Real Analysis II
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Lecture - 35.1
Multiple Riemann Integration

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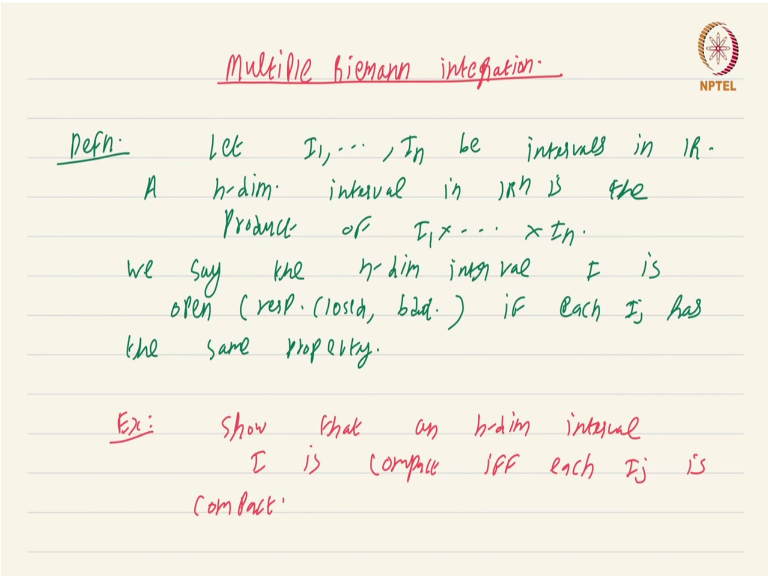

In this video, we shall see a short and brief development of Multiple Riemann Integration. All the hard work has been done way back and real analysis I, when we studied the Riemann integral in terms of upper sums and lower sums. The theory in several dimensions is not that different. We begin with the definition.

Definition; Let I 1 dot dot dot I n be intervals in R. So, these intervals I 1 to I n could be open, close, bounded, unbounded, half open, half close it really does not matter, they could even be points we even allow that. So, let I 1 to I n be intervals in R. A n dimensional interval in R n is the product of I 1 dot dot dot I n.

So, a n dimensional interval is nothing, but a product of intervals in R. We say the n-dimensional interval I is open respectively, closed, bounded if each I j has the same property; has the same property, ok. So, we consider n dimensional intervals which are just products of intervals in R.

So, as an interesting and simple exercise. Show that an n dimensional interval I is compact if and only if each I j is compact. This is actually a trivial exercise and it follows rather immediately from the theory of compact as we have developed elaborately for metric spaces ok.

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Defn: (Measure of an interval). Given an interval $I = I_1 \times \cdots \times I_n$, we define the measure of I

$$\mu(I) := \mu(I_1) \times \mu(I_2) \times \cdots \mu(I_n)$$

where $\mu(I_j) :=$ length of the interval.

Defn: Let $I = I_1 \times \cdots \times I_n$ be an interval. A partition of I is just a product $P_1 \times \cdots \times P_n$ where P_j is a partition of I_j.

So, given So, another definition this is the important definition of measure of an interval. So, henceforth for the sake of brevity I will not keep saying n dimensional interval I will just say interval I and leave it to you to infer from context, whether it is an interval in R or a general n dimensional interval.

Given an interval, I equal to I 1 cross dot dot dot I n, we define the measure of I mu I to be as you can guess mu of I 1 into mu of I 2 into dot dot dot mu of I n, where mu of I J is just the length of the interval, length of the interval. Since we allow intervals to degenerate to single points and also allow in finite length intervals this product could be 0, this product could be in finite as well.

Note that if one of the intervals is a single point, but another interval is an interval of length infinity, then the product interval will still have measures 0. According to this definition that sort of makes sense, because what will happen is if one of the intervals degenerate to a point when you take the product you do not get an n dimensional object. You get sort of a lower dimensional object. So, defining the measure of such a lower dimensional object to be 0 makes perfect sense ok.

So, we have now defined what the measure or length of a general n dimensional interval is. Once this is done the definition of a Riemann integral is rather simple. So, we will all have some more sequence of definitions, some more definitions. Definition; let I equal to I 1 dot dot dot I n be an open interval, ok sorry let I 1 to I n be any interval it does not matter be an interval.

