Real Analysis II
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Lecture – 3.2
Equivalent Metrics and Product Spaces

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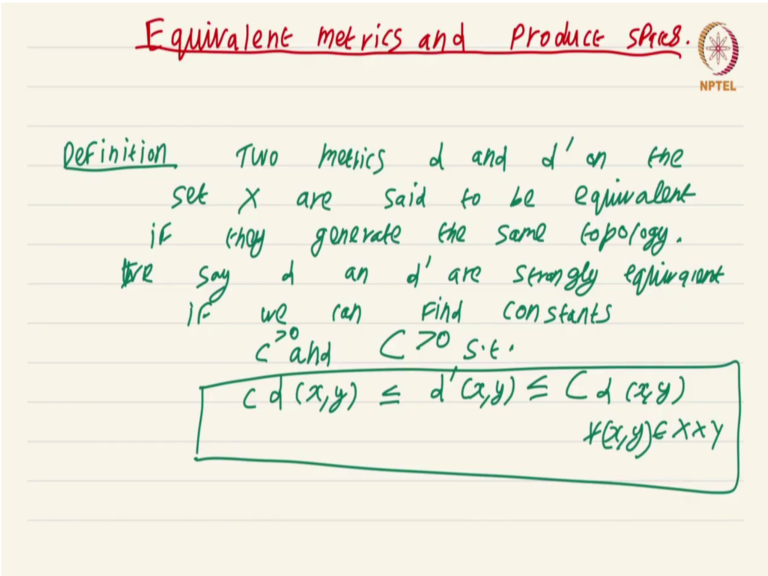

Our objective in this video is to study product spaces. Given two metric spaces x and y, there are several natural ways to make x cross y the Cartesian product into a metric space. The issue is that since there are several ways to do this, which one is the correct one. To address this we will first begin by studying the notion of equivalent metric spaces.

We note that this is different from two metric spaces being isometric or something like that. This is a given metric space or rather a given set with two different metrics. And, we want to study when these two metrics can be considered the same. So, we begin with a simple definition. This is the notion of equivalence of two metric spaces from the perspective of topology. Two metrics d and d prime on the set X; on the set X are said to be equivalent, are said to be equivalent, if they generate the same topology.

So, the open sets in both metric spaces are exactly the same. Since, convergence of sequences as well as continuity can be characterize entirely in terms of open sets, this is a natural definition of two metrics being equivalent on a given set. If, rather we say d and d prime are strongly equivalent, are strongly equivalent.

If, we can find constants, if we can find constants small c and capital C, such that we have the following inequalities c d x, y is less than or equal to d prime x, y is less than or equal to capital C d x, y and this should be true for all x, y in x cross y. So, let me just put it as a pair ok.

So, of course, c is greater than 0 and capital C is also greater than 0. So, the notion of strong equivalence actually subsumes the case of equivalent metric spaces. That means, if this condition is satisfied if this is true, then I ask you to show as an exercise that both d and d prime will generate the same topology. However, this notion of strong equivalence is in fact, stronger as the name suggests to see that work out this exercise.

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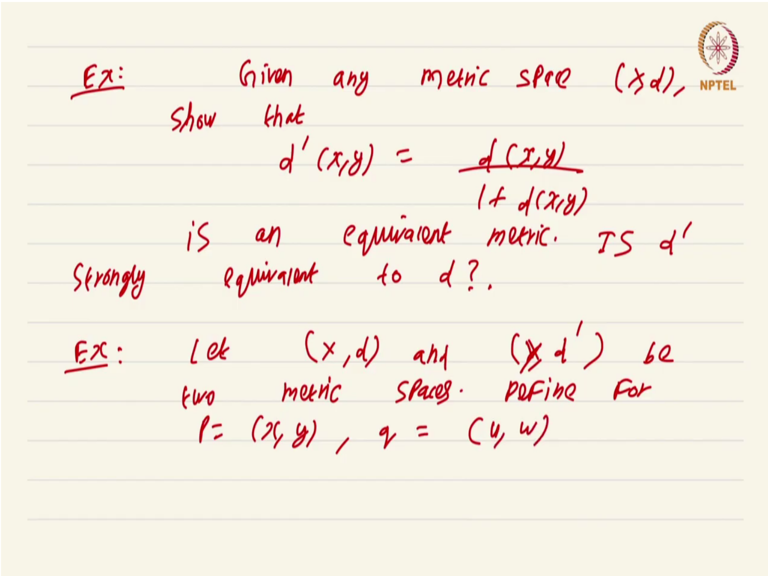

