Real Analysis II
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Lecture - 4.1
Completeness

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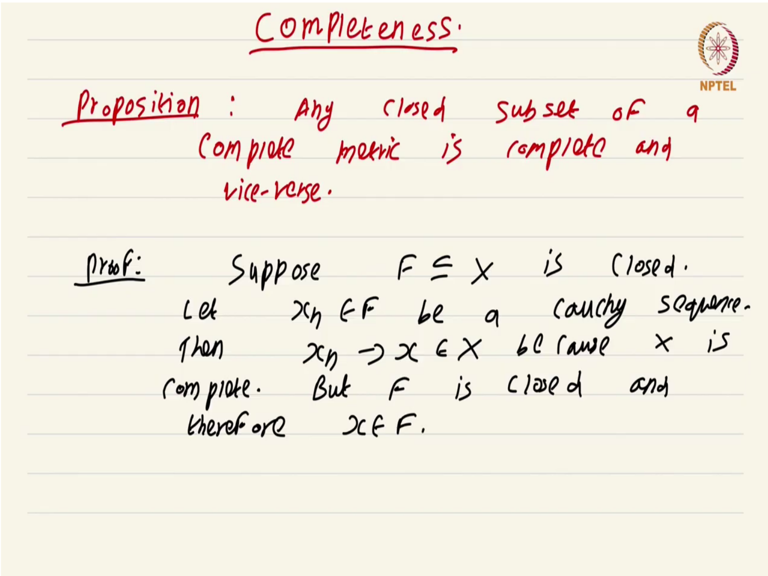

In the next set of videos, we will discuss the all important notion of completeness. Recall that a metric space X is said to be complete if any Cauchy sequence converges to a point in the metric space. The most important result in this section would be Baire's category theorem and also a fact that any metric space can be completed.

In fact, any metric space can be isometrically embedded within a Banach space; that I will leave it to you as an exercise to prove in a long chain of simple problems that will finally end with the result. I will sketch a proof of the fact that any normed vector space can be put inside a Banach space. So, let us begin with some simple consequences of completeness.

So, we begin with the simple proposition the proof is rather easy proposition any close subset of a complete metric space, complete metric space is complete and vice versa that is any subset of a complete metric space which is also complete is going to be a close set and vice versa, or rather I could have stated it like this subspace of a complete metric space is complete if and only if it is closed.

So proof, let us begin with the proof. Suppose, F subset of X is closed. Let x n in F be a Cauchy sequence be a Cauchy sequence, then x n converges to x in X because X is complete because X is complete. But F is closed, F is closed, and therefore x is an element of F ok.

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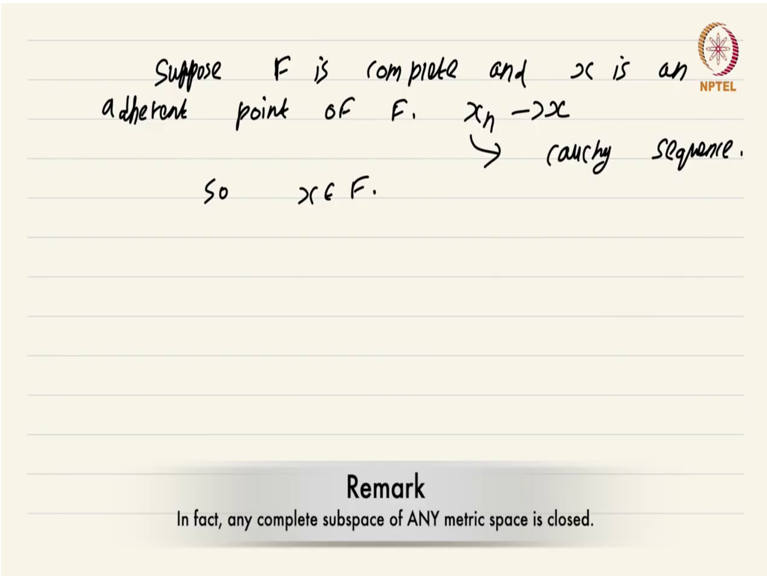

Now, on the other hand, suppose, F is closed sorry that is the case that we dealt with suppose F is complete suppose F is complete and x is an adherent point of F, we have to show that x is in F but that is obvious because we have a sequence x n that converges to x that is the definition of an adherent point, and this x n is a Cauchy sequence, this x n is a Cauchy sequence right.