A partition of I is just a product P 1 cross dot dot dot P n, where P j is a partition of I j. So, a partition of an n-dimensional interval is nothing more than a product of partitions of the intervals in R. So, a simple picture will illustrate what is going on. Suppose we are taking a two-dimensional interval that is just a product of two intervals.

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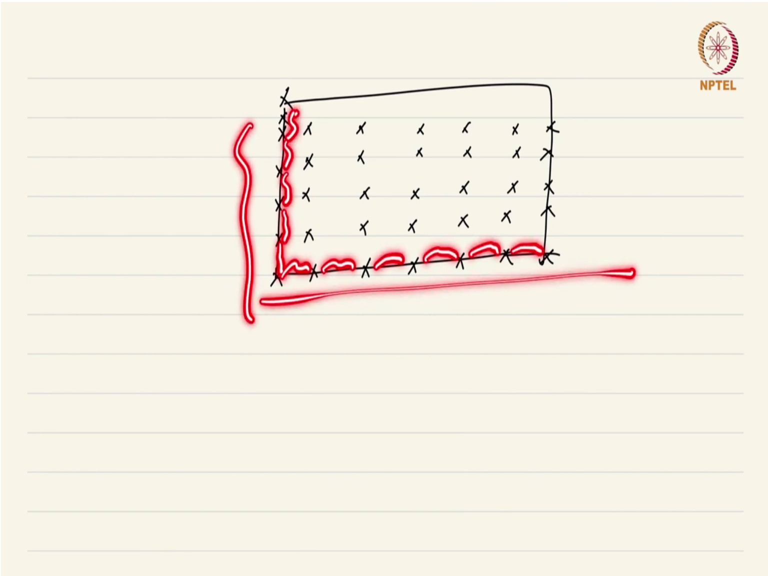

So, in general it will look like a rectangle. So, a partition is just given by a product of partitions of the two interval. So, let us take a partition P 1 of the interval, which I have drawn on the x axis. And another partition, which is there on the y axis. Then together you take the product. So; that means, of course, the endpoints are always there in the partition.

So, what will happen is you will consider a sort of grid like this, when you take the partition, when you take the product of the partitions you get something like this. These are all the points that will be there in the partition. So, yeah to prevent you from going to sleep I will not complete this picture.

So, note that each of these partitions in each one of these axis determines some sub intervals. So, you have these sub-intervals determined by this and sub intervals determined by this ok. And the product of those would give you sort of smaller rectangles like this.

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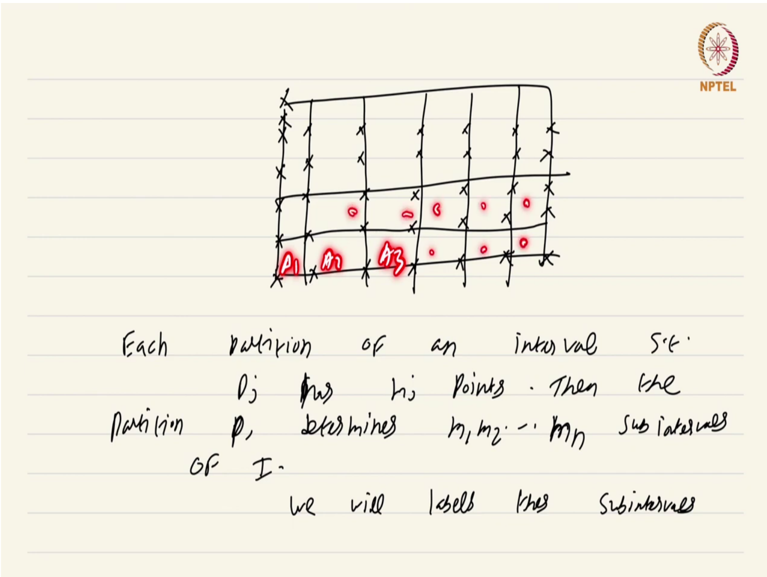

So, that I can draw. So, you will have sort of rectangles determined by this partition. So, let me make a remark, each partition of an interval such that, P j has m j points. That is I am writing the partition P as the product P 1 cross dot dot dot P n, where each P j is a partition of I j. Assume that P j as m j points, then the partition P determines m 1, m 2 dot dot dot m j intervals or rather sub intervals of I.

