Real Analysis II
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Lecture - 34.2
Convergence in L^2 (I)

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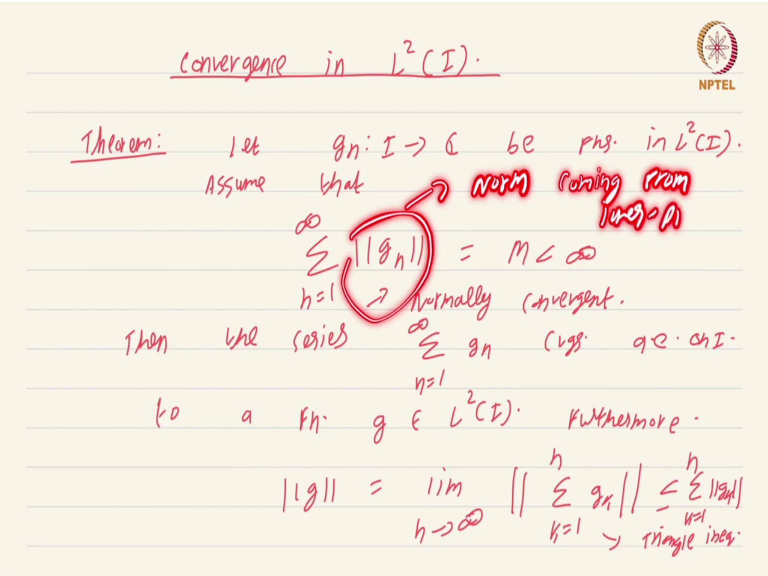

In this video we are going to show a convergence theorem for functions in L 2 I complex valued. And in the next video we are going to use this convergence theorem to prove the all famous Riesz Fischer theorem which states that the space L 2 I is actually complete. The Riesz Fischer theorem is sort of indispensable in the study of Fourier analysis. So, this is sort of a very very major theorem which is a consequence of the theory we have developed.

So, let us state and prove this convergence theorem, this is sort of like an analog of the dominated convergence theorem that we have seen for the space L of I. So, the setup is as follows let g n from I to C be functions in L 2 of I. Assume that summation n equals 1 to infinity norm g n is equal to M less than infinity. So, this is sort of called normally convergent ok. Actually, the reason why its called normally convergent is sort of stupid the sum of the norms converges.

So, it is sort of called normally convergent. Then the series summation n equals 1 to infinity g n converges almost everywhere on I to a function; to a function g that is there in L 2 of I. Furthermore the norm of this function g is going to be equal to limit and going to infinity of non summation K equals 1 to n g K and which is of course, less than or equal to summation K equals 1 from to n norm g K ok. The last part is just the triangle inequality ok.

So, what this theorem is trying to say is that the moment the sequence or rather the series of norms of a sequence of functions converges then the series itself converges almost everywhere to a function in L 2 I. And moreover the norm of the limit function is nothing but the limit of the sums the norm of the finite partial sums.

Of course, I must mention, so that there is no confusion the norm here is the norm is the norm coming from the inner product coming from inner product. In particular this is not the sup norm or anything do not make that mistake. This is the norm coming from the inner product that we defined for complex valued functions in L 2 I last time ok, excellent, so let us prove this.

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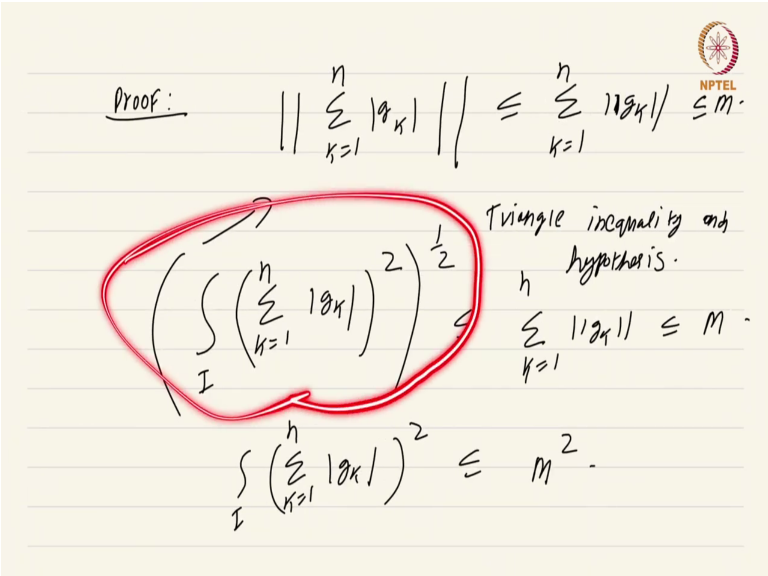

Proof, so what is the triangle inequality say? Well, the triangle inequality says that summation K equal to 1 to n mod g K the norm of this is less than or equal to summation K equals 1 to n norm g K ok. And this will be less than or equal to M, so this is just triangle inequality and hypothesis and hypothesis ok. Now, expanding out the left hand side, what is this?

