Sobolev Spaces and Partial Differential Equations

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Lecture – 1

Test Functions – Part 1

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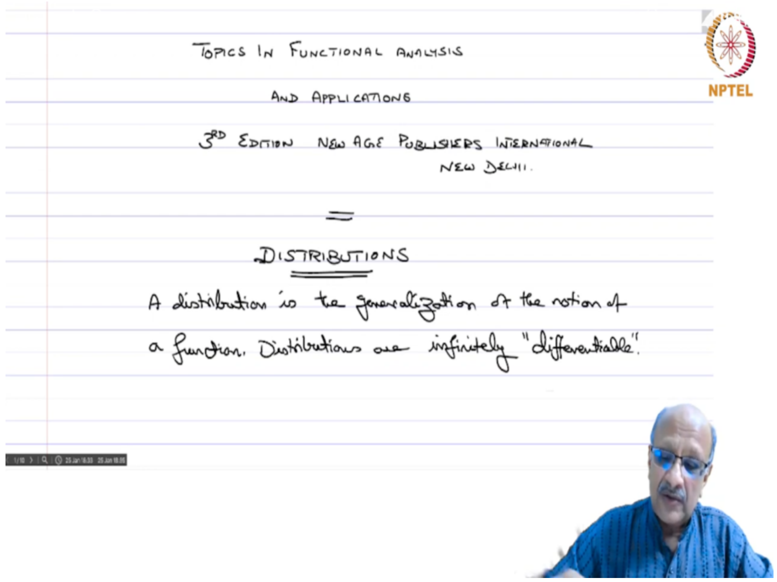

Welcome to this course on Sobolev Spaces and Partial Differential Equations. So, the book which I will be following almost faithfully is one by myself. It is called Topics in Functional Analysis and Applications. It is now in the third edition and published by New Age Publishers International; they are based in Delhi.

So, what is this course about? So, we will be studying functional analytic methods for the study of partial differential equations. This is something which revolutionized the study of PDEs; and especially suitable for numerical computations with the advent of high-speed computers. So, many modern numerical methods depend highly on functional analytic approach. And therefore, we will be studying all these things in this course.

So, to start with we the starting point of all this is the theory of distributions. And what is a distribution? So, so, what is a distribution? A distribution is the generalization of the notion of a function; and distributions are infinitely differentiable. Functions as you know will not be infinitely differentiable, even differentiable continuous functions which are nowhere differentiable; whereas, distributions are infinitely differentiable, and they generalize the notion of a function.

So, why do we want to generalize the notion of a function? This is not just for the fun of generalization; but it has some specific purpose. So, when we want to solve the partial differential equation, generally, by what is a classical solution we mean. The solution which is continuously differentiable at least as many times as the order of the equation. And that this function satisfies the differential equation at every point in the domain of consideration.

Now, if you stick to this view, many interesting problems will cease to have solutions. And we will not be able to study very many physically relevant things, which turn up all the time in Physics and engineering. Many such differential equations we cannot study. So, let me give you just one simple example.

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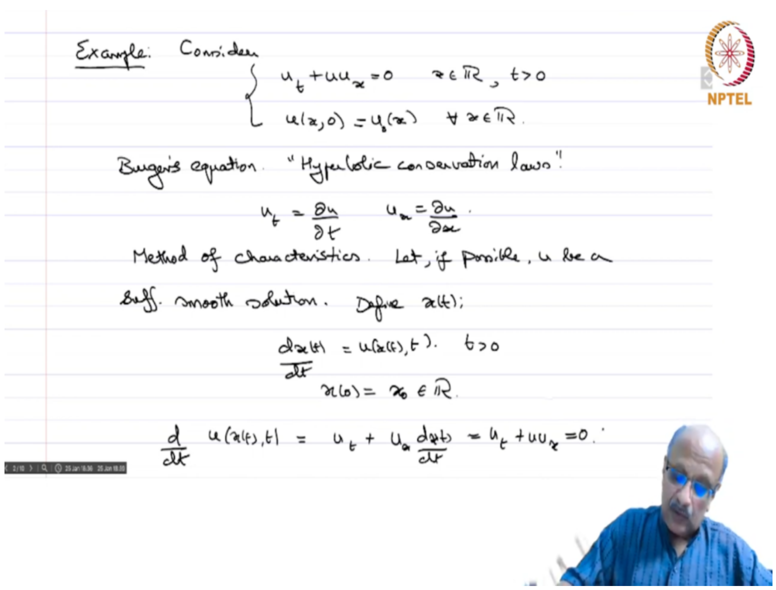