So, if you have a partition P write it as a product P 1 cross dot dot dot P n. If P j contains m j points, then there are m 1 times m 2 times dot dot dot, sorry this is not m j this is m n, this is m n sub intervals determined by this partition. So, each sub interval will be determined in the way we have defined in this picture ok.

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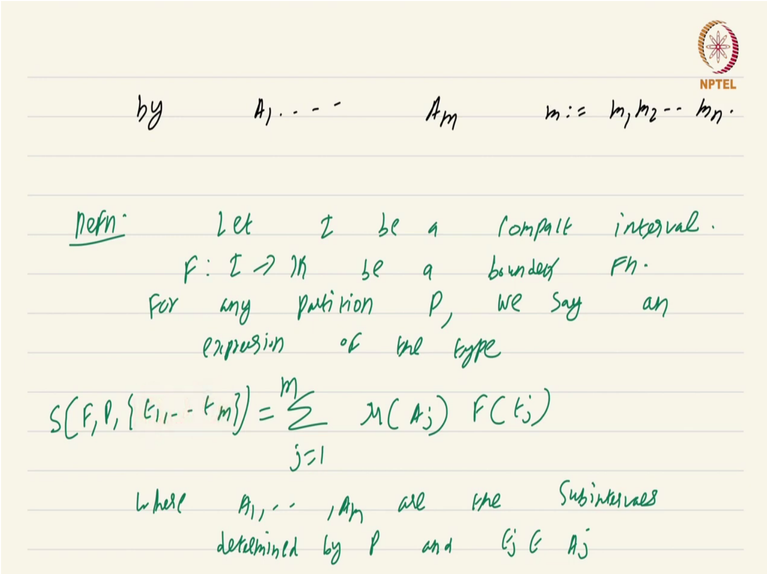

Now, usually what we will do is we will just, we will label these sub intervals by A 1 dot dot dot A m, where m is just m 1, m 2 dot dot dot m n ok. So, now that we have defined what it means for a partition to determine sub intervals we can move on to the next step and define the Riemann integral. Definition; let I be a compact interval, ok. Let F from I to R be a bounded function.

For any partition P, we say an expression of the type summation j running from 1 to m mu of A j times F of t j, where A 1 to A m are the sub intervals determined by P sub intervals determined by P. And t j is some point in A j. An expression of the type summation j running from 1 to m mu of A j F of t j, where these A j's are the various sub intervals that are determined by the partition so, each A j for instance in this case this can be A 1, A 2, A 3.

You ordered them in some way it really does not matter, there will be finitely many of them. You order them in some they call them A j's you take the measure of A j and multiply it by F of t j, where t j is some point.

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is called a Riemann Sum.

We say F is Riemann integrable on I, if we can find a number A s.t. for each $\varepsilon > 0$, there is a partition P_ε satisfying :-

For any finer partition $P \supseteq P_\varepsilon$ we have the Riemann sum coming from P to be $< \varepsilon$.

$|S(P, F, \{t_1, \dots, t_m\}) - A| < \varepsilon$ for any choice of $t_j \in A_j$.

So, an expression of this type is called a Riemann sum. So, a Riemann sum is just, you consider a partition you sample points from each sub interval determined by the partition. Evaluate the given function at that particular set of points and then multiply it by the corresponding measure of the sub intervals and take the sum. That is called the Riemann sum ok, it is called a Riemann sum.

We say F is Riemann integrable on I if we can find if we can find a number capital A, such that for each epsilon greater than 0, there is a partition P epsilon satisfying the following somewhat complicated looking condition, for any finer partition P. So, finer partition just means this partition P contains P epsilon ok.

So, you have essentially added more points to each one of the sub intervals, not each one of the sub intervals you have added more points to each one of the P j's determined by P epsilon. So, for any finer partition P, we have the Riemann sum the Riemann sum coming from P to be less than epsilon.

So, there is a lot unpack here. Let us do in an effort to make our understanding clear what we can do is we can introduce some notation. Usually notation seems to overburden things, but sometimes having a clean notation can make us understand what is going on. So, let us give an expression for this. We will call this the Riemann sum so we let us call it S F P. And since this Riemann sum depends on the choice of points, we can call it t 1 to t m ok. Let us give it this notation.

