**Real Analysis II**
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**Lecture - 35.2**
**Multiple Lebesgue Integration**

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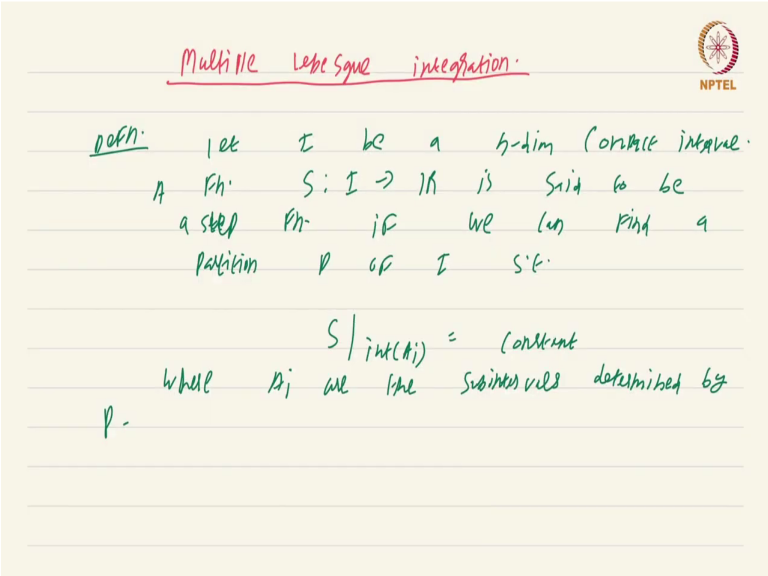


In this final video of this course we are going to discuss Lebesgue multiple integration or Multiple Lebesgue Integration. The definition and the various properties of this integral are exactly the same as what we did for one dimensional Lebesgue integrals, so we will be very very brief.

So, we begin with the definition of a step function. So, let I be a n-dimensional interval, as I remark sometime in the earlier video I will not keep repeating n-dimensional interval. And sometimes just say interval leaving you to infer from context what exactly that interval is; is it just an interval in R or an interval in R n.

So, let I be an n dimensional compact interval and ok, let it be like this. Let I be an n dimensional compact interval, a function S from I to R is said to be a step function if we can find we can find a partition p of I such that S restricted to A i is constant. Or, rather S restricted to interior of A i is constant where A i are the sub intervals determined by p.

So, a step function in R n is nothing but a function defined on a compact interval such that when you restrict that step function to each one of the sub intervals determined by this partition p, rather the interior of the sub intervals determined by the partition p you get a constant value.

As in the one dimensional case we do not really care what the value of S is on the boundary of A i it could be anything, it could be completely badly behaved on the boundary of each one of these A is. All we want is the function should be constant on each sub interval determined by these partition by these partition ok.

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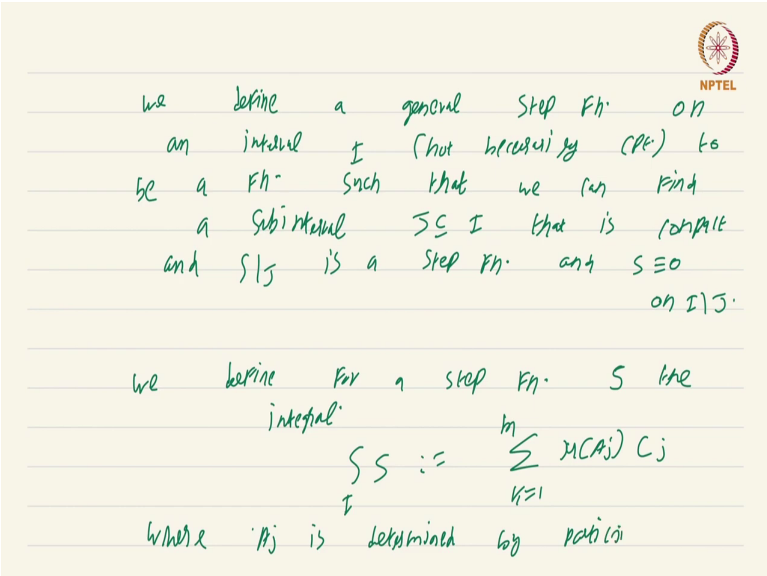


Similarly, we define a general step function on R n on sorry, on an interval I not necessarily compact not necessarily compact to be a function such that we can find we can find a sub interval J subset of I that is compact that is compact. And S restricted to J is a step function as defined before and S is identically 0 on I minus J.