Consider the problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, \ t > 0,$$

$$u(x, 0) = u_0(x), \ \forall \, x \in \mathbb{R}.$$

So, this is sometimes called Burger's equation, and in some literature, this is the limiting case of Burger's equation; there will be a small term which goes to 0. And it is a typical prototype of what are called hyperbolic conservation laws. So, this is an example of a hyperbolic conservation law.

So, here of course u_t means the partial derivative. So, $u_t = \frac{du}{dt}$ and $u_x = \frac{du}{dx}$. So, this is notation which we use sometimes for shortcut. So, we will try to solve this equation by what is called the method of characteristics. What is that? Let, if possible, u be a sufficiently smooth solution.

So, now you define $x(t)$ in the following way:

$$\frac{dx(t)}{dt} = u(x(t), t),\ t > 0,$$

$$x(0) = x_0, \text{ where } x_0 \in \mathbb{R}.$$

So, if u is sufficiently smooth, then the theory of ODE says that this problem has a solution. So, let us now try to differentiate the $u(x(t), t)$ with respect to t. So, this will become:

$$\frac{d}{dt}u(x(t), t) = u_t + u_x \frac{dx(t)}{dt} = u_t + uu_x = 0.$$

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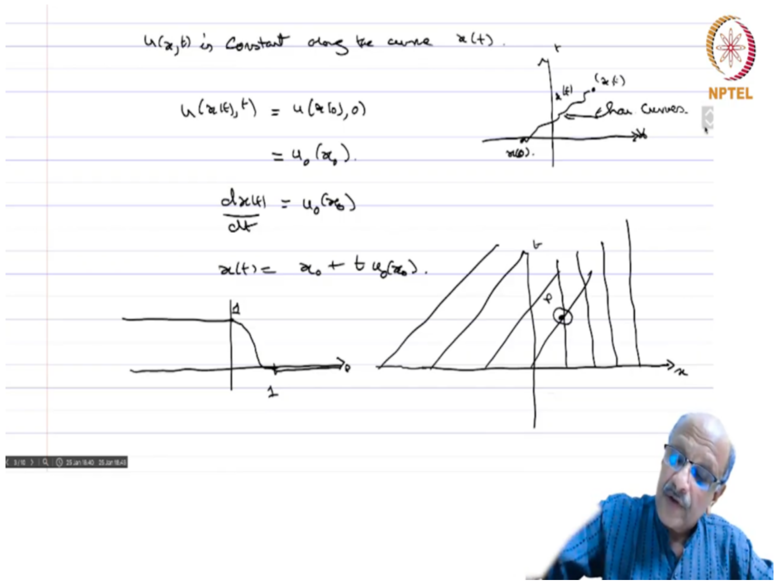

So, this means that the function $u(x,t)$ is constant along the curve $x(t)$. So, if you have for instance, x and t, and then you have a point (x,t) here, so if you want you find this curve $x(t)$; so, it will start at $x(0)$, which is in $\mathbb{R}$. Therefore, it will go like this, and if you want to find the solution $u(x,t)$ here; all you have to do is find the line passing through this point (x,t); and slide down the line, and what is the value here:

$$u(x(t),t) = u(x(0),0)$$

$$= u_0(x_0).$$

So, you have solved the equation theoretically; by method the, these curves are called the characteristic curves. So, all you have to do is to find the characteristic curves and given any point, you find the characteristic curve passing through it. And then slide down till you meet the initial x-axis, and then that value of the initial function will give you the solution here.