Exercise given any metric space, any metric space, X, d show that, d prime x, y is equal to d x, y divided by 1 plus d x y is an equivalent metric; is an equivalent metric. Is d prime strongly equivalent to d; is d prime strongly equivalent to d? Now, what happens is that this new metric that we have defined d x, y by 1 plus d x, y has the special property that its a bounded metric.

What I mean by that is no matter what set you take including the whole of x itself, its going to be of bounded diameter. The diameter of any set in x under d prime is going to be bounded, whereas, that might not be the case in d. So, you can take an infinite set and you can find this equivalent metric d prime in which the set will become a finite diameter.

Now, why this is interesting is Cauchy sequences between d prime and d may not be the same. It could happen that a particular sequence is Cauchy in d, but not in d prime or vice versa. I want you to investigate such aspects. Nevertheless if x n converges to x in d then x n will converge to x in d prime also. So, convergence of sequences will be the same, but not necessarily Cauchy sequences.

So, strongly equivalent metrics in some sense preserve all properties of metric spaces including things like boundedness and completeness, which we will come to in the next future videos. But, a strongly equivalent metrics will preserve all such properties whereas, equivalent metrics will preserve only the topological properties, those that can be characterized entirely using open sets ok.

So, there are these dual notions depending on what you are interested in if your primary interest is topology, then equivalent metrics are enough. On the other hand if you are interested in aspects that are more analytic in nature like completeness you might need to consider only strongly equivalent metrics ok.

Now, let us see a concrete example rather I will leave it as an exercise to you exercise again. Let X, d and X, d prime or sorry Y, d prime be two metric spaces be two metric spaces.

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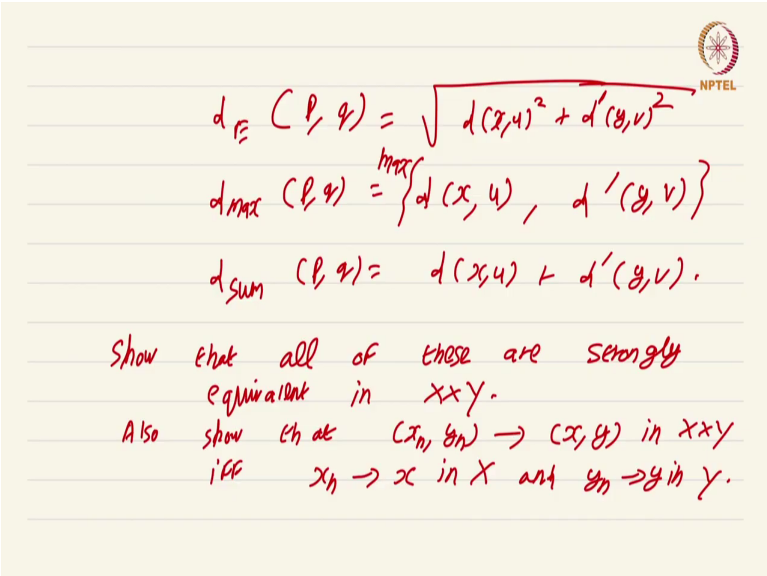

Define for P equal to x, y and q equal to u, w the following; d E of P comma q is under root d of x, y squared plus d of sorry there is a slight mistake here, this is under root d of x, u squared plus d of y, v square ok. E here is supposed to be an abbreviation to suggest Euclidean, because this is the way the standard Euclidean metric is defined on R 2 starting with the absolute value on R.

The second one d max of P, q is just d of x, u sorry maximum of d of x, u comma d prime here again it should be d prime here. So, just note that there was a small mistake here this is supposed to be d prime. So, max of d x, u comma d prime y, v you just take the maximum of the two as the name suggests. And, the third one is the d sum of P, q is as you could expect d of x, u plus d prime of y, v.