So, this I could now be anything it could be the whole of R n as well we say a function S is a general step function on this I. If you could find a sub interval J which is compact such that when restricted to J it is a step function as defined before, but outside of J its 0 ok. Now, we define for a step function for a step function S the integral of I S by definition to be just summation K equals 1 to m mu of A j times this C j where A j is determined by the partition p determined by partition p.

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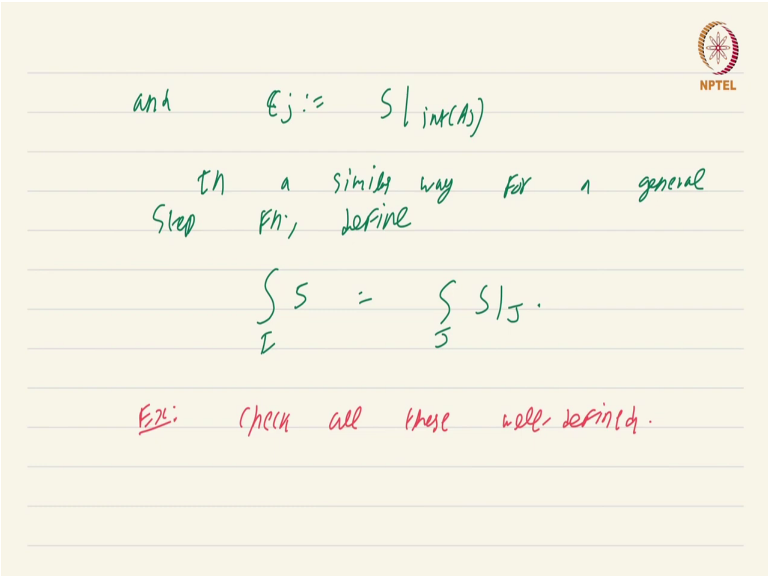


And C j is just by definition the value when you restrict S to A j. Note not A j interior of A j, note S restricted interior of A j is constant. So, you just look at the constant value and multiplied by the measure of the corresponding sub interval A j. In a similar way, in a similar way for a general step function, for a general step function define integral of S on I to be nothing but integral of on J S restricted to J.

J is a compact interval s restricted to J is still a step function. So, the previous definition will work. So, there are number of simple things to check exercise check all these are well defined wherever there are choices involved.

And I remark that we have already done all this in the one dimensional case, really there is nothing happening in the several variable case also it is exactly the same proofs. One of the reasons of reusing the notation capital I for the interval even though something like saying that it is a cube or a hypercube or something like that saying something like that is a better choice of terminology.

But calling it intervals and calling it I has the benefit that the same proofs that we did in the one dimensional case will move without any further change to the higher dimensional case as well ok. So, once you have defined the integrals of step functions it is a simple matter to define upper functions and Lebesgue integrability.

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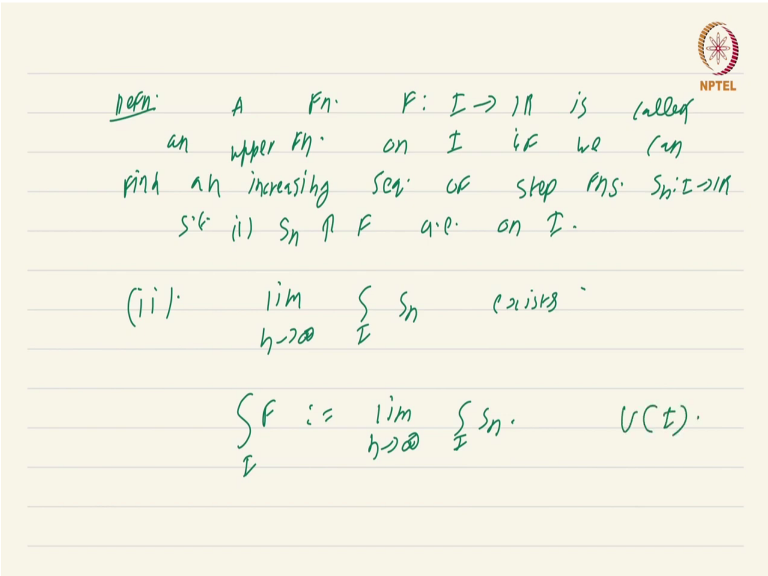