So, this seems to be a good situation; so, you are able to solve the problem very easily. But, now let us look at this, so, if this is a constant along the curve; then this means that

$$\frac{dx(t)}{dt} = u_0(x_0)$$

$$\Rightarrow x(t) = x_0 + tu_0(x_0).$$

So, these characteristic curves are straight lines; so, then this makes it even more pleasant we hope. And now let us see, let us for instance take x here; and let us take a function u_0 which is 1 up to this point and after 1, it is the function 0, and in between it is something smooth; which we can make it like this. And then let us try to draw the characteristic curves; now, if you draw the characteristic curves.

So, if you have the negative side, you have $\frac{dx}{dt} = 1$. This means you have all straight lines; which are therefore parallel to this. On the positive real axis, you have $\frac{dx}{dt} = 0$; remember the x is here and t is here. Therefore, the characteristic curves will be perpendicular to the axis.

So, now I told you that to find the solution at any point, all you have to do is to find the characteristic passing through the point, and then slide down. But, now let us take a point P, which is here; now you have two characteristic curves and so how do you, which down which should you slide.

So, therefore in a very short time once the characteristics start intersecting; you find that the function is not well defined. In other words, the solution is not even continuous, it is multivalued. And therefore, it contradicts the fact that we started with a smooth solution. And therefore, if you stick to the notion of a characteristic of a classical solution, then you will not be able to study equations like this and many others.

And for instance, this Burger's equation or hyperbolic conservation laws, a study of shock waves; which is very important in aeronautics, and therefore one should be able to capture the discontinuities of a solution. But, then for that we must first know how a discontinuous solution can be a solution of a partial differential equation?

So, what we do is generalize the notion of a solution of a function to what are called distributions. These distributions will be infinitely differentiable and therefore we can look at solutions to differential equations in the class of distributions. And thereby we will be able to study discontinuous solutions also.

And these we have to interpret, because what do we mean by a discontinuous function being solution of a differential equation?. So, we have to interpret them in a suitable way; and for all this we have the theory of distributions which is very useful. And for this, Lauren Schwartz got the fields medal. So, how do we generalize the notion of a function?

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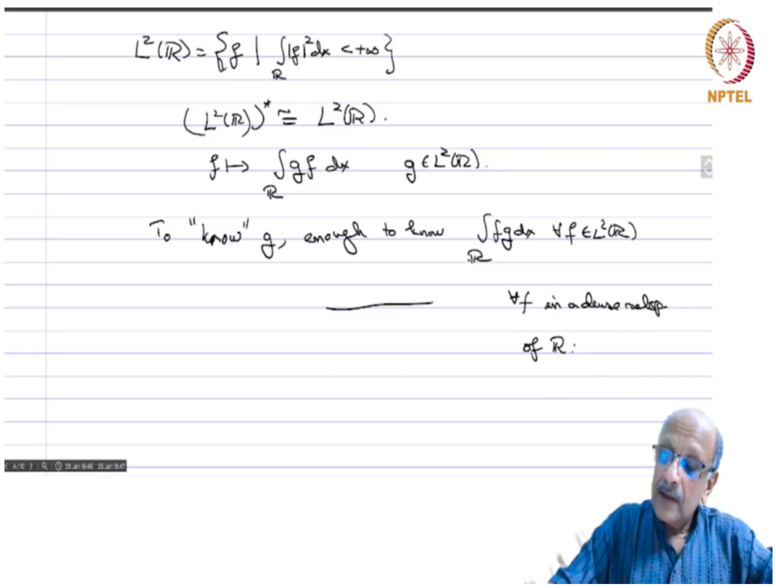

