Sobolev Spaces and Partial Differential Equations

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Lecture 10
Exercises – Part 1

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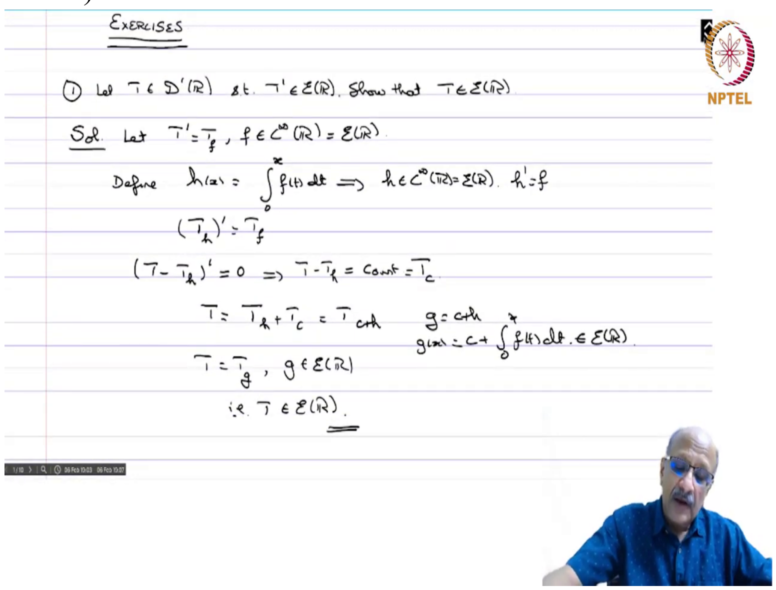

We will now do some exercises, so that we understand what we have learned so far.

Exercises:

(1) Let $T \in D'(\mathbb{R})$ $s.t.$ $T' \in E(\mathbb{R})$. Show that $T \in E(\mathbb{R})$.

solution. Let $T' = T_f$, $f \in C^\infty(\mathbb{R}) = E(\mathbb{R})$.

Define $$h(x) = \int_0^x f(t)dt \Rightarrow h \in C^\infty(\mathbb{R}) = E(\mathbb{R}), h' = f.$$

$$T_h' = T_f.$$

Therefore, you have

$$(T - T_h)' = 0 \Rightarrow T - T_h = \text{const.} = T_c$$

So, $T = T_h + T_c = T_{h+c}$.

And therefore, if you write g equal to c plus h, what is g of x? G of x is equal to c plus integral 0 to x f t dt. And this itself is again a C infinity function. Therefore,

$$T = T_g,\ g \in E(\mathbb{R}).$$

$$\text{i.e., } T \in E(\mathbb{R}).$$

So, it will also be the classical derivative in this case.

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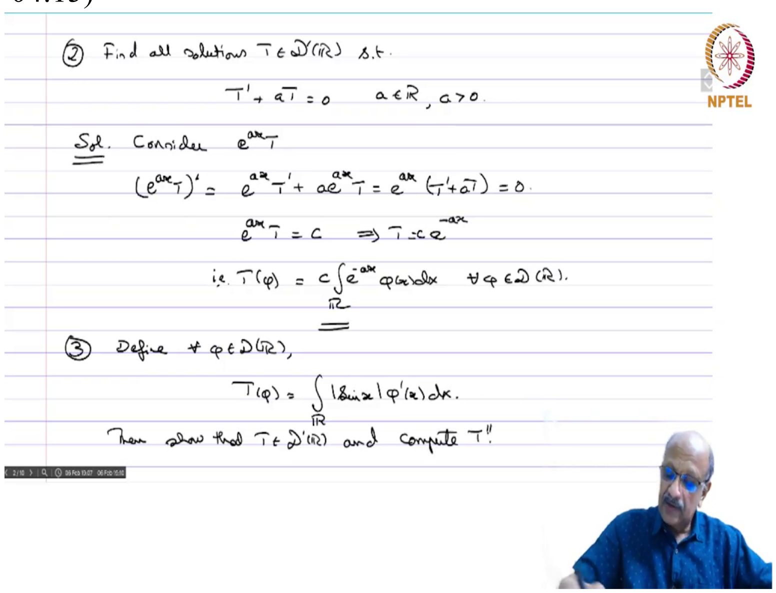