So, here the understanding is that you take the sum of the products of the intervals with the values of F at these various points in the final slot ok. So, that is the notation here. What this is saying is, for any partition P we have the Riemann sum coming from P to be less than epsilon that just means S P F t 1 to t m is less than epsilon for any choice for any choice of t j in A j ok.

So, what this is saying is no matter how you sample the points in the sub intervals the Riemann sum always turns out to be sorry I made a mistake, what I want to say is I completely forgot the integral value minus A is less than epsilon ok. So, what I want to say is, given any finer partition P of this P epsilon, no matter how you sample the points t 1 to t m from this partition P. When you take the Riemann sum with respect to that choice of points, the absolute value of that minus A is less than epsilon ok.

Then we say that F is Riemann integrable on A on I and we say integral of I F is equal to A.

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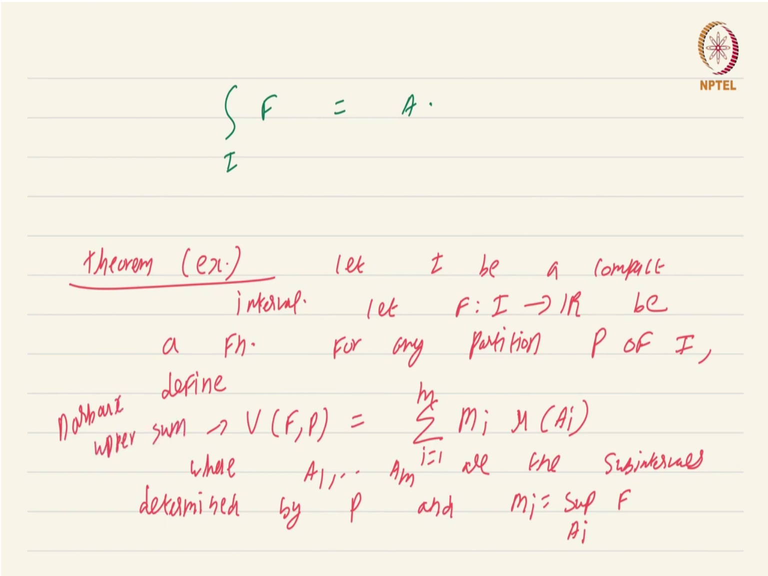

So, the Riemann integral in several variables exist, if you can make the values of Riemann sums arbitrarily close to a particular value, which we have called A ok. So, there is really nothing much happening you would have seen a similar definition of the Riemann integral in a standard course on Real Analysis, or at least you would have seen it equivalent definition in terms of upper sums and lower sums.

So, let me just leave you with somewhat elaborate exercise that will really test your understanding of basic analysis. So, this I am going to call it a theorem, but it is actually an exercise for you to work out. This will really test your understanding. So, let I be a compact interval be a compact interval.

Let F from I to R be a function. For any partition P of I define U F P to be summation capital M I mu of A I, where A 1 to A m so, let I run from 1 to m. A 1 to A m are the sub intervals determined by P sub intervals determined by P. And capital M i is supremum of F on A i ok. So, this is called the upper sum. The Riemann upper sum or rather the Darboux upper sum Darboux upper sum ok.

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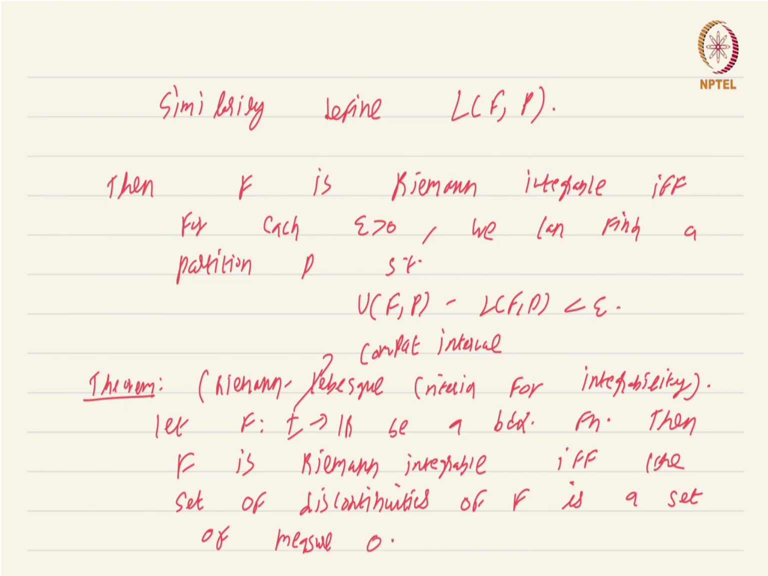

Similarly, define L F P, their Darboux lower sum I am going to leave it to you to define L F P, its exactly like what we did in Real analysis I, ok. Then F is Riemann integrable if and only if for each epsilon greater than 0 we can find a partition P, such that U F P minus L F P is less than epsilon.