So, definition a function F from I to R is called an upper function on I if we can find, we can find an increasing sequence increasing sequence of step functions S n from I to R such that S n increases to F almost everywhere on I. Here almost everywhere has the same meaning; that means, outside a set of measures 0 and we have defined what a set of measure zero is in the previous video. So, condition 1 S n increases to F almost everywhere on 1.

Condition 2 limit n going to infinity integral of I S n exists ok. In this scenario we just define integral of I F to be by definition to be this limit ok. So, the set of upper functions is denoted by U of I same notation as in the one dimensional case.

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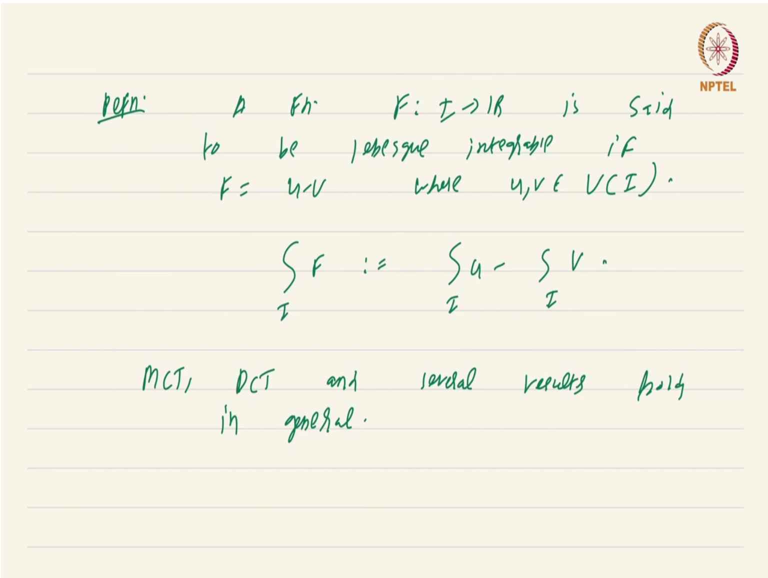


Now, we make the final definition a function F from I to R is said to be Lebesgue integrable, said to be Lebesgue integrable if F is equal to u minus v where u comma v are upper functions on I. And in this scenario we define integral of I of F to be integral I u minus integral I v ok.

So, the theory in the several variable case is almost exactly the same as the theory in the one dimensional case. Word for word the same definitions we have used, same notation we have used. As you can expect the monotone convergence theorem dominated convergence theorem and several other results several results hold in general.

And the proofs are all exactly the same really nothing much to do. So, all you have to do is go through the proofs in the one dimensional case 1 again sit under a tree look at the sky and think over why those proofs generalize in a straightforward manner to the several variable case.

So, this formally concludes this course on Real Analysis at a future point in time I shall present a course in Vector Analysis. So, vector analysis will sort of tie up many loose ends in this course, the starting point will be the theory of manifolds which we have already briefly dealt with in this course Real Analysis II. The aim is to developed integration on such manifolds.

And we will do that by using Lebesgue integrals. So, in this next course we will prove something called the change of variables which you have already used definitely when you had a course on calculus of several variables. These are the formulas relating the integral value on spherical coordinates and cylindrical coordinates and all that.

So, there is a general theorem that tells you how to transform integrals from one coordinate system to the other and the natural setting for to prove that theorem is on the on manifolds. Once we prove the change of variables theorem for multiple Lebesgue integrals we will be able to define a measure on a manifolds and also define integration on manifolds.

Once you define integration on manifolds we will be able to prove some of the classical theorems that you have studied like Stokes theorem in the general context of manifolds. So, that is at a future course in vector analysis where it will sort of directly proceed from what we have studied in Real Analysis II.

This is a course on Real Analysis and you have just watched the video on Multiple Lebesgue Isntegration.