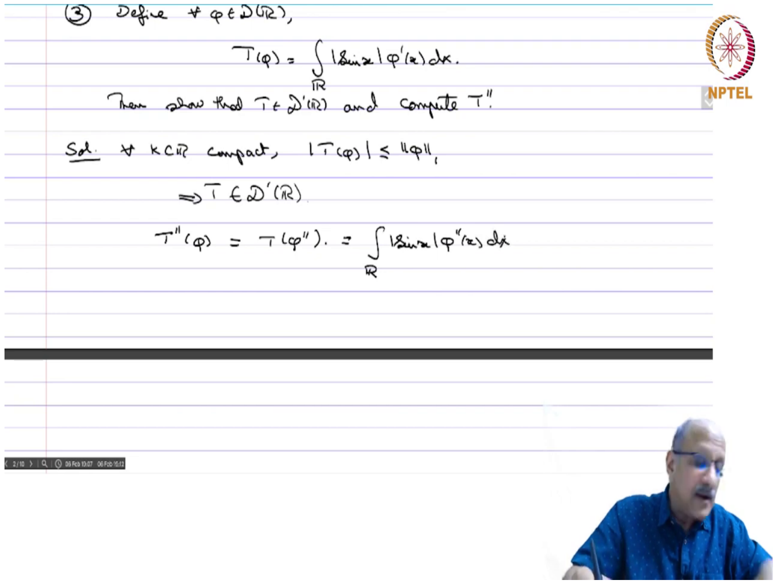

(2) Find all solutions $T \in D'(\mathbb{R})$ such that,

$$T' + \alpha T = 0, \quad \alpha \in \mathbb{R},\ \alpha > 0.$$

Let us say a positive, it does not matter a, a can be anything. So, we are looking at a differential equation now. And differential equations, we are looking for solutions in the class of distributions. So, these are called distribution solutions of the differential equation. So, we want to know what are all the distribution solutions, which will, which can (())(05:07).

solution. So, we do exactly as we do in the classical case, what, this is a linear differential equation. What do you do? You multiply by the integrating factor. So, let us consider T, e power ax into T. So, this you know what it means, because you are multiplying T by a C infinity function in the front. So, let us take the power ax T dash. Then we saw that this satisfies the usual product rule.

So, you will get e power ax times T dash plus a e powers ax times T. So, that is equal to e power ax into T dash plus a T, but T is a distribution solution of this equation and therefore, that is equal to 0. So, you have the derivative, distribution derivative of something equal to 0. So, then we know that e power ax T equals a constant. Again, recall that we mean that this is generated by the constant function.

So, then this implies the T equals e power minus x times c or the e equals C e power minus ax to be (())(06:30). So, now what does this mean? That means, that T of phi equals c times integral e power minus x times phi of x dx over R are all phi in D of R, this what we mean. So, these are all this, there is only one solution. That is in fact, the generated, the distribution generated by the classical solution of this differential equation.

(3) Define $\forall\phi \in D(\mathbb{R})$,

$$T(\phi) = \int_{\mathbb{R}} |sin(x)|\phi'(x)dx.$$

Then, show that $T \in D'(\mathbb{R})$ and compute T''.

solution. for every $K \subset \mathbb{R}$ compact, we have $|T(\phi)| \leq ||\phi||_1$.

$$\Rightarrow T \in D'(\mathbb{R}).$$

So, this implies that T is a distribution, from the other characterization which we saw. So, it is a distribution whose order is 1. So, now we want to compute T double prime. So,

$$T''(\phi) = T(\phi') = \int_{\mathbb{R}} |sin(x)|\phi''(x)dx.$$

I have just used the definition.

