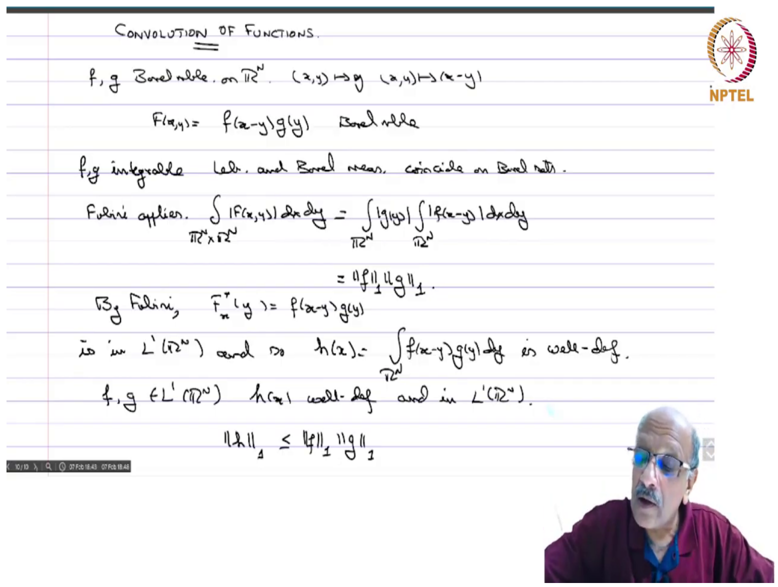
Sobolev Spaces and Partial Differential Equations
Professor. S. Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Lecture 11
Convolution of Functions Part - 1

(Refer Slide Time: 00:20)



We will now discuss a very important notion namely that of convolution. We will first discuss it for functions and later extend it to classes of distributions. So, let us take f and g Borel measurable functions on $\mathbb{R}^N$ and then you have let $(x, y) \to y$ and $(x, y) \to x - y$ are continuous functions. And therefore, if you compose then you have

$F(x, y) = f(x - y)g(y)$ is a Borel measurable function.

Now, if you look at if f and g are integrable, if you combine compose continuous and Borel measurable you will get Borel measurable. So, if f and g are in addition integrable then Lebesgue and Borel measure the coin side on Borel sets and therefore, you can apply Fubini theorem, therefore you have that integral.

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |F(x, y)|dxdy = \int_{\mathbb{R}^N} |g(y)| \int_{\mathbb{R}^N} |f(x - y)|dxdy = ||f||_1 ||g||_1.$$

Now, this is equal to integral over Rn mod gy integral over Rn mod f of x minus y dx and then dy and since everything is non-negative you can do this. And now, Lebsegue measure this translation invariant and therefore, this will give you equal to norm f1 in Rn and then that leaves you only with mod g y and that for that will give you norm g1 as well. So, this means by Fubini theorem again you have that the x section namely

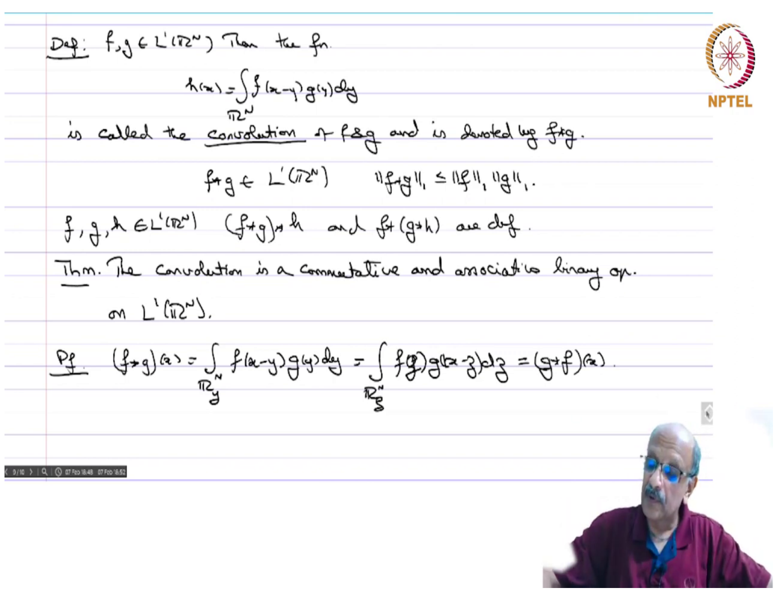
$$F^{*}_{x}(y) = f(x - y)g(y) \text{ is in } L^{1}(\mathbb{R}^{N}).$$

