Real Analysis II
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Lecture - 34.1
Norms and Inner-Products on Complex Vector Spaces

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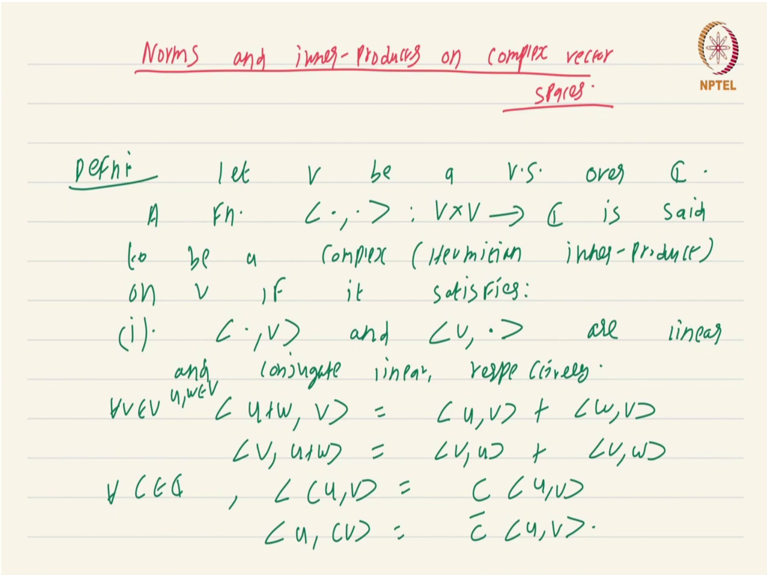

In this short video our goal is to define L 2 of I where the functions could possibly be complex value. To do this we first need to define what norm and inner product are on a complex vector space and the definition has only one slight twist, which you will understand in the moment.

Let V be a vector space over the complex numbers a function which we will denote by the same notation we use for inner product from V cross V to C is said to be a complex or

Hermitian inner product on v, if it satisfies condition 1 dot comma v and v comma dot are linear and conjugate linear respectively. What does this mean?

Well, the linearity part is rather easy to say, what it means is for all V in V and u comma w in V we have u plus w comma V is u v plus w v as expected and similarly V comma u plus w is v u plus v w as expected. And we also have for all C in the complex numbers C u comma v is C u comma v again as expected. And the twist comes this conjugate linearity twist comes in this portion when you write u comma C v you actually get C conjugate u v ok.

So, if you fix the second slot what you get is a linear mapping from V to C. On the other hand, if you fix this first slot what you get is not quite linear because when you multiply by a complex scalar what comes out is the conjugate of that complex scalar such things are known as conjugate linear or anti linear or something like that ok. Now, this was the first property, now you should convince yourself that this is not so unnatural.

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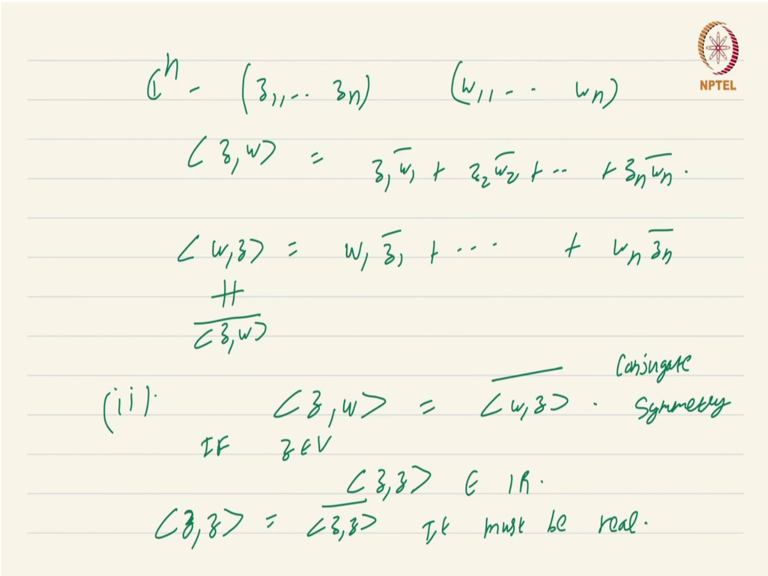

Because you must be familiar that in C n if you take two vectors z 1 to z n and w 1 to w n, then the standard inner product on C n of z comma w is z 1 w 1 conjugate plus z 2 w 2 conjugate plus dot dot dot z n and w n conjugate. So, this is a standard inner product on the complex space C n.

Now, observe that for this standard inner product on C n when you interchange the role of w and z what you get is w 1 z 1 conjugate plus dot dot dot w n z n conjugate, which is not quite equal to inner product z comma w, but its rather equal to the conjugate ok. So, this motivates the second property we have inner product z w equal to inner product w z conjugate.

Now, once you put this condition this is sort of like conjugate symmetry, once you put this condition it becomes clear that if z is in the vector space V, then inner product z z has to be real number; has to be a real number. Why is this the case? Well, observe that when you interchange z comma z you are supposed to get z z conjugate by the conjugate symmetry condition. Which means that the inner product of z and z is a complex number whose conjugate is itself therefore, it must be real; it must be real, ok.

