + he_i - y) - f(x - y))g(y)dy$$

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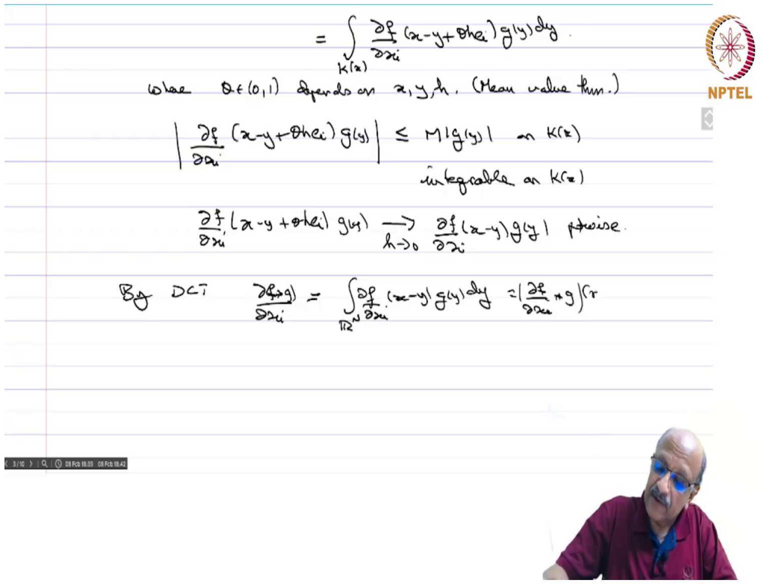

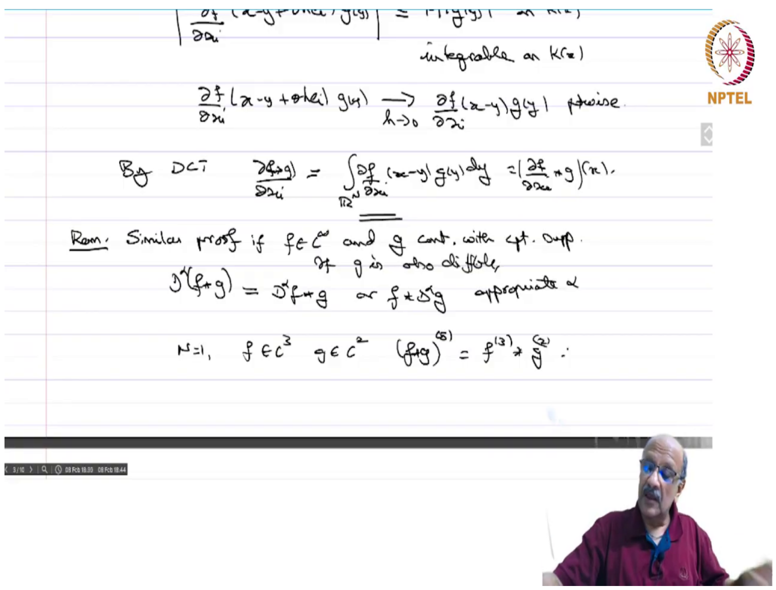

$= \int_{K(x)} \frac{\partial f}{\partial x_i}(x - y + \theta h e_i) g(y) dy,$ where $\theta \in (0, 1)$ depends on $x, y,\ h$. (Mean value Theorem).

$$\frac{\partial f}{\partial x_i}(x - y + \theta h e_i) g(y) \rightarrow \frac{\partial f}{\partial x_i}(x - y) g(y) \text{ pointwise as } h \rightarrow 0.$$

Therefore, by the dominated convergence theorem, we get that

$$\frac{\partial}{\partial x_i}(f * g)(x) = \int_{\mathbb{R}^N} \frac{\partial f}{\partial x_i}(x - y)g(y)dy = \left(\frac{\partial f}{\partial x_i} * g\right)(x).$$

Remark. Similar proof in fact, it is even easier if f is C^∞ and g continuous with compact support. It does not matter where the support is, one of them should be of compact support that is okay.

And therefore, when we can, so in fact, if f is, suppose f is you can also show that

$$D^\alpha(f * g) = D^\alpha f * g \; or \; f * D^\alpha g, \text{ for appropriate } \alpha.$$

If $f \in C^3$ and $f \in C^2$ for instance, then the $(f * g)^{(5)} = f^{(3)} * g^{(2)}$.

You can partition and define, differentiate like this. So, you can differentiate in any number of ways. So, that is the beauty of this theorem, that namely you can by convolving a smooth function with a rough function you produce a smooth function.