And so, we can and so, h of x equal to integral f of x minus y gy dy over Rn is well defined. Now, if f and g are integrable then f and g are almost everywhere equal to Borel measurable functions and therefore, since integration does not depend, when you have functions almost everywhere the integration is the same and therefore, h x well defined again and we also have the

$$||h||_{1} \leq ||g||_{1}||f||_{1}$$

by the preceding calculations. So, therefore, you have this well defined. Now, this is called the convolution.

(Refer Slide Time: 05:01)



Definition: $f, g \in L^{1}(\mathbb{R}^{N})$. Then the function

$$h(x) = \int_{\mathbb{R}^N} f(x - y)g(y)dy$$

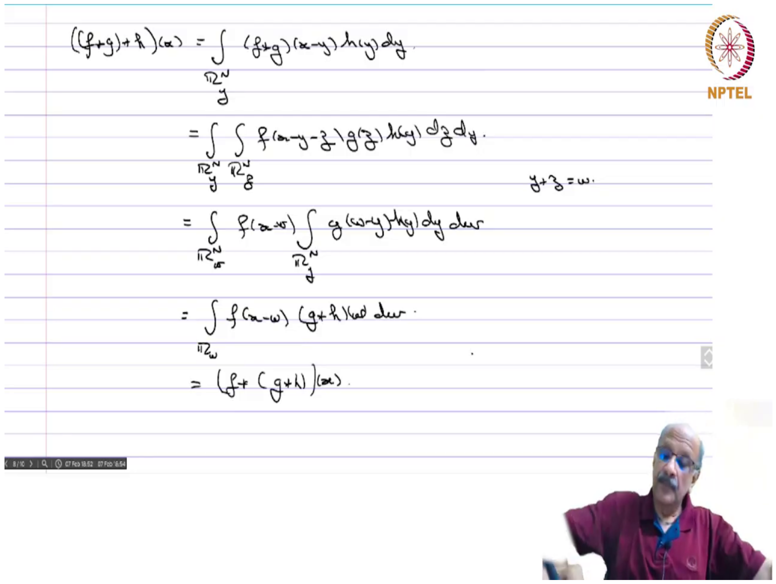
is called the convolution of f and g and it has been noted by $f * g$. And we have the f star g belongs to L1 of Rn and we saw a norm of f star g in L1 is less than equal to norm f in L1 norm g in L1. So, norm 1 is nothing but the L1 norm in Rn.

So, since the convolution is well defined and you produce again an L1 function, we can define f star g star h. Now, this h so, f g h (())(06:38) L1 of Rn. So, this h is not to be confused with this h here which is a temporary notation which I used. So, you can define f star g star h because this L1 this also L1 and f star g star h are defined. So, now we have the following theorem:

Theorem: the convolution is a commutative and associative binary operation on $L^1(\mathbb{R}^N)$.

proof: So, if star g of x is equal to integral Rn I will write with respect to y, so, just to tell you, which is the variable of integration, so, fy, gy, dy. So, I am going to make a change of variable. So, this becomes Rn of z. So, I am going to put x minus y equals z. So, this becomes f of z times g of x minus z and then by the change of variable factor, this will just give you dz and that is in fact equal to g star f of x. So, we have just used the change of variable and linear change of variable.