Show that all of these; all of these are strongly equivalent in X cross Y right. So, given two metric spaces x d and y d prime here are three natural ways to make the product into a metric space it turns out that all three of them are strongly equivalent. So, from the perspective of studying sequences continuity, Cauchy sequences completeness etcetera. All of these metrics can be interchangeably used and we will use that metric which is most convenient for the situation.

Please note that these three definitions we have already given in the context of R n Euclidean space ok. These three metrics; obviously, have extensions to finite products, you extended in the most straightforward and obvious way, all these are thankfully strongly equivalent. So, when we are doing analysis in the Euclidean space R n we can use any of these metrics, depending on the, what the situation demands? Ok. And of course, as you can see as I have mentioned before and which you should check is that Cauchy sequences in X cross Y under all these three coincide ok.

Now, there is another property, which is interesting with these a choice of metrics. Also show that, also show that, x n, y n converges to x, y in X cross Y if and only if x n converges to x in X and y n converges to y in Y. This is true under any of the three metrics that we have defined ok.

So, what this is essentially showing is that, when you choose these strongly equivalent metrics you can characterize convergence of a sequence in terms of the behavior of its components ok. Also show the same thing for Cauchy sequences I am not writing out that exercise its there in the notes, show that x n y n is Cauchy if and only if x n is Cauchy and y n is Cauchy ok.

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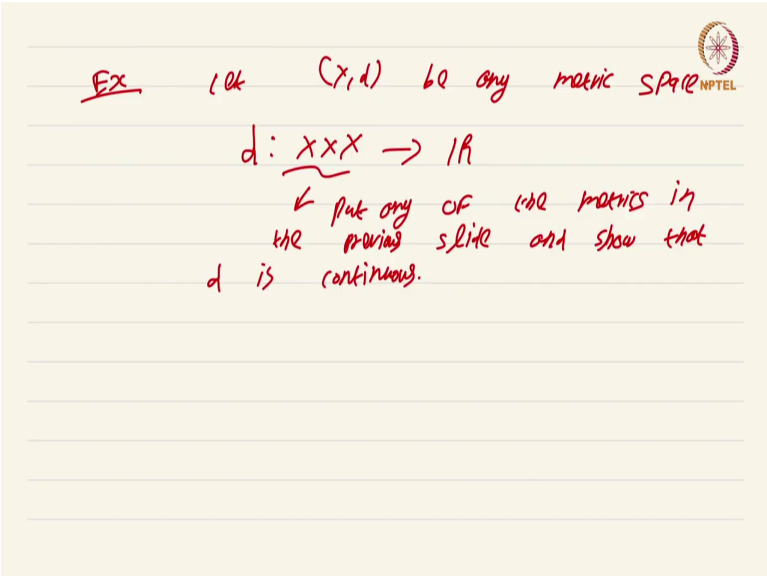

So, finally, one more exercise to reinforce you the understanding of equivalent and product metric spaces. Let X, d be any metric space; any metric space. Look at the function d from X cross X to R ok. The actual metric is essentially a function from the product space to R. Now, put any of the standard any of the metrics in the previous slide, in the previous slide, the previous slide and show that d is continuous, show that d is continuous.

So, when you consider X Cartesian product with itself you have these three natural metrics that we have defined, under any of them the metric itself d would be a continuous function ok. Now, we come to the central theorem of this short video, it is essentially saying that the choice of metric does not matter at least from the topological perspective.

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Theorem: Let (X, d_1) and (Y, d_2) be two metric spaces. Also we equip $X \times Y$ with a metric d with the property that

$(x_n, y_n) \to (x, y)$ in $X \times Y$

iff $x_n \to x$ and $y_n \to y$.

Then the open sets in $X \times Y$ are precisely the union of products $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open.

So, here is the theorem, here is the theorem, let X, d 1 and y, d 2 be two metric spaces. Also we equip X cross Y with a metric d with the property; with the property, property that x n, y n converges to x y in X cross Y, if and only if x n converges to x and y n converges to y ok.