So, let us take the example of, say $L^2(\mathbb{R})$:

$$L^2(\mathbb{R}) = \left\{ f\colon \int_{\mathbb{R}} |f|^2 < \infty \right\}.$$

And you know that this is a Hilbert space and therefore, so $\left(L^2(\mathbb{R})\right)^{\star}$, its dual the space of continuous linear functional, is identified with $L^2(R)$ itself, i.e.,

$$\left(L^2(\mathbb{R})\right)^{\star} \simeq L^2(\mathbb{R}).$$

And so, any continuous linear functional is given by

$$f \mapsto \int_{\mathbb{R}} gf\,dx, \quad \text{where } \; g \in L^2(\mathbb{R}).$$

So every function in $L^2(\mathbb{R})$ produces a unique continuous linear functional; and therefore, you can think of functions in $L^2(\mathbb{R})$ as continuous linear functionals on $L^2(\mathbb{R})$.

So, this is a new way of looking at the notion of a function. So, to know a function in L^2, it is enough to know its inner product with namely all these integrals for every $f \in L^2$. So, to know g, it is enough to know integral $\int_{\mathbb{R}} gf\,dx$, for every $f \in L^2(\mathbb{R})$. In fact, it is enough to know for every f in a dense subspace of $\mathbb{R}$.

So, it is enough to know that once you know, then the function is uniquely fixed. And this not just a function is only defined almost everywhere as you know; and therefore anyway, so it is just you have to know this, so, we will adopt this approach. The way we will generalize functions is to look at them as continuous linear functionals on a suitable space. This is the way we want to generalize the notion of a function.

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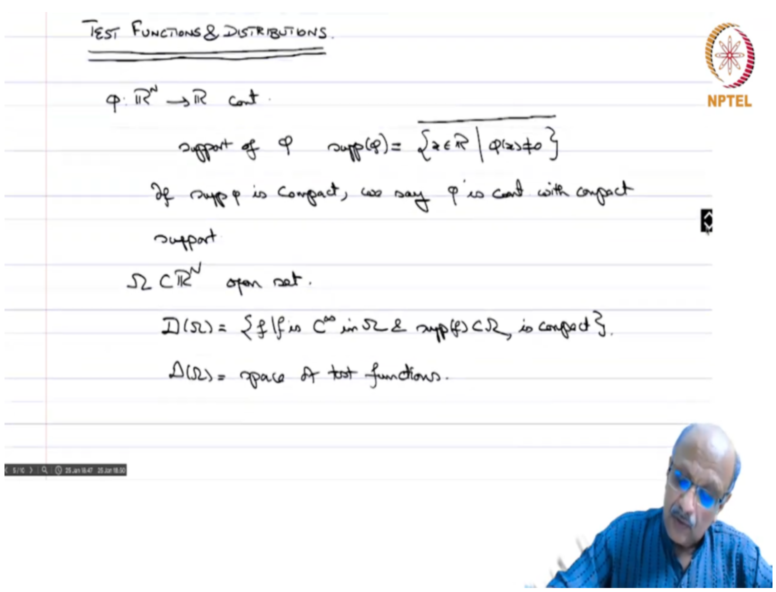