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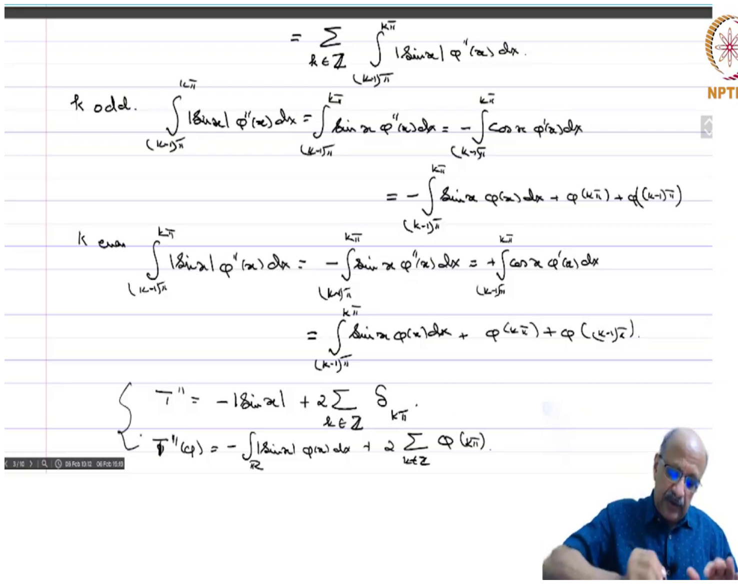

$$T''(\phi) = \sum_{k\in\mathbb{Z}} \int_{(k+1)\pi}^{k\pi} |sin(x)|\phi''(x)dx.$$

k odd: $\int_{(k+1)\pi}^{k\pi} |sin(x)|\phi''(x)dx = \int_{(k+1)\pi}^{k\pi} sin(x)\phi''(x)dx = -\int_{(k+1)\pi}^{k\pi} cos(x)\phi'(x)dx$

$$= -\int_{(k+1)\pi}^{k\pi} sin(x)\phi(x)dx + \phi(k\pi) + \phi((k+1)\pi).$$

k even: $\int_{(k+1)\pi}^{k\pi} |sin(x)|\phi''(x)dx = -\int_{(k+1)\pi}^{k\pi} sin(x)\phi''(x)dx = \int_{(k+1)\pi}^{k\pi} cos(x)\phi'(x)dx$

$$= \int_{(k+1)\pi}^{k\pi} sin(x)\phi(x)dx + \phi(k\pi) + \phi((k+1)\pi).$$

So, you have

$$T'' = -|sin(x)| + 2\sum_{k\in\mathbb{Z}} \delta_{k\pi}.$$

$$T''(\phi) = -\int_{\mathbb{R}} |sin(x)|\phi(x)dx + 2\sum_{k\in\mathbb{Z}} \phi(k\pi).$$

And each time you will, everything will be repeated twice. If you write, because it is summation over k over all integers. So, every integer, every one of these terms will come once in the pre, odd interval, once in the even interval neighborhood, neighboring interval.

And therefore, you will get two times sigma sorry, k in z of. So, yeah the evaluating phi at k pi and there is nothing but the distribution delta k pi. So, T double dash is this. What does this mean? This means, that T double dash. So, an alternative way of writing this is minus integral mod sine x phi x dx over R and then plus twice sigma k in z phi of k pi. This is how explicitly the definition is, this is the same. These two statements are one and the same.

So, we can express a distribution, whenever we express it we either give a formula like this or we just give the in terms of phi, but there should be no derivatives coming in here. So, only it should be an action on phi which is what we expect in the answer.

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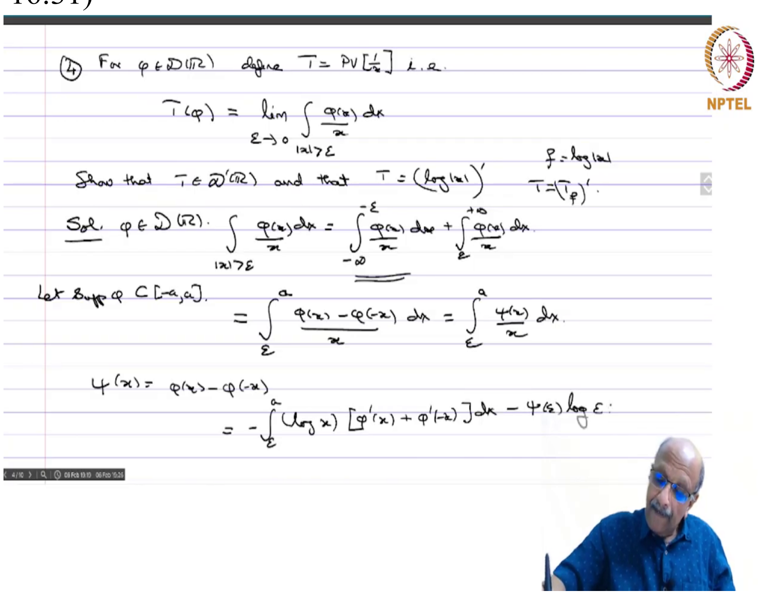