Then, the claim is that you can completely characterize the topology of X cross Y just from the data, that you have put a metric with this characteristic property. The natural one that you would expect from a product space that a pair of sequences converge if and only if component wise they converge. Then, the open sets, in X cross Y are precisely the unions of products, products U cross V where U subset of X and V subset of Y are open.

So, what is this theorem trying to say, it is just saying that if you take a product of two metric spaces and put a metric on that product, which has this characteristic property that you would expect that x n, y n converges to x y if and only if x n converges to x and y n converges to y.

Then, you can immediately write down what the open sets in X cross Y are going to be they are precisely the products U cross V, where U is coming is an open set in X and V is an open set in Y.

Consequently, what this is indirectly showing or rather directly showing is that you could have put whatever metric you want on X cross Y. As long as it has this property, it is going to be equivalent to any other metric with that property. So, two metrics on X cross Y that have this characteristic property they are equivalent to each other.

So, in some sense it really does not matter what metric you put on the product space from the perspective of topology, simply because they are all going to lead to exactly the same open sets.

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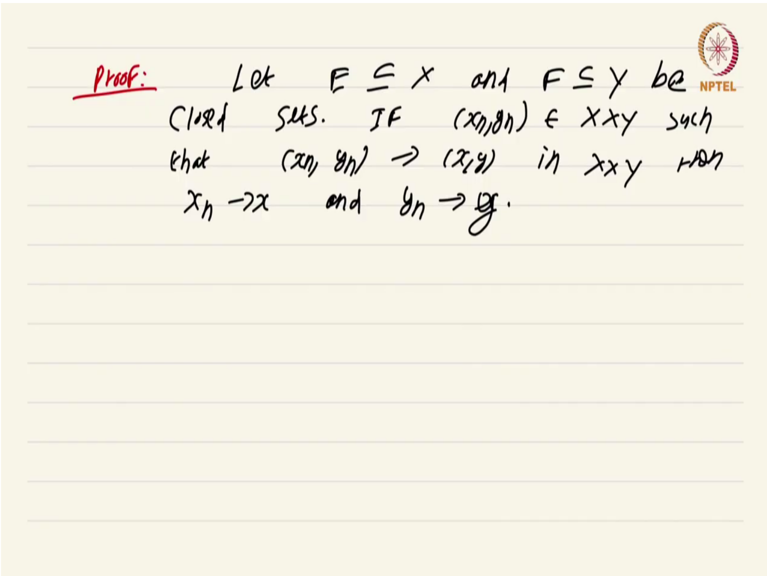

Now, the proof has some nice ideas. So, please I mean grab a hot beverage and put on your seat belts. So, what we do is the following. Let E subset of X and F, subset of Y be closed sets. You will understand in a moment why we are choosing closed sets, its sort of understandable, because we have the key property in terms of sequences and closed sets are in fact, defined using sequences. So, its sort of natural to expect that close sets will have a role to play.

So, if x n comma y n is a sequence; is a sequence in X cross Y such that, x n, y n converges to x, y in X cross Y; then well we have the characteristic property x n converges to x and y n converges to y sorry y n converges to y ok. Now, let me just make a small change x n, y n is in E cross F. Otherwise, what was the point of picking these close sets ok.

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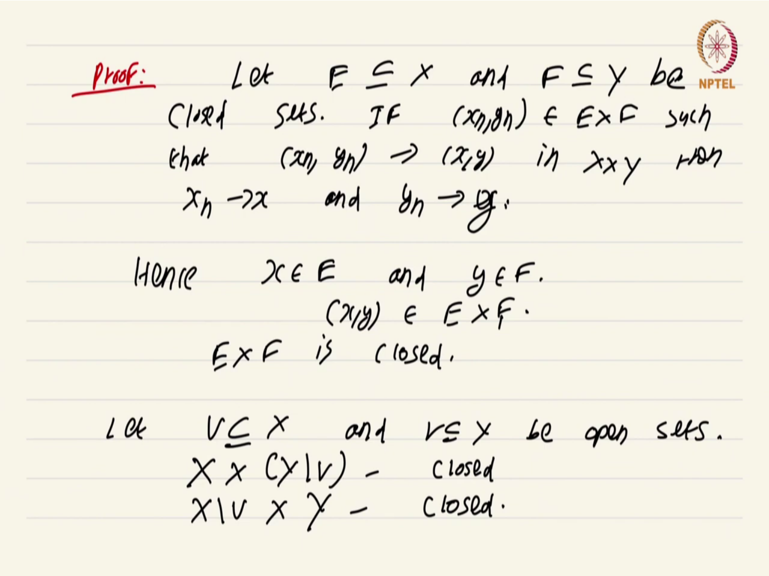