So, by completeness, the limit of x n should also be in F. So, x is in F. So, a close subset of a complete metric space is complete if and only if sorry subset of a complete metric space is complete if and only if it is closed ok.

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Suppose F is complete and x is an adherent point of F. $x_n \to x$

$\hookrightarrow$ Cauchy sequence.

So $x \in F$.

Cantor's intersection theorem. Let X be a complete metric space and let F_n be nested closed non-empty sets; with $\operatorname{diam}(F_n) \to 0$. Then $F := \bigcap F_n$ is a singleton set.

Now, what we do is we generalize Cantor's intersection theorem, Cantor's intersection theorem to metric spaces. And again completeness will play a role and that will become very clear during the course of the proof. I am going to prove the version that is most useful in applications the version where you have the intersection to be a single point ok. Let X be a complete metric space, complete metric space and let F n be nested closed non-empty sets ok.

So, what you do is you consider nested close sets that are non-empty with diameter of F n converging to 0. So, this is essentially the shrinking case of the Cantor's intersection theorem. Then F which is by definition equal to intersection of F n is exactly I should not write not equal to is exactly is a single point is a singleton set. So, the intersection of closed non-empty shrinking sets in a complete metric space is going to be a singleton set.

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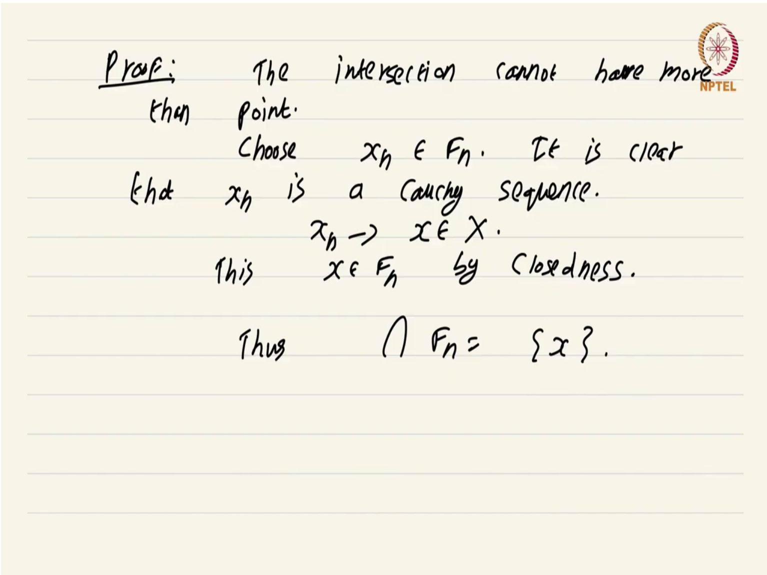

And the proof is again rather easy because we have already seen proofs of several versions of this Cantor's intersection theorem; the metric space situation poses no additional difficulty. Now, the fact that diameter F n converges to 0, clearly shows that the intersection cannot have more than one point that is obvious, the intersection cannot have more than one point have more than one point. Now, all that needs to be shown is the intersection is non-empty ok.

Now, what you do is the following choose x n in F n ok. It is clear that this x n is a Cauchy sequence, it is clear that x n is a Cauchy sequence ok. And because we are in a complete metric space, x n must converge to x in X, but each F n is a closed set and therefore, this x must be an element of F n. So, this x must be an element of F n by closeness.

And of course, this is a rather easy argument that we have seen before you just ignore the first few terms of the sequence, then the rest of the sequence will be contained in F n therefore, the limit x should also be contained in x n F n ok. Thus intersection of F n is nothing but this singleton set x. This proof was rather easy and is modeled on the proofs of Cantor's intersection theorem that we have seen earlier ok.

Now, the next order of business in this line of generalizing facts that we have already seen in the context of real numbers to metric spaces by essentially doing the same argument replacing the absolute value by the metric d is Baire's category theorem ok.

