Real Analysis II

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Lecture – 3.1
3.1 Continuity in Metric Spaces

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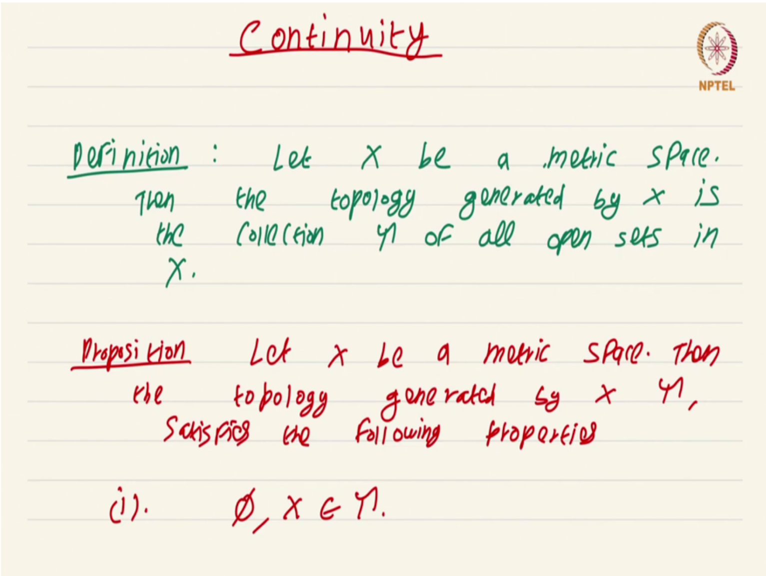

We are now going to begin the serious study of continuity. Our approach will be through what is known as topology. Open sets completely characterize continuity of a function between metric spaces. So, we pause and first define the topology in a metric space.

Definition: Let X be a metric space, then the topology generated by X is the collection τ of all open sets in X.

So, you take every open set and shove it all in this collection, and you get this topology typically denoted by τ. Now, I will prove a very basic property about the topology associated with a metric space.

And this proposition is rather simple to prove, and you have seen a very similar proposition in the context of real numbers.

Proposition: Let X be a metric space. Then the topology generated by X τ satisfies the following properties.

(i) The empty set and the whole space X are elements of τ. So, this is a natural property that follows immediately actually.

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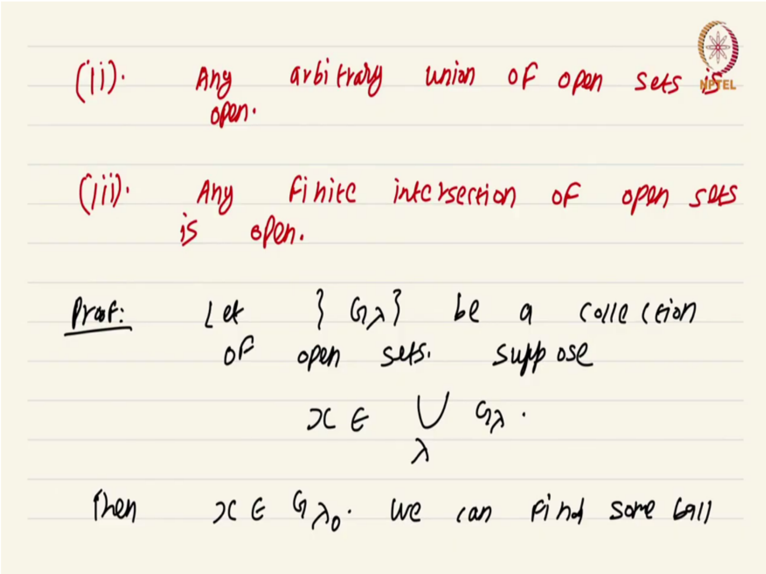

(ii) Any arbitrary union of open sets is open.

(iii) Any finite intersection of open sets is open.

You have seen all these properties in the context of the metric ℝ, and the proofs are more or less the same. So, again I am going to be very brief.

Proof: Part 1 is utterly trivial. So, I am not even going to bother proving it.

Let $\{G_\lambda\}$ be a collection of open sets. Suppose

$$x \in \cup_\lambda G_\lambda.$$

Then $x \in G_{\lambda_0}$. It has to be at least in one of the G_{λ_0}. Therefore, we can find some ball.