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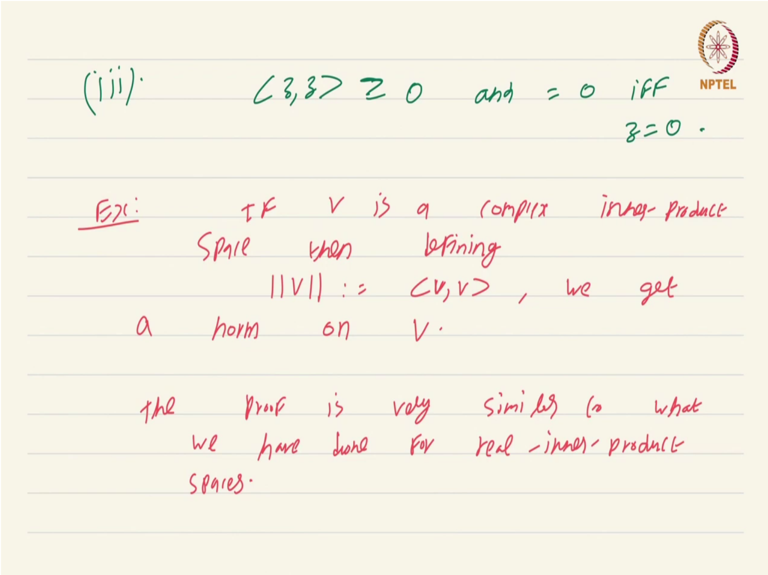

And the 3rd condition as you can expect is that inner product z z must be greater than or equal to 0 and equal to 0 if and only if z is equal to 0. So, these are the three conditions that are there in the definition of a complex inner product, they sort of mirror the three conditions that we had for a real inner product space ok.

Now, a trivial fact that you can check trivial fact you can check if V is a complex inner product space; inner product space then defining norm V to be equal to inner product V v inner product v v we get a norm on a V. So, using a complex inner product you get a norm.

So, this will involve proving things like Cauchy Schwarz inequality and all that for complexity product spaces and the proofs are very similar. The proof is very similar to what we have done; what we have done for real inner product spaces real inner product spaces. Little bit of twisting around needs to be done to take care of the conjugate linearity and conjugate symmetry and it is not really that hard ok. Now, that we have a definitions of complex inner product spaces we can define a norm one second.

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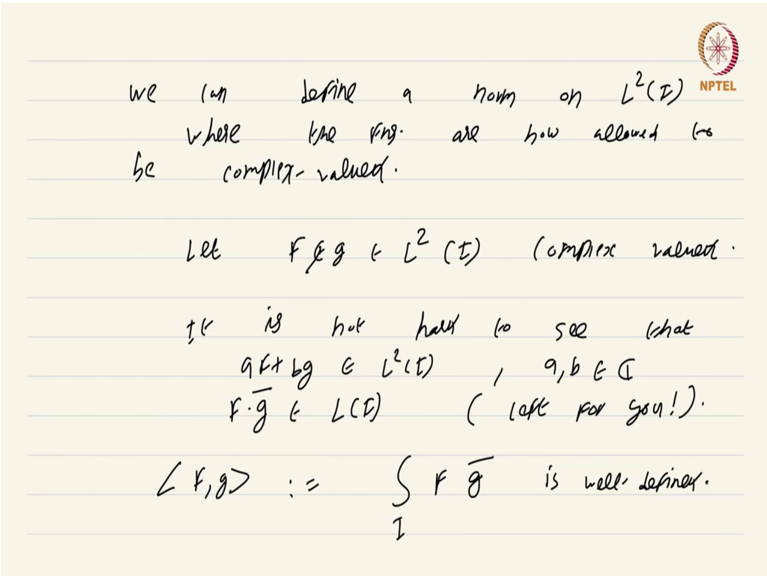

We can define a norm on L 2 of I where the functions are now allowed to be complex valued are now allowed to be complex valued. Well, the definition is as follows let F be F comma g be in L 2 I complex valued then as we have seen before it is not hard to see it is not hard to see that aF plus bg is also an L 2 I where a comma b are now complex numbers. And so and F into g is in L of I ok.

So, these two are left for you they are easy straightforward checks look up the proofs of what we did for the real valued case in the last few videos and adapt the arguments to the complex valued case you will get F comma aF plus bg is in L 2 I and so, is the product F F g ok. So, you define the inner product of F comma g to be by definition integral of F g conjugate. In fact, I should say F g conjugate is an L of I. So, check that this is satisfied that integral F g conjugate over I is well defined.

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$$||f|| := \left(\int_I |f|^2\right)^{\frac{1}{2}}.$$

All the properties of a complex inner product will be satisfied

$\langle f,f\rangle = 0$ iff $f=0$.

$L^2(I)$ complete in this "semi-norm".

And of course, norm of a vector F would be just integral of mod F squared over I ok. Now, the fact that mod F squared is going to be in one second this is whole power half how did I forget that. So, the fact that this is well defined will follow in exactly the same way you did for the real valued case.

So, and this all the properties all the properties of a complex inner product inner product will be satisfied will be satisfied with the one standard exception it need not be true that norm F is 0, not norm F is 0 it may not be true that inner product F comma F is 0 if and only if F is 0 this need not be true.

What will happen is if you can have functions which are almost everywhere 0 such that what will happen is the inner product will still be 0, but the function is not identically 0 ok. So, as I mentioned before the Cauchy Schwarz inequality and the triangle inequality are both satisfied in this scenario as well because this definition gives an inner product.

So, now, what we are going to do is we are going to show in the next videos that L 2 of I now complex valued is sort of complete in this norm; complete in this norm in this semi norm to be precise because all properties are not there. The property that norm F equal to 0 implies F equal to 0 is not true. So, we will show that L 2 of I is actually a complete metric space. This is a course on Real Analysis and you have just watched the video on Norms and Inner Products on Complex Vector Spaces.