(Refer Slide Time: 08:36)



So, now let us take three functions

$$((f * g) * h)(x) = \int_{\mathbb{R}^N_y} (f * g)(x - y)h(y)dy$$

$$= \int_{\mathbb{R}^N_y} \int_{\mathbb{R}^N_z} f(x - y - z)g(z)h(y)dzdy$$

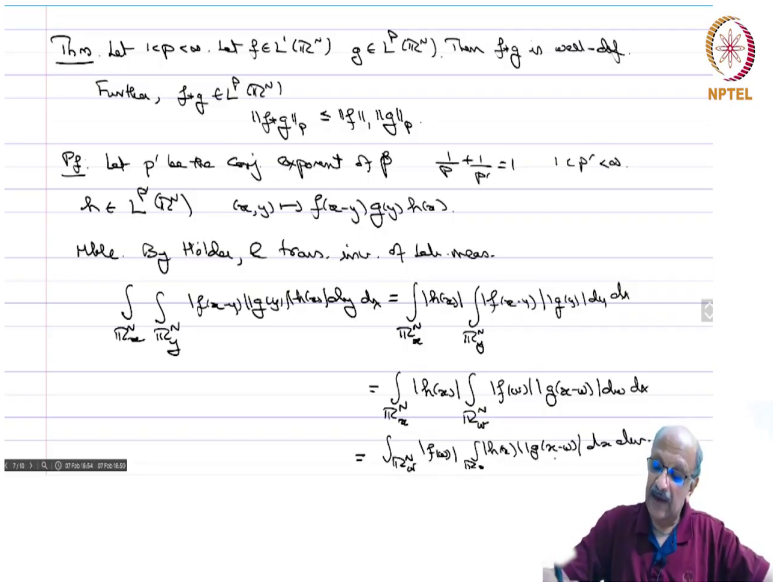
$$= \int_{\mathbb{R}^N_w} f(x - w) \int_{\mathbb{R}^N_z} g(w - y)h(y)dwdy$$

$$= \int_{\mathbb{R}^N} f(x - w)(g + h)(w)dw$$

$$=(f * (g * h))(x).$$

Therefore, this is an associative operation.

(Refer Slide Time: 11:17)



So, now we can extend this definition. So, the next theorem:

Theorem: Let $1 < p < \infty$ and $f \in L^1(\mathbb{R}^N)$, $g \in L^p(\mathbb{R}^N)$. Then f * g is well defined. Further f *g $\in L^p(\mathbb{R}^N)$ and

$$||f * g||_p \leq ||f||_1 ||g||_p.$$

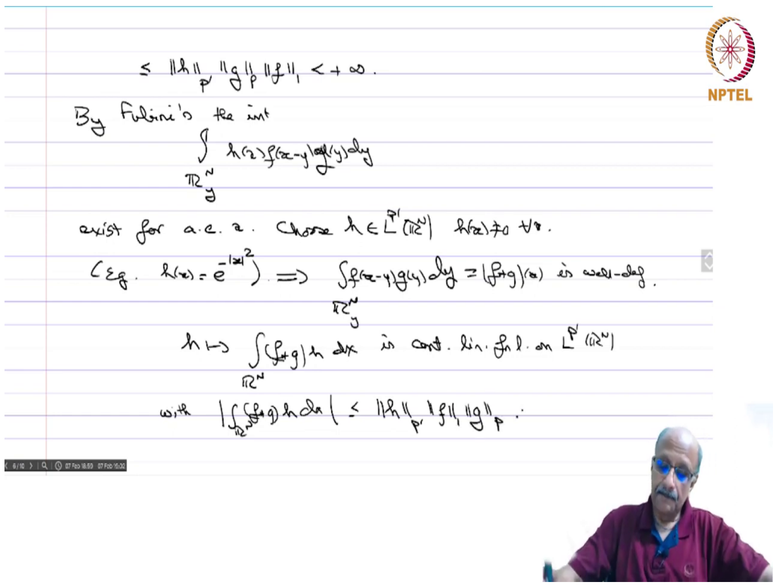
proof: so, if p equals 1 which we have excluded here we have already seen f and g are both in L1 then f star g seen L1, f is norm f1 norm g1 that is what we already proved.