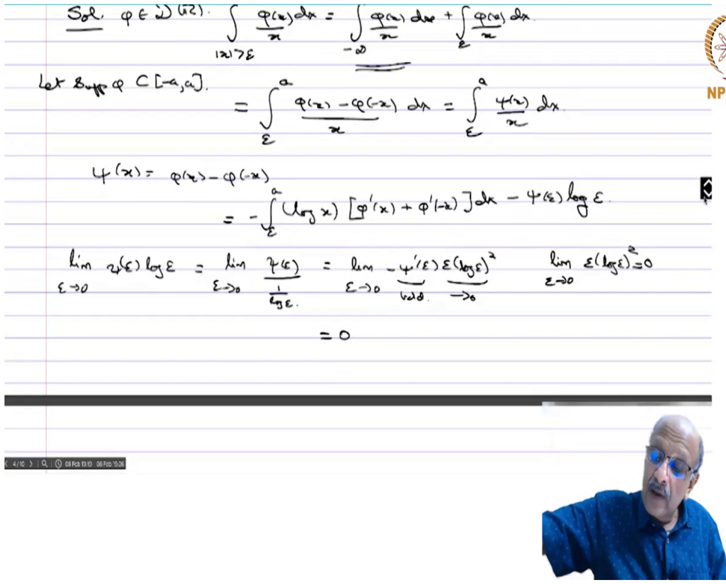

So, this is an important exercise. So, we have seen that locally integrable functions can be made into distributions. Now, even functions which are not locally integrable by suitable limit processes sometimes we can convert them into a distribution. So, one such example is 1 by x, 1 by x is not integrable in any in a neighborhood of the origin. And therefore, it is not really a locally integrable function on R.

So, what we are trying to do is for, phi in D of R, define T of equals PV of 1 by x. So, this is called the principal value of 1 by x, this is a technical notation. So, that where that is T of phi equals limit epsilon tending to 0 of integral mod x greater than epsilon of phi, phi x by x dx. So, what we are doing is the only singularity of 1 by x is at the origin. So, we are excluding, puncturing out, cutting out a neighborhood of the origin then integrating this function.

As though it is 1 by x, and then try to take the limit as epsilon goes to 0. So, and we have to show that this limit exists. So, the T phi is well defined and gives you a distribution in T, and then we will. So, the rest of the problem is show that T belongs to the prime of R and that T is equal to log mod x prime. That means, log mod x is a local integrable function. So, if you take f equals log mod x, then we say that T is equal to Tf dash.

That is the meaning of the statement here, T is equal to. So, log mod x is an locally integrable function which you can check yourself. So, solution so, let phi belong to D of R. And you look at integral mod x greater than epsilon phi x by x dx is equal to integral minus infinity to minus epsilon phi x by x dx plus integral epsilon plus infinity phi x by x dx. Now, phi has compact support. So, let us say support of phi is contained in minus a.

So, this if you make a change of variable x going to minus x in this, in this integral. So, if so, this will become equal to the integral epsilon to a of phi x minus phi of minus x by x dx. So, you can verify that there was this elementary change of variable. So, let us call psi of x, it was phi x minus phi of minus x. So, this is equal to the integral epsilon to a psi x by x dx. So, now, this we can write.

So, we are going to integrate this in part. So, this will be minus integral epsilon to a of log x. So, the derivative of log x is 1 by x away from the origin. So, we have no problems whatsoever. So, this is log x and it is a nice smooth function, differentiable n number of times. And therefore, the distribution derivative is the same as classic. Anyway, we are just doing integration by parts. So, and then we have here, a psi dash of x.

So, that means that is the phi dash of x plus the phi dash of minus x because there is a minus here, there will be one more when you pick up by the function of function rule, dx. And then you have to take the limit, but at the limit a phi and (())(21:41) a and minus a phi vanishes and therefore, you will have minus psi epsilon, log epsilon. So, let us see what happens to psi epsilon, log epsilon when you are taking limit epsilon going to 0 of psi epsilon log epsilon.