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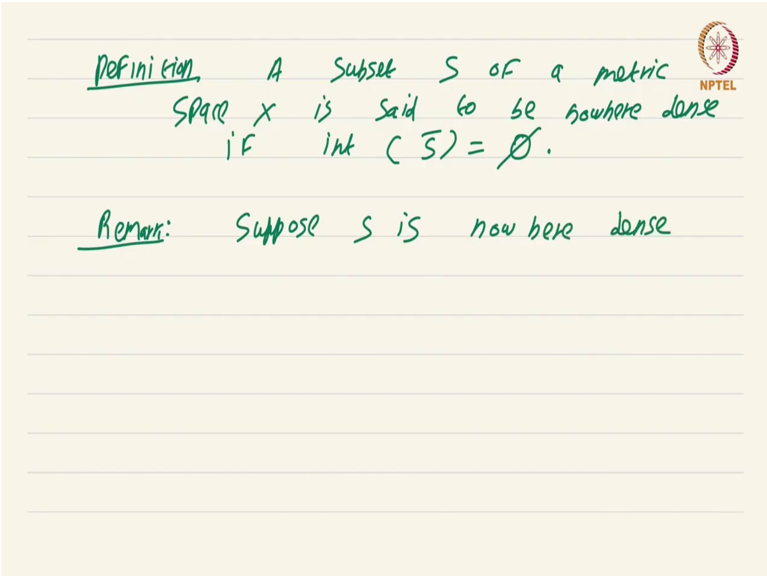

So, for that, we need a definition we need a definition. And the definition is that of a nowhere dense set which we have already studied. A subset S of a metric space of a metric space X is said to be nowhere dense if nowhere dense if interior of S closure is the empty set ok. Now, I am going to leave a very simple exercise for you. I am not even going to bother calling it an exercise because it is rather easy to show.

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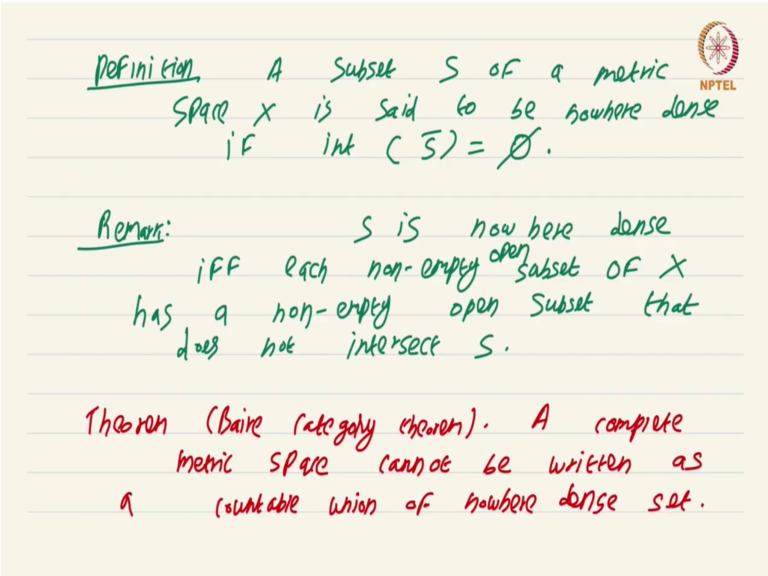

Remark, suppose S is nowhere dense, suppose S is nowhere dense or rather let me phrase it as an if and only if condition S is nowhere dense if and only if; if and only if each non-empty subset of X each non-empty subset of X has a non-empty each non-empty open subset sorry about that each non-empty open subset of X has a non-empty open subset that does not intersect S ok.

Please prove this exercise, remark, please prove the content of this remark as an exercise is rather easy to see set S is nowhere dense if and only if no matter what non-empty open subset of X you take, you can find an even smaller non-empty open set that does not intersect S ok.

Now, we are going to state and prove the Baire category theorem Baire category. What Baire category theorem says is the following. A complete metric space, complete metric space cannot be written cannot be written has as a countable union of nowhere dense sets. So, you cannot write you cannot write a complete metric space as a countable union of nowhere dense sets.