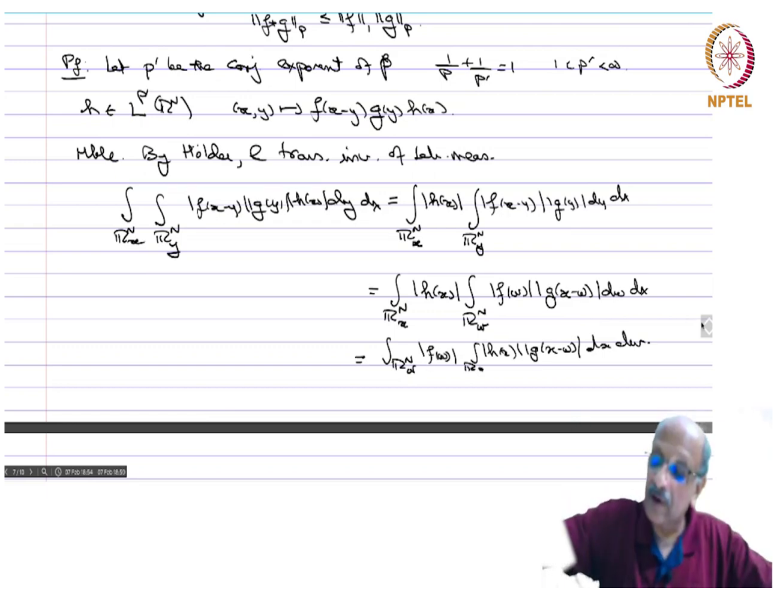
Now, we are going to extend this result. So, when p is strictly bigger than 1 but less than infinity. So, let p dash be the conjugate exponent that means 1 by p plus 1 by p dash equal to 1. So, you have 1 less than p dash less than infinity as well. So, then you take h in Lp dash Rn. Now, you look at the map x y going to f of x minus y gy hx. So, this is measurable and by Holder's inequality.

So, measurable by Holders and translation invariance of Lebesgue measure we have integral Rn x integral Rn y of mod f of x minus y gy, hx dx dy, dy dx. So, this is equal to integrals Rn x so, everything is positive so, I cannot worry about any integrability etc, I can use Fubini theorem straight away integral Rn y f of x minus y mod gy dy dx. Now, that is equal to integral r and x mod hx integral Rn w again I am making a change of variable of mod x y in this way I am going to put as w so I get mod f of w g of x minus w dw dx.

Now, I apply the Fubini theorem I take, I bring out the w. So, this is c equal to integral Rn of w mod f of w, you can keep out and now you write Rn of x mod hx mod gx minus w dx and dw. Now, what do you have F as in L1 G is in Lp h is in Lp dash, they are conjugate coefficients. So, by the Reese representation theorem.

(Refer Slide Time: 15:45)





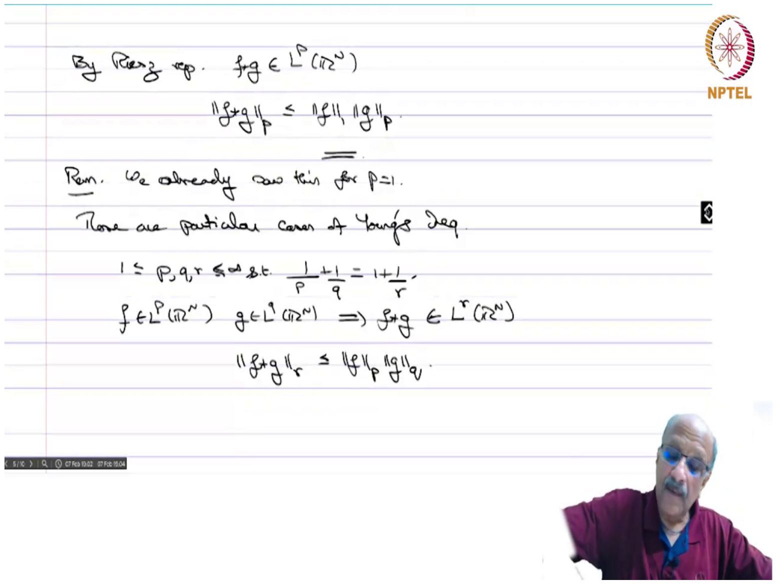
So, this is less than or equal to and you have the translation invariance of the Lebesgue measure. So, you norm h in the Lp dash of Rn, norm g, into Lp. Lp of rn so, I am bringing out so, this because of w is fixed, this is just translation by w and therefore, this is the same norm as the nominal p. So, norm g in Lp norm h in Lp dash that is Holder inequality, and then those come out remaining is just norm f in L1.