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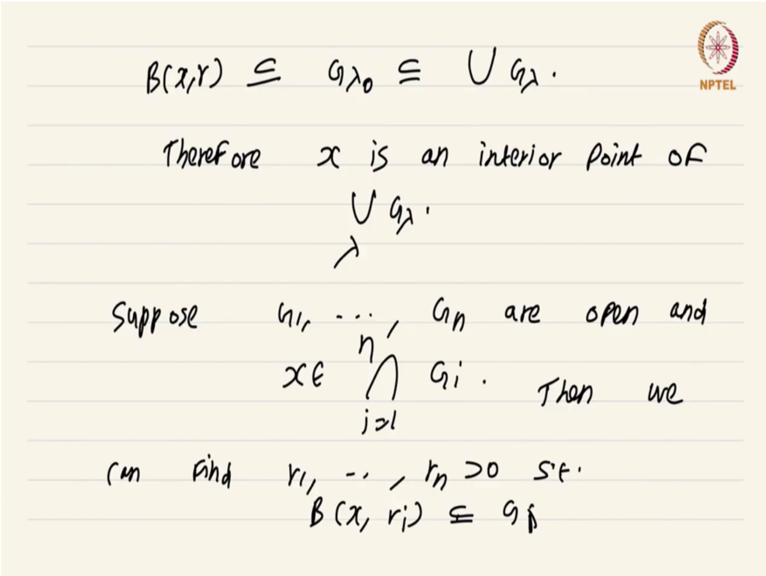

$$B(x,r) \subseteq G_{\lambda_0} \subseteq \cup\, G_{\lambda}.$$

Therefore, x is an interior of the $\cup\, G_{\lambda}$. That was rather easy. The intersection case is equally easy.

Suppose $G_1, \ldots, G_n$ are open and

$$x \in \cap_{i=1}^{n} G_i.$$

Then we can find, $r_1,\ldots,r_n > 0$ such that

$$B(x, r_i) \subseteq G_i.$$

This is just because each G_i is open. Therefore, each x is an interior point of G_i and hence we can find some ball of positive radius centered at x which is fully contained in G_i.

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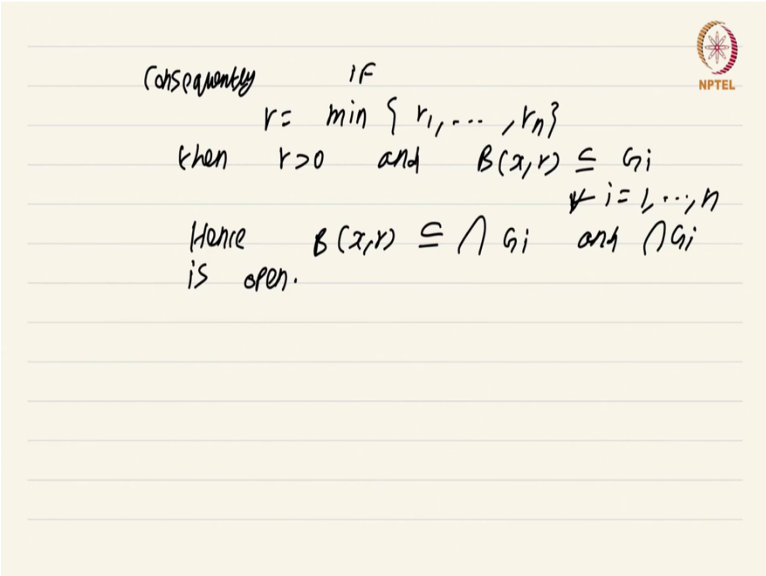

Consequently, if

$$r = \min\{r_1, \dots, r_n\},$$

then $r > 0$ and $B(x,r) \subseteq G_i, \forall i = 1, \dots, n$. Hence $B(x,r) \subseteq \cap\, G_i$ and $\cap\, G_i$ is open. Very very easy proofs; rather, these are proofs that are sort of automatic. You just start writing down what it is that you must prove, and the proof just falls into place, ok.

You have an analogous result for closed sets. I leave it to you to think about closed sets and what you can say about unions and intersections of closed sets. Now, let me move to the next page because we will introduce the notion of continuity. As usual, instead of just having already seen the definition, we will now characterize it just like what we did in $\mathbb{R}$ in several ways this notion of continuity. One key change is that our focus will now be primarily on open sets.

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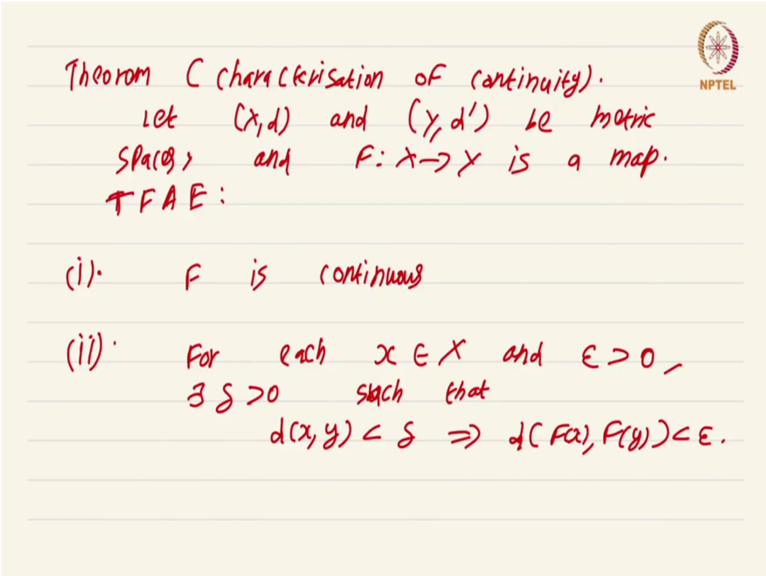

So, we have this theorem, and this is a rather important theorem because it will help you understand continuity in multiple ways.