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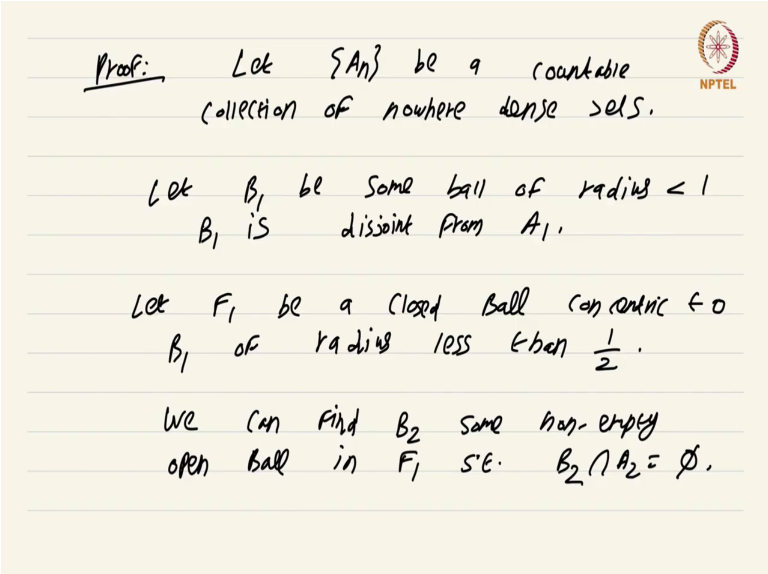

On to the proof the statement might look somewhat sophisticated compared to the statement of Baire category theorem that we saw earlier in the context of the real numbers. But the proof is so short and the proof is so similar to what we see what we have seen in the context of the real numbers that you will have no difficulty in understanding what is going on.

So, let A n be a countable collection be a countable collection of nowhere dense sets nowhere dense sets. Now, the aim is to show that we can find a point x in the metric space which is not there in the union A n. So, what we are going to do is this; let B 1 be some ball of radius 1, any ball it does not really matter just choose some ball of radius less than 1 with the key property with the key property that B 1 is disjoint from A 1.

Why can you find a ball of radius less than 1 that is disjoint from A 1? Well, you have to solve this remark; this content of the remark precisely says this set is nowhere dense if and only if for each non-empty subset, we can find an even smaller open subset that does not intersect. As the moment you solve this exercise this will be clear to you that you can find a ball of radius less than 1 that is disjoint from A 1 ok.

Now, what you do is let F 1 be a closed ball concentric to B 1 to B 1 of radius less than half concentric just means that this F 1 has the same center as B 1 ok. Now, we can find again inductively, we can find B 2 some non-empty open ball in F 1 such that B 2 intersect A 2 is empty. Again this just uses the fact that A 2 is a nowhere dense set and F 1 I mean or at least the interior of F n is an open set, therefore, we can find some possibly smaller ball that does not intersect A 2 ok.

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Choose F_2 to be a concentric closed ball to B_2 of radius less than $\frac{1}{4}$.

Choose $B_3 \subseteq F_2$ some ball s.t.

$$B_3 \cap A_3 = \emptyset.$$

Repeat. We get a sequence of non-empty nested closed sets.
By Cantor's intersection thm.

$$\{x\} = \bigcap F_n \neq \emptyset.$$

This point $x \in B_n$. $x \in \bigcup A_n$.

Now, the argument is should be rather clear choose F 2 to be a concentric closed ball to B 2 of radius less than 1 by 4, less than 1 by 4. Now, choose B 3 choose B 3 to be contained in F 2 some ball such that B 3 intersect A 3 is empty rinse and repeat. Repeat this argument we get a sequence we get a sequence of non-empty nested closed sets nested closed sets ok.

By Cantor's intersection theorem, by Cantor's intersection theorem this intersection of all these F ns is going to be non-empty intersection of all the F ns is going to be non-empty. Of course, we have used the fact that diameter of F n is going to converge to 0 that is clear by construction ok. Now, this point x which is exactly the intersection is going to be in each B n that means x cannot be in the union x cannot be in the union of A n's it cannot be in any A n in fact ok.

So, this proves Baire's category theorem in the context of matrix spaces. Now, I am not going to give any applications of Baire's category theorem in this course, but it is extensively used in many places including functional analysis, and you will no doubt come across such applications in your future studies.

This is a course on Real Analysis, and you have just watched the video on Completeness.