So, we can use the function rule. I mean, l'hopital's rule. So, you get, you can write this as limit epsilon going to 0 of psi epsilon by 1 over log epsilon. So, infinity where 0 by 0 form and so, this will give you a limit epsilon going to 0 of psi dash epsilon minus psi dash epsilon into epsilon log epsilon squared. Now, this is a bounded function and this goes to 0. So, again limit, you have to check this again you can use l'hopital's rules once, once more.

So, an epsilon log epsilon square is equal to 0. Because log epsilon is something which goes much slower than epsilon. So, even though it goes to infinity, minus infinity as epsilon goes to 0 it is much slower than any polynomial. And therefore, you have this. So, you can again check this using. So, this limit is 0. So, you can check that again using l'hopital's rule.

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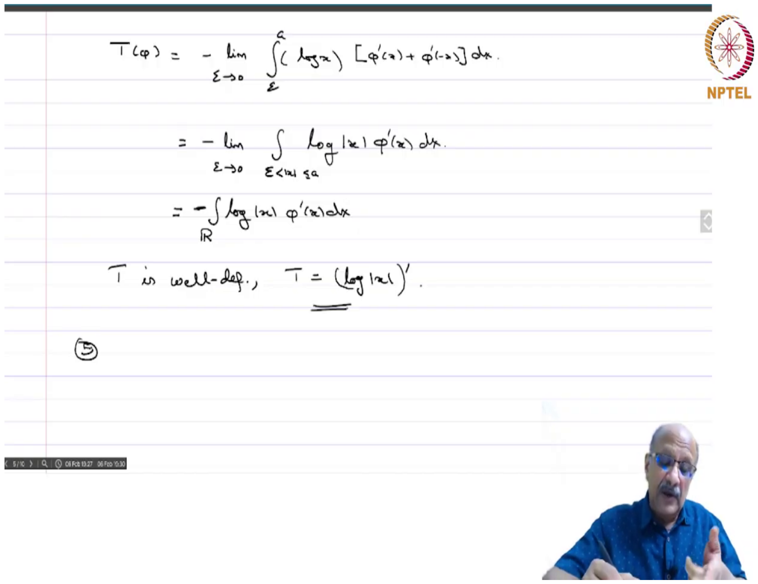

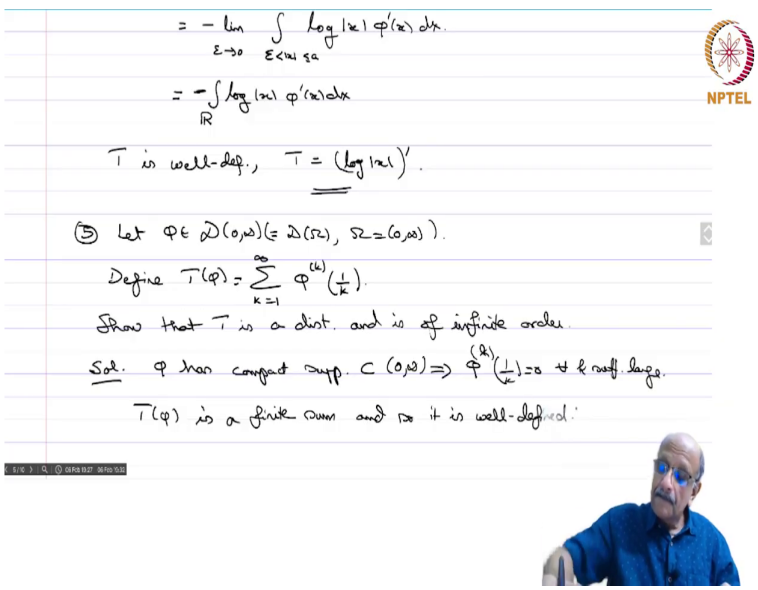

So, we have T of phi equals minus limit epsilon tending to 0 of epsilon to a log x into phi dash x plus phi dash minus x dx. So, this is what we have. So, again you make a change of variable for this integral y equals minus x and therefore, you will get this is equal to minus limit epsilon tending to 0 of integral epsilon less than mod x less than or equal to a. So, this will be two integrals, the second integral will go from minus a to minus epsilon.