So, you would if you are that sort of student who has a very good memory you will remember that we proved that a function F in the real line or defined on a compact interval in a real line is integrable if and only if this is true. However, there we are defined the notion of Riemann integrability also in terms of upper sums and lower sums.

In fact, what we do is we take the infimum of all upper sums very infimies over every possible partition and you take the supremum of all lower sums, where you take the supremum over all possible partitions. And you say the function F is Riemann integrable if and only if the supremum of the lower sums agree with the infimum of the upper sums.

So, there the definition was slightly different, here the definition is in terms of choosing points from each sub interval sampling points and sort of considering the product of the measure and the value at that point. So, there is a subtle difference between the earlier result in one-dimension and the result we have here.

Nevertheless, this exercise requires a bit of work, but it is really going to reinforce your understanding of this entire course Real analysis I and Real analysis II. So, I want you to work out this exercise in detail please refer to Real analysis I to get some ideas of how to tackle this ok.

So, the final thing I want to say about multiple Riemann integration is another theorem, which is not that hard to show, all the hard work has been done in Real analysis I. This is sort of the Riemann Lebesque criteria for integrability ok. So, let F from I to R be a bounded function be a bounded function, then F is Riemann integrable of course, here I is a compact interval F is Riemann integrable if and only if the set of discontinuities of F is a set of measure 0. What do I mean by a set of measures 0?

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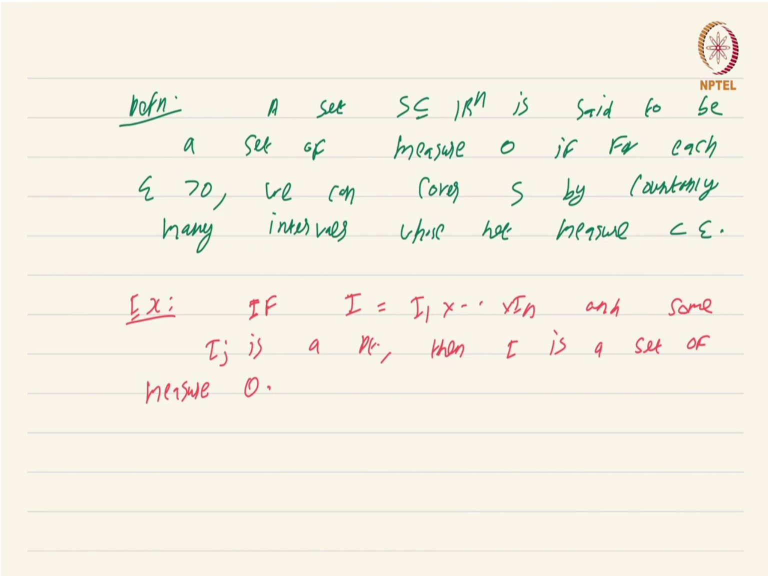

Well that is the final definition of this video. Definition a set S subset of R n is said to be a set of measure 0 0, if for each epsilon greater than 0 we can cover we can cover S by countably many intervals, whose net measure whose net measure is less than epsilon.

Just as before you can show that any function sorry any set, which is countable is going to be a set of measures 0 and you can also show this interesting exercise, which is related to the remark I made before, if I equal to I 1 to I n and some I j is a point then I is a set of measure 0, you can try to show this ok.

So, a set of measures 0 is something that can be covered by sets such that, the net measure of those sets is going to be less than epsilon and these sets have to be intervals, because we have defined measure only for intervals so far. So, this was a brief overview of multiple Riemann integration. In the next and final video of the course we will have a brief overview of multiple

Lebesque integration. This is a course on Real Analysis and you have just watched the video on Multiple Reimann Integration.