This is just integral summation K equals 1 to n mod g K the whole squared the power half this is what that left hand side is by definition. And, what this is trying to say is that this is less than or equal to summation K equals 1 to n norm g K which is less than or equal to M ok. So, squaring what we get is integral over I summation K equals 1 to n mod g K squared this is less than or equal to M square ok.

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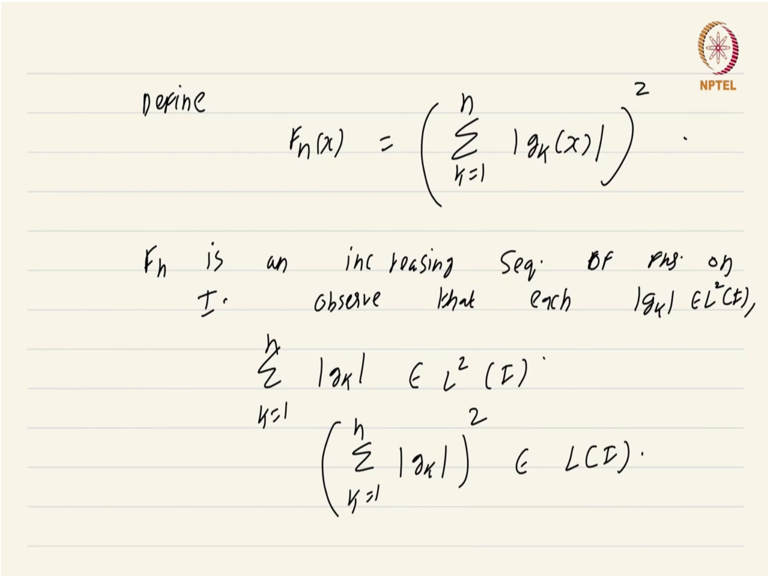

Now, what you do is define this new sequence of functions F n x to be the finite partial sums of the absolute values of the functions g Ks. So, you define F n of x to be summation K equals 1 to n mod g K x the whole square ok. Now, what can we conclude? First of all its obvious that F n is an increasing sequence, F n is an increasing sequence of functions on I.

Furthermore observe that each mod g K is in L 2 of I. Therefore, this summation K equals 1 to n mod g K this is also in L 2 of I, which just means that summation K equals 1 to n mod g K the whole squared is an L of I, this is just a translating the definitions ok.

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$F_n \in L(I)$.

$\int_I F_n \leq M^2$.

→ real numbers.

F_n increasing $\left\{ \int_I F_n \right\}$ is bounded above

By Levi's MCT, F_n must cvg. a.e. on I to a fn. $F : I \to \mathbb{R}$.

What does that tell us? It tells us that F n is an element of L of I. Furthermore, the equation that we have here that the integral of the sum of mod g K squared power half is less than or equal to M. Or, in other words integral summation K equal to 1 to n mod g K square less than or equal to M square this just becomes; this just becomes integral over I F n is also less than or equal to M square.

So, what do we have? We have F n increasing, we have F n increasing and we have integral I F n this set of all the integral values is bounded above. Note, that these are all real numbers, these are all real numbers because we got rid of the complex valued nature of the functions g K by going to the absolute value, excellent. So, by the by Levis monotone convergence theorem F n must converge almost everywhere on I to a function F from I to R ok excellent. Now, we have got a limit function.