Theorem (characterization of continuity): Let (X, d) and (Y, d') be metric spaces, and $f: X \to Y$ is a map. Then the following are all equivalent.

(i) f is continuous.

(ii) For each $x \in X$ and $\epsilon > 0, \exists \delta > 0$ such that

$$d(x,y) < \delta \Rightarrow d\big(f(x), f(y)\big) < \epsilon.$$

The same epsilon delta definition that had given us nightmares when we studied continuity in $\mathbb{R}$, the same nightmares will now repeat for metric spaces.

The only difference between this version of epsilon-delta and what you have seen earlier is that the distance between x and y has replaced the absolute value. As I have repeatedly emphasized, much of this study of metric spaces would be just taking the definitions and theorems that we have already seen in the chapter on the taste of topology and wherever you see absolute value just replace it by $d(x, y)$, and this is sort of going to illustrate that.

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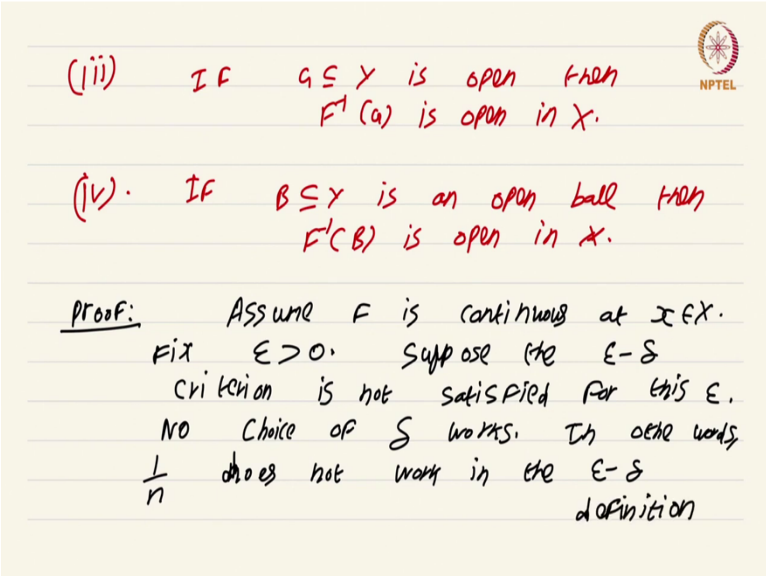

The next condition we have not studied in depth when we studied continuity in the real numbers; we had just dealt with it briefly. This characterizes continuity entirely in terms of topology. And this is going to be very important because it is this equivalent definition of continuity that generalizes to the context of topological spaces.

(iii) If $G \subseteq Y$ is open, then $f^{-1}(G)$ is open in X.

So, of course, the first open is in Y, or that is obvious. So, you can briefly say that continuous functions pull back open sets to open sets. And the fourth condition is just a minor variant of this

(iv) If $B \subseteq Y$ is an open ball, then $f^{-1}(B)$ is open in X.

So, you need not check that the pullback of every single open set and Y is an open set in X. You need to do this only for open balls that suffice.

So, conditions (iii) and (iv) characterize continuity entirely in terms of open sets. And when you come to the no study of topological spaces where you cannot measure the distance between two points and you will also learn that sequences are also not adequate to characterize such things in a topological space, conditions 3 and 4 will save the day. We can characterize continuity entirely in terms of open sets.

So, on to the proof, some parts will be very familiar to you; the arguments will be very similar to what we have done; only a little bit of novelty is involved now.

Proof: Assume f is continuous at $x \in X$. Remember, this means that whenever you have a sequence x_n converging to x, the sequence $f(x_n)$ converges to $f(x)$. This is the definition of continuity we are taking. So, fix $\epsilon > 0$.

Now, we are essentially going to show that the sequential definition of continuity automatically implies the epsilon-delta version of continuity. Now, suppose the epsilon-delta criterion is not satisfied for this epsilon. I am going to show that this is simply not possible.

Given any fixed epsilon, the epsilon-delta criterion has to be satisfied; if not, we are going to get a contradiction. What is this mean? This means that no choice of delta works. In other words, $\frac{1}{n}$ does not work in the definition of epsilon-delta.