So, what is left is nothing but the integral of f 1. So, this is internorm F in L1 and that of course, everything is finite. So, since all this is finite, so, by Fubini's theorem when you have everything for the modulus is finite. So, the integral over Rn y hx f of x minus y, gy dy exists for almost every x. So, now you choose h in Lp dash Rn so, set hx is not equal to 0 for all x. So, for example you can take hx equals e power minus mod x square.

Now, this belongs to all Lp spaces and this is strictly positive for all x and therefore, you can take like that. So, hx is not a constant as far as this integral is concerned and therefore, it comes out. So, you can divide by it and therefore, so, this implies that integral f of x minus y gy dy over Rn y exists and further we have shown that h going to, so the and therefore, f star g is defined.

So, therefore, we can write equations f star g of x is well defined. So, further we have shown that h going to Rn f star g times h dx is a continuous linear functional on Lp dash Rn by this inequality which is here and therefore, with norm with mod integral over Rn f star g h dx less than or equal to norm h p dash norm if 1 norm g.

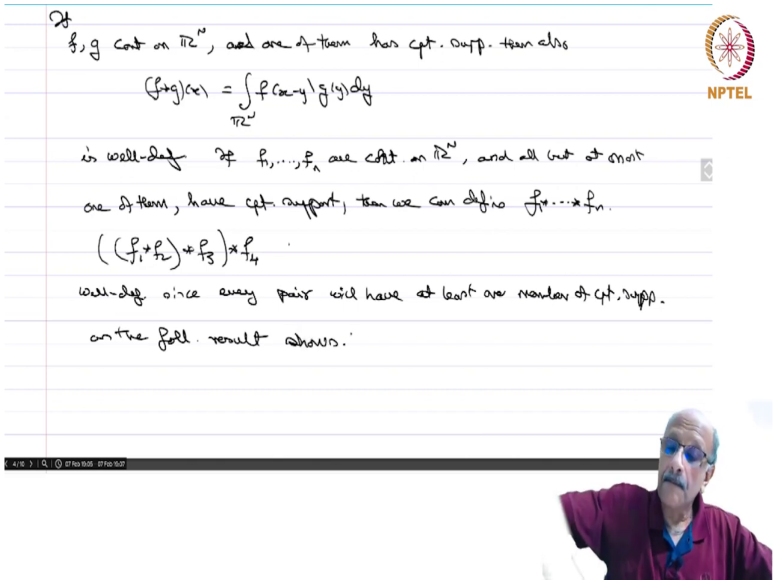
(Refer Slide Time: 19:23)



So, what does this mean by the Riesz representation theorem we have that f star g must belong to the dual of Lp dash which is the Lp of Rn and the norm of f star g in Lp is less than equal to norm f1 norm gp. So, that completely proves this theorem. So, remark so, we already proved this for so we already showed this solve this for p equals 1, that is one thing and then these are particular cases of Young's inequality. What is young's inequality 1 less than or equal to p q r less than infinity.

So, you have 1 by p plus 1 by q equals 1 plus 1 by r such that. Then if f is in Lp of Rn g is in Lq of Rn then you have that f star g belongs to Lr of Rn and norm, f star g in a Lr is less than or equal to norm f in Lp norm, norm g Lq. So, this is a particular case of this.

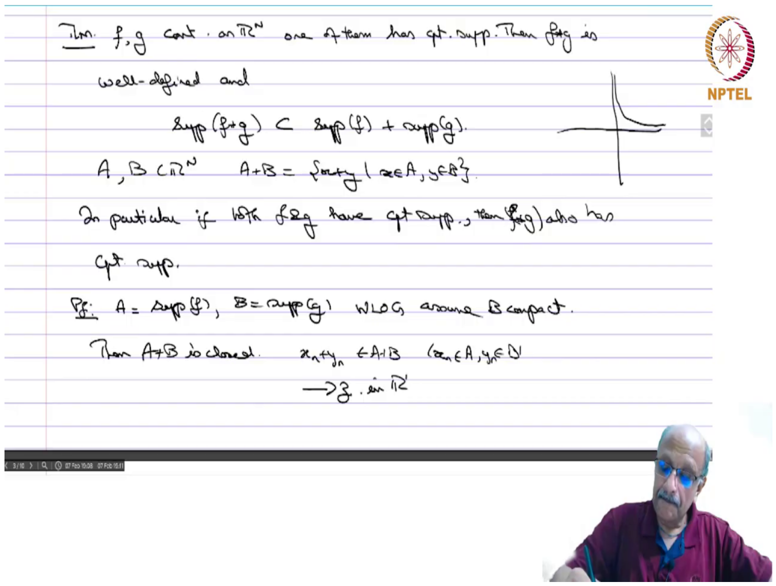
(Refer Slide Time: 21:27)