What is this? So, because E and F are closed we must have x is in E. Hence, x is in E and y is in F that is just because E and F are closed sets. Consequently x cross x comma y is there in E cross F ok. What this shows is that E cross F is closed. The product of a closed set in x and a closed set in y is going to be closed, simply because we have shown that if you take an adherent point x, y of E cross F, then that adherent point is there in E cross F, ok.

Now, how does this help? We have shown that the product of two closed sets is close, how does this help well, let us consider U subset of X and V subset of Y be open sets take two open sets. Now, the first goal is to show that U cross V is also open. Ultimate aim is to show, that given any open set it can be written as a union of sets U cross V; first let us see that U cross V is open to begin with, ok.

Now, the way to do that is we know that closed sets continue to be close when you take product. So, we are given open sets. So, its natural that we try to consider complements and take open sets and see what happens? Now, X itself is a closed set, you can take X cross Y set minus V ok. Now, V is an open set therefore, Y set minus V is a closed set. So, this is closed, this is closed. In an entirely similar way we have X set minus U cross Y is closed ok. Both these sets are closed ok.

Now; that means, the complements of these two sets must be open right? What are the complements of X cross Y set minus V and X minus U cross V, well they are nothing, but X cross V and U cross Y are both open taking complements, ok.

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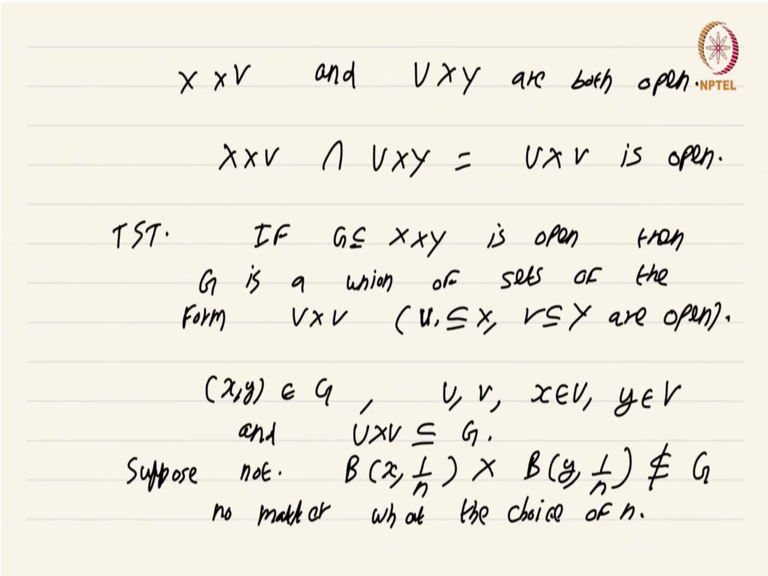

Well; that means, the intersection X cross V intersect U cross Y, which is nothing, but U cross V is open ok. That was rather easy, what we have done is by using the fact that closeness is preserved by taking products, we have shown that openness is preserved by taking products ok. Now, what is our goal to show that, if G subset of X cross Y is open, then G is a union of sets of the form; of the form U cross V, u subset of x, V subset of Y are open ok.

This is our ultimate goal. Now, what we are going to do is the following? We are going to take a pair x comma y, which is there in G, which is there in G and what we are going to do is we are going to find U comma V, x is in U, y is in V, and U cross V is subset of G ok.

Just a 30 second thought will convince you that this is more than enough. All we are doing is given any point in G, we are squeezing a product U cross V between that contains this point and is contained in G. And, this will show that G can be written as a union of sets of the form U cross V. Suppose not, well what is this mean? This means that, if you consider B x 1 by n Cartesian product B y 1 by n, these are the open balls in x and y respectively, this is not going to be a subset of G, no matter what n we choose right.