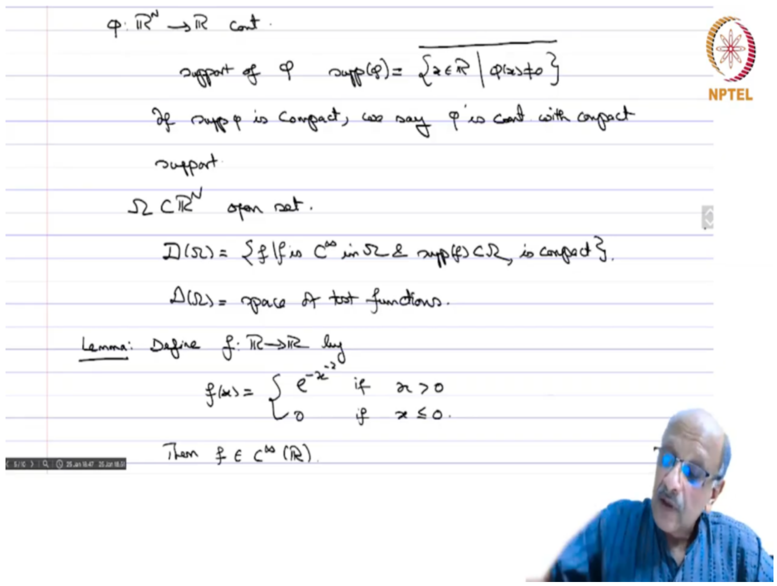

So, with this preamble, let me now proceed to the next sections, which is the test functions and distributions. So, let $\phi: \mathbb{R}^N \mapsto \mathbb{R}$ and it is continuous. Then what is the support of ϕ? So, I recall, support of ϕ, denoted as $\mathrm{supp}(\phi)$, defined as:

$$\mathrm{supp}(\phi) = \overline{\{x \in \mathbb{R}^N : \phi(x) \neq 0\}}.$$

So, this is always a closed set.

So, if $\mathrm{supp}(\phi)$, which is always a closed set, is in addition compact, we say ϕ is continuous with compact support. So, we now use this idea, so let $\Omega \subset \mathbb{R}^n$ be an open set. And then we denote by

$$D(\Omega) = \left\{f:\ f \text{ is } C^\infty \text{ in } \Omega \text{ and } supp(\phi) \subset \Omega \text{ is compact}\right\}.$$

This space is called the space of test functions. So, $D(\Omega) =$ space of test functions. The base field can be real or complex; most of the time, I will talk only of real valued functions.

But all that I say will be applicable to complex values; and if there is any change, I will especially mention it. So, we will assume without mentioning it that we are talking of real valued functions, and all vector spaces which we talk about, is the space of real valued functions. So, are there such functions which are continuous with compact support?. So, we want to give examples of functions which are continuous with compact support.

So, with that we start with the Lemma.

Lemma. Define $f: \mathbb{R} \longmapsto \mathbb{R}$ by

$$f(x) = e^{-x^{-2}}, \text{ if } x > 0,$$

$$= 0, \quad \text{ if } x \leq 0.$$

Then $f \in C^{\infty}(\mathbb{R})$.

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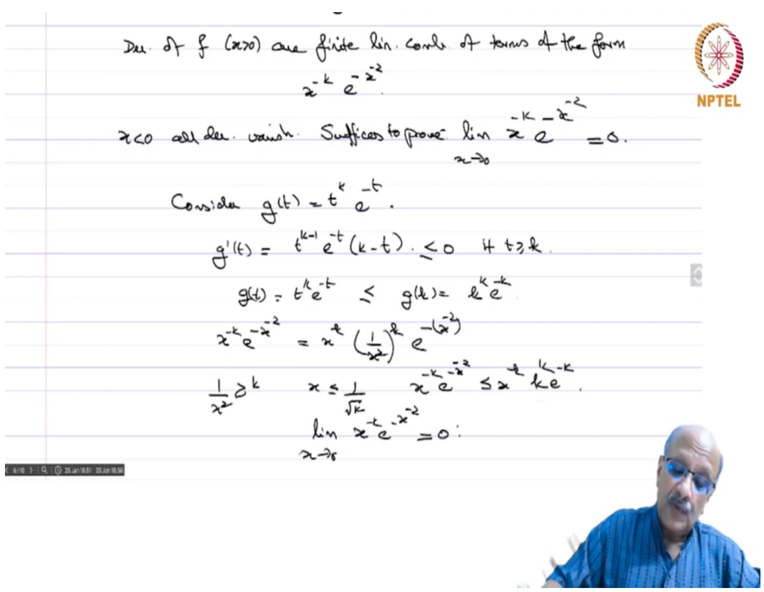

Proof. If $x \leq 0$, then $f(x) = 0$ and therefore, it is a C^{∞} function. If $x > 0$, then $f(x) = e^{-x^{-2}}$ therefore this is also a C^{∞} function. And therefore, you have nothing to be worried. So, we have to check continuity and differentiability at $x = 0$. So, this is the only point where we have to look at the function. So, now what are the derivatives? So, derivatives of $f(x > 0)$ are finite linear combinations of terms of the form

$$x^{-k} e^{-x^{-2}}.$$

So, if you just take this and apply the product rule and keep differentiating, you will get repeatedly terms of this kind, and all the derivatives up to any order will look like this. So, we will end on the left: if $x \le 0$, all derivatives vanish.