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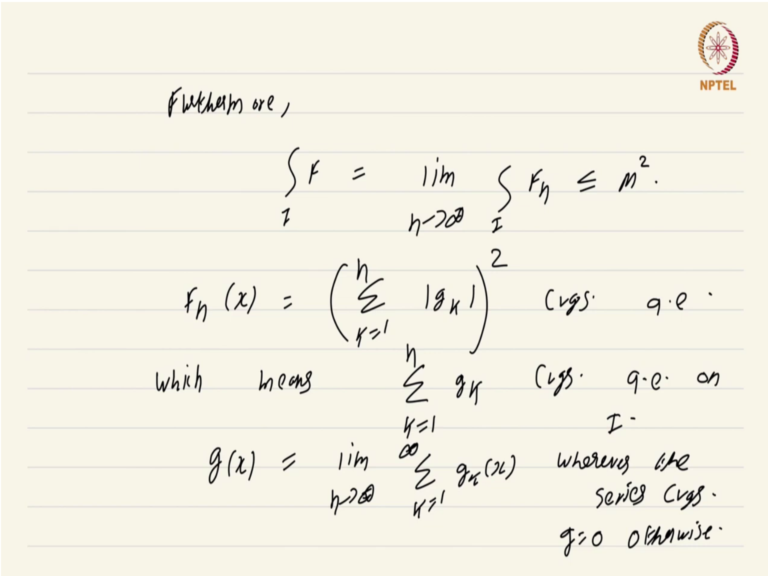

Furthermore, the monotone convergence theorem also say something about the integral values integral of I F is a limit n going to infinity integral of I F n which is going to be less than or equal to M square. This is also another consequence of the monotone convergence theorem. So, what have we concluded? We have concluded that this sequence of functions F n of x equal to summation K equals 1 to infinity mod sorry, K equals 1 to n mod g K squared sorry, mod g K the whole squared converges almost everywhere ok.

Which means that, the series without the absolute value also converges, which means summation K equals 1 to n g K converges almost everywhere on I right. You can see this immediately by just taking square roots on both sides and observing that when you take square roots also you have convergence. And therefore, you have that the series of absolute values converge which means that the series itself should converge almost everywhere on I ok.

So, you just define g, g of x to be limit n go into infinity summation K equals 1 to infinity g K of x wherever the series converges, wherever the series converges ok. And you can just set g to be 1 otherwise, it really does not matter what value you set on the points of a set of measure 0 ok.

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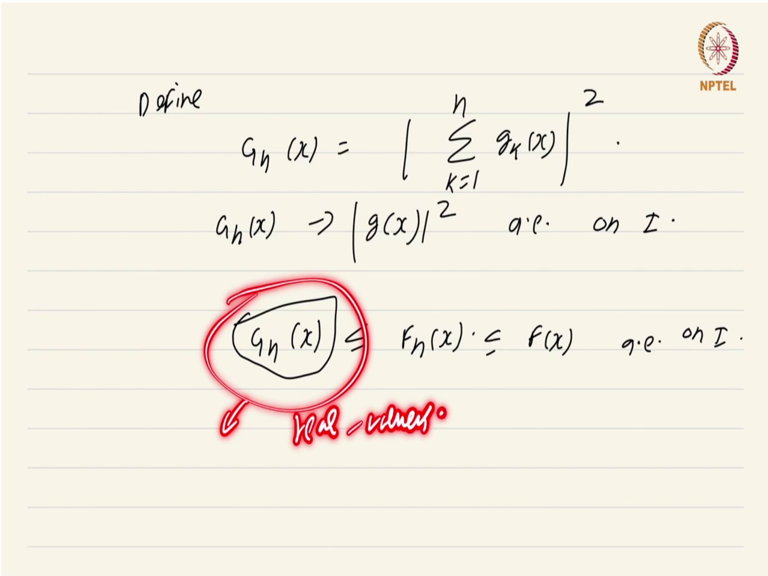

Now, define capital G n of x to be the absolute value K equals 1 to n of g K of x the whole square ok. So, again by what we have established we know that G n of x converges to g of x modulus squared almost everywhere on I. This is because the sequence inside the sorry the series inside converges to g of x almost everywhere. So, G n of x converges to mod g of x squared almost everywhere ok. We also have the following relationships G n of x is less than or equal to F n of x this is just the triangle inequality again.

This is absolute value of sum and this will be less than or equal to sum of the absolute values ok. And remember that f n was defined to be the sum of the absolute value square ok. So, G n of x will be less than or equal to F n of x and since F ns increased to F this will be less than or equal to F of x and all this will be true almost everywhere on I ok.

So, we have got this sequence of functions which are real valued again remember G n x are real valued the reason why we took absolute values is to make sure you end up with real valued stuff. So, that we can apply all the dominated convergence theorem and etcetera which we have shown only for real valued functions ok.