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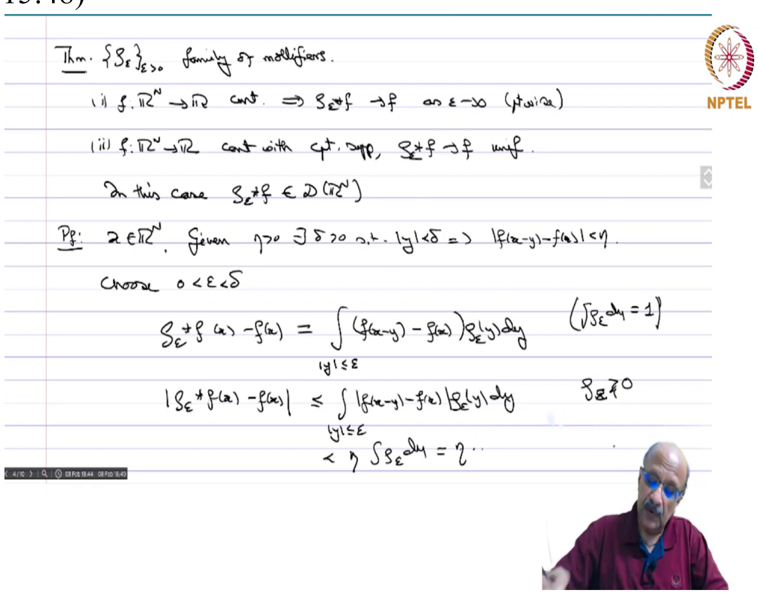

And this has several important applications and we will prove a lot of density theorems based on these things. So, the first application of this theorem is

Theorem: $\{\rho_\epsilon\}_{\epsilon>0}$ family of mollifiers.

(i) $f: \mathbb{R}^N \to \mathbb{R}$ continuous $\Rightarrow \rho_\epsilon * f \to f \; as \; \epsilon \to 0$ (pointwise).

(ii) $f: \mathbb{R}^N \to \mathbb{R}$ continuous with compact support $\Rightarrow \rho_\epsilon * f \to f \; as \; \epsilon \to 0$ (uniformly).

In this case $\rho_\epsilon * f \in D(\mathbb{R}^N)$.

proof: Let $x \in \mathbb{R}^N$. Given $\eta > 0$, $\exists\, \delta > 0 \; s.t. \; |y| < \delta \Rightarrow |f(x-y) - f(x)| < \eta$.

Choose $0 < \epsilon < \delta$. Then

$$|\rho_\epsilon * f(x) - f(x)| \le \int_{|y|\le\epsilon} |f(x-y) - f(x)||\rho_\epsilon(y)|dy$$

$$< \eta \int_{|y|\le\epsilon} \rho_\epsilon dy = \eta.$$

This proves the continuity.

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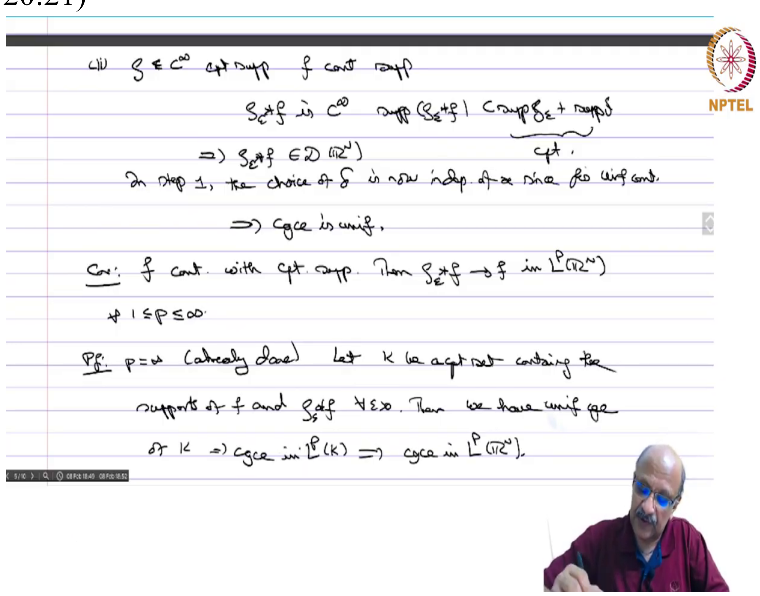

So, in the second step, so $\rho_\epsilon \in C^\infty$ with compact support. And f is continuous with compact support. So, $\rho_\epsilon * f \in C^\infty$. And supp($\rho_\epsilon * f$)⊂supp($\rho_\epsilon$)+supp(f).

$$\Rightarrow \rho_\epsilon * f \in D(\mathbb{R}^N).$$

And in step 1, the choice of delta is now independent of x. Since, f is uniformly continuous, f is continuous with compact support, therefore, it is uniformly continuous.

And therefore, you have that this works. So, the convergence therefore, the convergence is uniform.

Corollary: f continuous with compact support and then $\rho_\epsilon * f \to f\ in\ L^p(\mathbb{R}^N)$.

proof: $p = \infty$ is already done. Because we have proved uniform continuity and that is exactly convergence in $p = \infty$.

So, let K be a compact set containing the supports of f and supp. of $\rho_\epsilon * f$ for all $\epsilon > 0$. Then we have uniform convergence on K ⇒ convergence in $L^p(K)$. And since everything is 0 outside K ⇒ convergence in $L^p(\mathbb{R}^N)$.

So, its measure is finite and consequently, it is automatically convergent in L^p as well. So, using this theorem, we will now go prove some important density theorems for L^p spaces and contain and C^∞ functions with compact support and we will see that next.