Because, we are assuming that it is not possible to squeeze a product of open sets within G, that contain the point x comma y. If you take these particular balls of radius 1 by n centered at x and y its not going to be contained within G. So, no matter what choice of n we choose, no matter what the choice of n ok.

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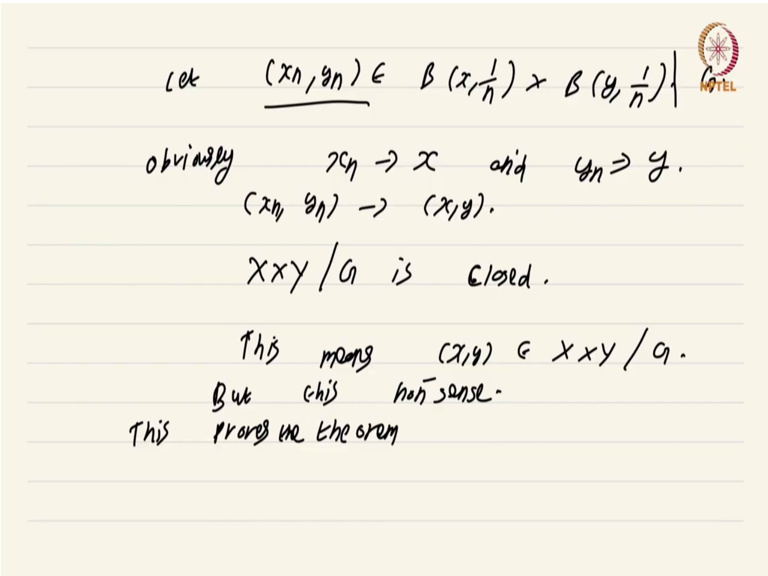

Now, what is this suggest well look at let x n comma y n be an element of B x 1 by n Cartesian product B y 1 by n. So, essentially we are taking a sequence, which is ultimately what we should be doing, because the hypothesis is entirely on the behavior of sequences.

So, take this sequence x n comma y n which is coming from the ball of radius 1 by n centered at x Cartesian product with the ball of radius 1 by n in y. Now; obviously x n converges to x and I made a slight error actually it is not an error it is more of an omission. I am do not take coming from B x n cross, B y 1 by n ok.

What I am saying is take it in let x n y n be an element of this set minus G set minus G right. I am taking a sequence, which is there in the product of these balls, but this sequence the elements of this sequence are not in G. Why does such an element exist well, because look at our hypothesis. We are saying that this product is never going to be a subset of G irrespective of the choice of n.

So, for each n we can at least find one element, which is there in B x 1 by n cross B y 1 by n, which is not there in G ok. So, choose this sequence x n y n, which is there in B x 1 by n Cartesian product B y 1 by n set minus G. Well; obviously x n converges to x and y n converges to y that is exactly the way the sequence has been constructed ok.

Now, our hypothesis says x n, y n must converge to x, y right. Whenever you have components converging then the product sequence must also converge ok. But, X cross Y set minus G is open, because is closed. Because, G is assumed to be open ok. So, x Cartesian product y set minus G is closed, but we have just seen that x n, y n is a sequence in x Cartesian product y minus G remember because that is the way we have chosen this sequence x n, y n.

That means, this means, x y is an element of x cross y set minus G right, because this is a closed set. But, this is nonsense, but this is nonsense. We started off with a point in G. And, we are ending up with the stupid conclusion that this point is not there in G ok. So, this proves the theorem. Our assumption that I mean more specifically, this assumption that you cannot find open sets U and V, such that U cross V is subset of G, and x comma y is there and U cross V is untenable and therefore, the result is proved.

So, we have this nice characterization of product metric spaces, when you have this key characteristic property that x n, y n converges to x, y if and only if x n converges to x and y n converges to y if we have this property ok. So, what does this tell us, this sort of tells us that, the metric of choice, does not really matter if you are really interested only in the topological properties or if you are rather interested only in the open sets, it really does not matter which metric you choose.

This is a course on real analysis and you have just watched the video on equivalent metrics and product space.