Therefore, it suffices to prove

$$\lim_{x \to 0} x^{-k} e^{-x^{-2}} = 0.$$

So this is what we need to prove. So, let us consider $g(t) = t^k e^{-t}$. Then

$$g'(t) = t^{k-1} e^{-t}(k - t) \le 0, \text{ if } k \le t.$$

$$\Rightarrow g(t) = t^k e^{-t} \le g(k) = k^k e^{-k}.$$

Now $x^{-k} e^{-x^{-2}} = x^k \left(\frac{1}{x^2}\right)^k e^{-x^{-2}}$. So, if you have $\frac{1}{x^2} \ge k$, i.e., $x \le \frac{1}{\sqrt{k}}$, then we have that

$$x^{-k} e^{-x^{-2}} \le x^k k^k e^{-k}.$$

Therefore, it follows that

$$\lim_{x \to 0} x^{-k} e^{-x^{-2}} = 0.$$

and this proves that all the derivatives vanish at the origin; and therefore, this is in fact C^{∞}-function.

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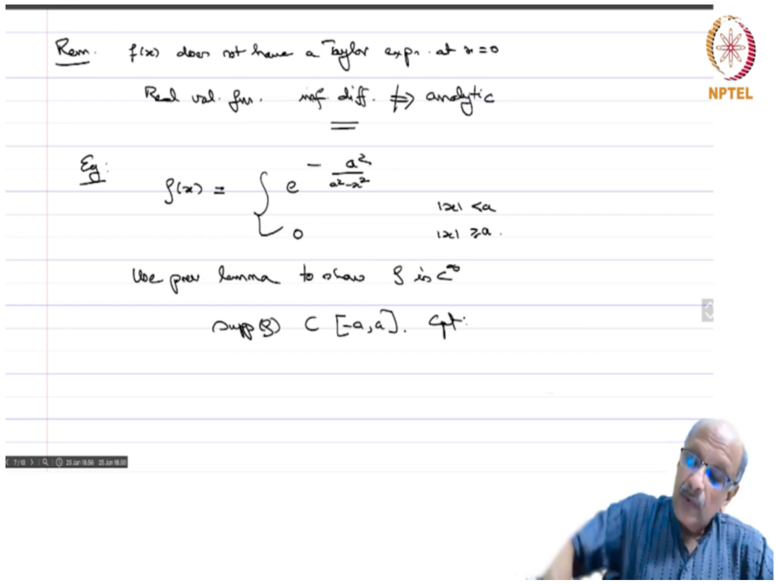

Now, this is also a very good example.

Remark. If you look at complex analysis, if you think a function is differentiable in the neighborhood, then it is infinitely differentiable automatically and it admits a power series expansion. And you know that the terms of the power series are nothing but the derivative successive derivatives of the function at that point.

Now, this function $f(x)$ does not have a Taylor expansion at $x = 0$; because all the derivatives at the origin are 0. So, if you wrote an infinite series, then you will get only the 0 series, but the function is not 0. Therefore, this function does not have a Taylor expansion. So, in real valued functions infinitely differentiable does not imply analytic, that means, it has a Taylor expansion in a neighborhood.

So, we have infinitely differentiable functions and real analytic functions are two different classes in the real valued functions, unlike the complex case. So, this is an example to show that these two classes are in fact different. So, now we will use this example to construct C^{∞} functions with compact support.