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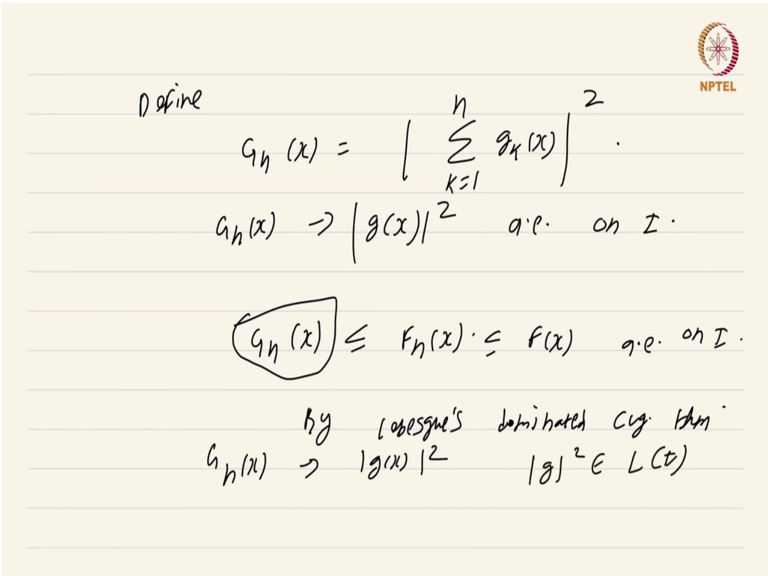

So, by Lebesgue's dominated convergence theorem, dominated convergence theorem G n x or rather the limit of G n x which is mod g of x squared this function is going to be mod g squared is going to be in L of I.

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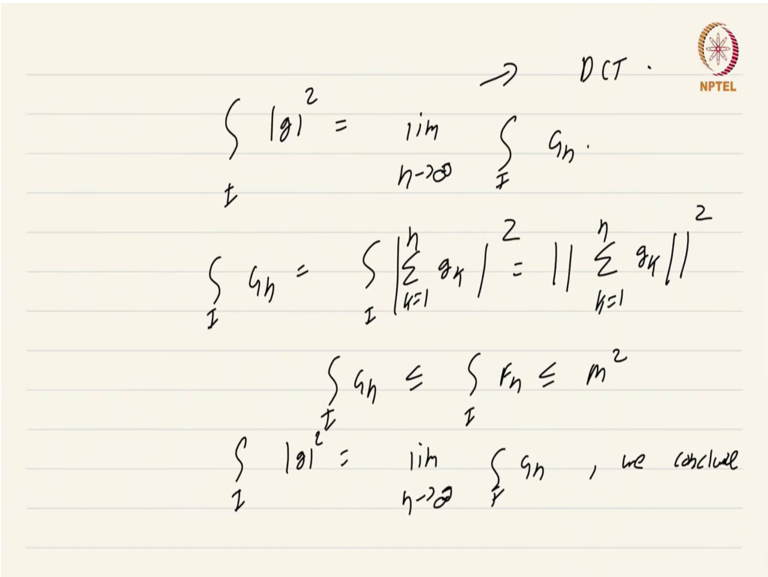

And furthermore, the integral of mod g squared is just going to be the limit n go into infinity of integral of I G n. So, this follows from dominated convergence theorem ok. Now, what is integral of I G n? By definition it is just integral of I summation g K mod square K equals 1 to n this is just by the definition. And this is by definition the norm of summation K equals 1 to n g K square ok and we also have integral of I G n is less than or equal to integral of I F n less than or equal to M square.

This just follows from the monotonicity properties or rather the behavior of the Lebesgue integrals under ordering. So, what can we conclude from the fact that integral of I mod g squared is limit n going to infinity integral of I G n we conclude that.

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that

$$\|g\|^2 = \lim_{n \to \infty} \left\| \sum_{k=1}^{n} g_k \right\|^2 \leq M^2$$

this finishes the proof.

That norm g squared is just limit n going to infinity of norm summation K equal to 1 to n g K squared which is less than or equal to M squared ok, and this finishes the proof; this finishes the proof ok excellent. So, now, we have got a sort of convergence theorem for series of functions in L 2 I complex valued. We are going to use this in the next video to prove Riesz Fischer theorem that says that L 2 I is actually complete. This is a course on Real Analysis and you have just watched the video on Convergence in L 2 of I.