And this first integral will be epsilon a. So, I am combining the two and writing it like this. And then, if you look at, when you change the integral, this (())(24:53) log of minus x in the negative side. So, you will get log mod x phi dash x dx. And now, log is a locally integral function, phi is the C infinity function with compact support. So, by the absolute continuity of the lebesgue integral. This is precisely equal to integral over R, of log mod x phi dash x dx.

I, that is, I have a remote a I have sent to infinity because phi whether it is over a or over R it does not matter. So, with a minus sign of course, this is the minus log x. And then, now this is a locally integral function and therefore, this is well defined. So, therefore, T is well defined. And in fact, T is equal to log mod x prime. Because what is the derivative of the distribution given by log mod x is minus log mod x acting on phi dash.

So, that is what we have.

(5) Let $\phi \in D((0, \infty))$. Define

$$T(\phi) = \sum_{k=1}^{\infty} \phi^{(k)}(\frac{1}{k}).$$

Show that T is a distribution and it is of infinite order.

So, here is the distribution of infinite orders. So, this is an example of a distribution with infinite order. So, T of phi is. So, phi has compact support contained in 0 infinity. So, this implies what?

Phi k of 1 by k will be 0 for all k sufficiently large. If k is sufficiently large, 1 by k will be outside the support of phi. And so, all the functions and all its derivatives will vanish. So, this will be (())(28:13). So, T phi is a finite sum. And so, it is well defined.

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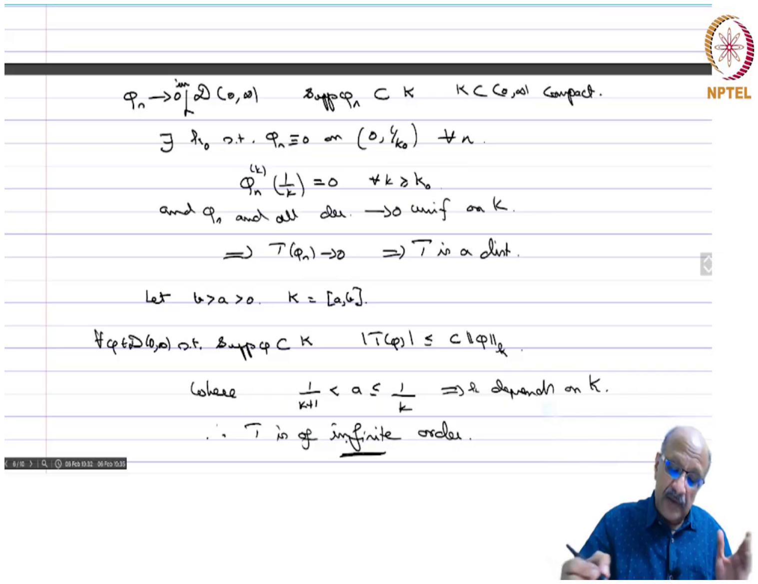

Once a $\phi_n \to 0$ in $D((0,\infty))$, $\text{supp}(\phi_n) \subset K$, $K \subset (0,\infty)$ is compact.

$$\exists\, k_0 \; s.t. \; \phi_n = 0 \; on \; (0, \frac{1}{k_0}), \; \forall\, n.$$

$$\phi_n^{(k)}(\frac{1}{k}) = 0, \; \forall\, k \geq k_0.$$

and ϕ_n and all derivatives go to 0 uniformly on K .

$$\Rightarrow \; T(\phi_n) \to 0 \Rightarrow T \text{ is a distribution.}$$

So, now let $b > a > 0$. $K = [a, b]$.

$$\forall \phi \in D((0,\infty)) \; s.t. \; supp(\phi) \subset K. \qquad |T(\phi)| \leq C||\phi||_k.$$

$$\frac{1}{k+1} \leq a \leq \frac{1}{k}$$

So, it depends on a. So, how many ks are outside this interval will entirely depend on how close a is to 0 and therefore, it depends on k. So, this implies k depends on capital K.

Therefore, you have T is of infinite order. So, we will stop here and we will do some more exercises later on, after we have covered some more theory.