So, let us take

$$\rho(x) = e^{-\frac{a^2}{a^2-x^2}}, \text{ if } |x| < a,$$

$$= 0, \qquad \text{if } |x| \geq a.$$

Then, one can use the previous lemma to show that ρis C^{∞}; because the only thing you have to do is check at $|x| = a$,i.e., $x = a$ and $x = -a$.

And this is precisely the notion the kind of function which we have been looking at, and therefore you can show by the same analysis that it is a C^{∞} function. Now

$$\text{supp}(\rho) \subset [-a, a]$$

and therefore $\text{supp}(\rho)$ is compact. So, this is a C^{∞} function with compact support. So, now we will slightly generalize this:

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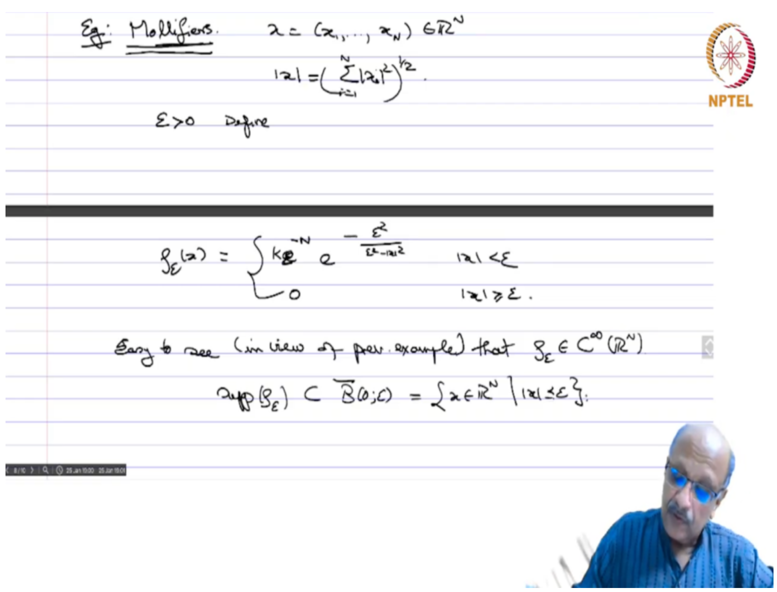

Mollifiers. We take $x = (x_1, x_2,..., x_N) \in \mathbb{R}^N$ and throughout this course I will use

$$|x| = \sqrt{\left(\sum_{i=1}^{N} |x_i|^2\right)}.$$

Let $\epsilon > 0$. Whenever I use symbol ϵ, I automatically think of it as a very small quantity very close to 0. And so, we define

$$\rho_{\epsilon}(x) = \kappa\epsilon^{-N} e^{-\frac{\epsilon^2}{\epsilon^2-|x|^2}}, \text{ if } |x| < \epsilon,$$

$$= 0, \qquad \text{if } |x| \geq \epsilon,$$

where $\kappa = \int\limits_{|x|\leq 1} e^{-\frac{1}{1-|x|^2}} dx$, dx is the Lebesgue measure in $\mathbb{R}^N$. i.e., $dx = dx_1 dx_2 ... dx_N$. Then it is easy to see in view of the previous example that $\rho_{\epsilon} \in C^{\infty}(\mathbb{R}^N)$, and supp $(\rho_{\epsilon}) \subset \overline{B(0, \epsilon)} = \{x \in \mathbb{R}^N : |x| \leq \epsilon\}$.So, this is again a C^{∞}-function with compact support. Now, it has some other properties. You also have that $\rho_{\epsilon} \geq 0$.