So, now we can go further So, fn g are continuous on Rn and one of them has compact support then also f star g of x equals integral over Rn f of x minus y gy dy is well different. So, now and you have commutativity and if f1 Fn are continuous on Rn and all but at most one of them has compact support then we can define f1 star, star fn by doing two at a time so, you can take f1 star f2 then you can take star f3, star f4, or you can do it in many ways it does not matter how we are going to pair them.

So, the pairing is unimportant, the order is unimportant because every convolution which we write one of them will have compact support as the following result shows. So, well defined since every pair will have at least one member of compact support as the following result shows.

(Refer Slide Time: 24:30)



Theorem: f, g continuous on $\mathbb{R}^N$., one of them has cpt. support. Then f*g is well defined and

supp(f*g)⊂ supp(f)+supp(g).

Where you know if A and B are subsets of Rn then you have A plus B instead of all x plus y such that x belongs to A y belongs to B. So, these are the algebraic sum of two sets, set of all x plus y x in A y in B.

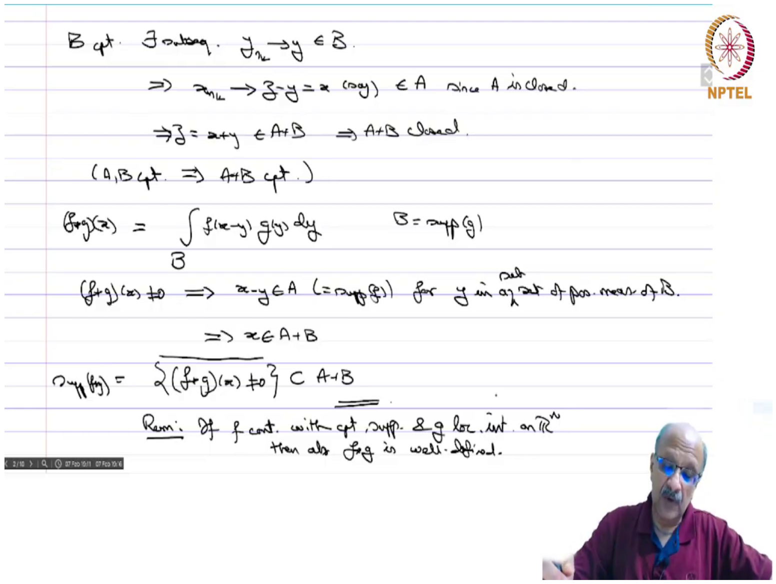
So, in particular if both f and g have compact support then f star g also has compact support.

proof. So, let $A = supp(f)$, $B = supp(g)$. WLOG assumes that B is cpt. Then

A+B is closed.

So, let $x_n + y_n \in A + B\ and\ converges\ to\ z \in \mathbb{R}^N$.

(Refer Slide Time: 28:15)



So, B is compact. So, there exists a subsequence $y_{n_k} \to y \in B$.

$$\Rightarrow x_{n_k} \to z - y = x\,(say) \in A$$

$$\Rightarrow z = x + y \in A + B \Rightarrow A + B \text{ is closed.}$$

So, now, if you look at the integral,

$$(f * g)(x) = \int_B f(x - y)g(y)dxdy \qquad B = supp(g)$$

$$(f * g)(x) \neq 0 \Rightarrow x - y \in A \text{ for } y \text{ in such that}$$

$$\Rightarrow x \in A + B$$

$$supp(f + g) = \overline{\{(f + g)(x) \neq 0\}} \subset A + B.$$

Remark: if f is continuous with compact support and g locally integrable on $\mathbb{R}^N$, then also f*g is well defined.

And in fact, you have the same support f*g is contained in support of f and supp(g). So, this is the starting point of a very interesting property of convolutions which really gives you the power of this operation and we will see that next time.