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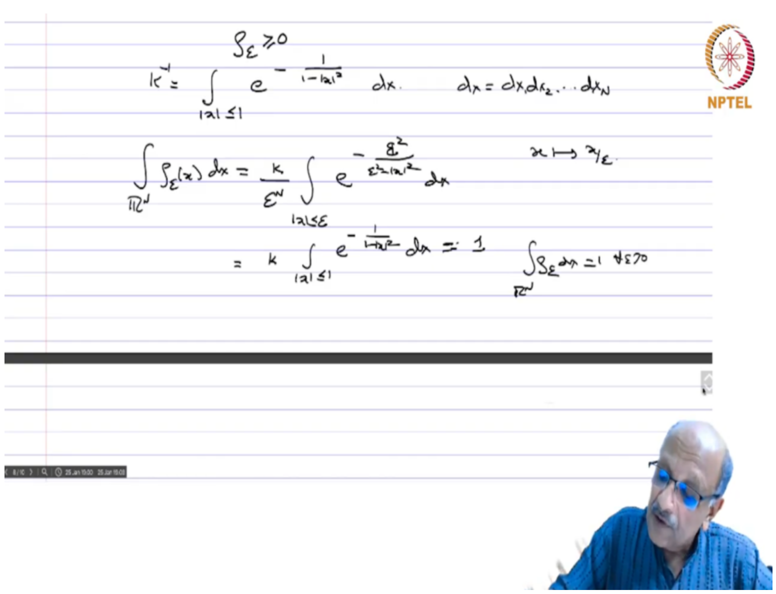

Now

$$\int\limits_{\mathbb{R}^N} \rho_{\epsilon}(x) dx = \frac{\kappa}{\epsilon^N} \int\limits_{|x|\leq\epsilon} e^{-\frac{\epsilon^2}{\epsilon^2-|x|^2}} dx$$

$$= \kappa \int_{|x|\leq 1} e^{-\frac{1}{1-|x|^2}} dx; \text{ by change of variable } x \to \frac{x}{\epsilon}.$$

$$=1, \quad [\text{by definition of } \kappa].$$

Therefore $\int_{\mathbb{R}^N} \rho_{\epsilon}(x)dx = 1$, for all $\epsilon > 0$.

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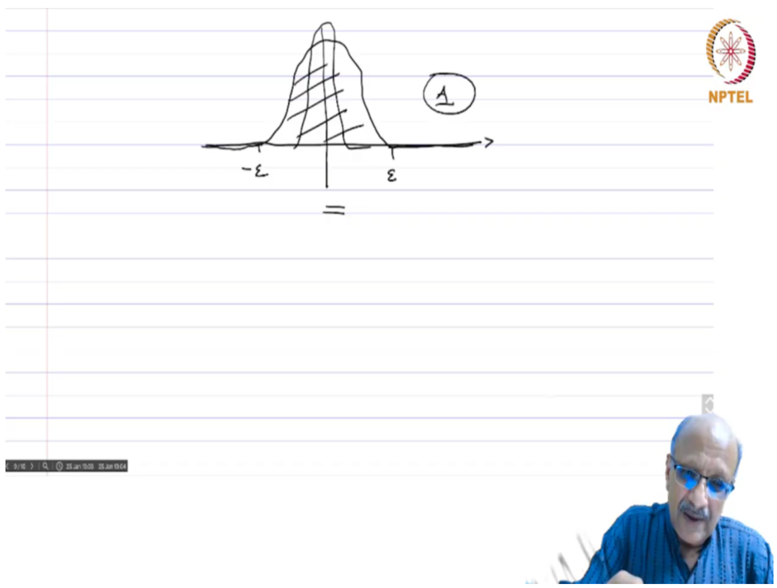

So, what are these functions looking like? You have probably seen them in some other situations; so, you have x here. So, you have $-\ \epsilon$and ϵ, and then this function is 0 outside this interval in between it; it is in fact this bell-shaped curve; so, the people who have studied probability, will be very well known and this area underneath the curve area equal to 1.

So, as ϵ becomes smaller, this function will become steeper and steeper. If you have another smaller epsilon, it will have to be something steeper like this. The area will always be equal to 1 underneath the curve and the function will have smaller and smaller supports. And these will be our friends for a long time; so, we will repeatedly come across these things. So, we have seen some examples of C^{∞}-functions with compact support. So, now we will see in fact that the space

$D(\Omega)$ is very rich; you can construct all kinds of functions that are C^{∞} with compact support. So, for that we need the following notion.