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I am sure you can anticipate where this is going. What this means is that given or not given rather.

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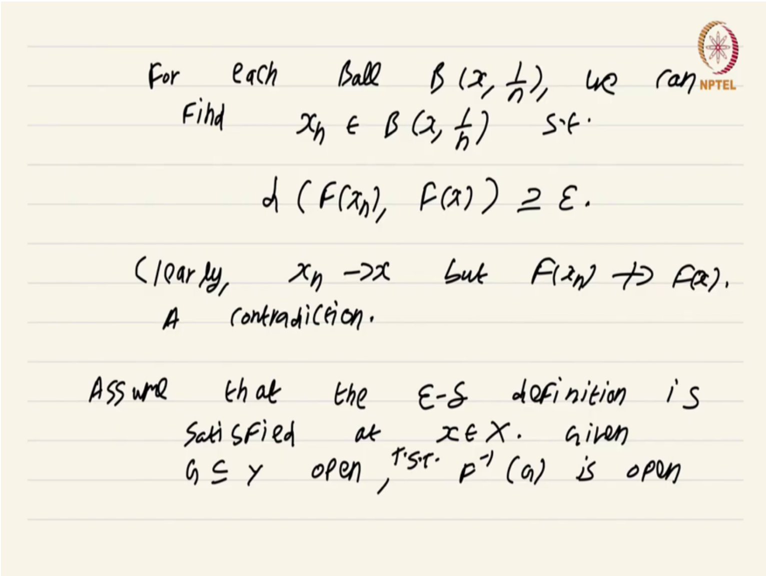

For each ball $B\left(x, \frac{1}{n}\right)$, we can find $x_n \in B\left(x, \frac{1}{n}\right)$ such that

$$d\big(f(x_n), f(x)\big) \geq \epsilon.$$

None of the choices $\frac{1}{n}$ work. Note that this $B\left(x, \frac{1}{n}\right)$ is precisely those points in the metric space X which are at the max $\frac{1}{n}$ distance away from the given point x.

So, this is the same as this is just another way of saying that

$$d(x, x_n) < \frac{1}{n}.$$

So, the fact that the epsilon-delta definition fails simply means that you can find a point x_n such that $d\big(f(x_n), f(x)\big) \geq \epsilon$. But clearly, x_n converges to x, but $f(x_n)$ does not converge to $f(x)$, this is a contradiction.

So, this shows that the sequential definition of continuity guarantees that the epsilon-delta definition will be satisfied ok. Now, we are going to go to the next part.

Assume that the epsilon-delta definition is satisfied. And even though I state this result as a global one that is continuity is assumed at all points x, I am just going to prove it locally.

So, the proof will say a bit more than what the statement is asking us to do, but it is useful to have this more general fact, if not in the statement, at least in the proof. So, assume that the epsilon-delta definition is satisfied at a particular point $x \in X$. Now, what I have to do is I have to show that given $G \subseteq Y$ open, I am just going to show that $f^{-1}(G)$ is also open. That is what we have to show. We have to show that $f^{-1}(G)$ is also open.

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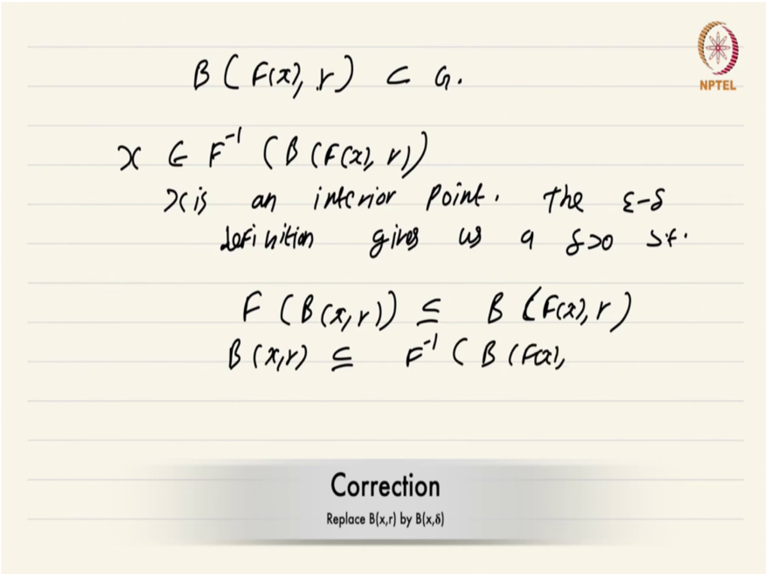

Note that because G is open, we can find some ball, $\mathrm{B}(\mathrm{f}(\mathrm{x}), \mathrm{r}) \subseteq \mathrm{G}$ which simply because G is an open set in Y. Now, what we are going to do is show rather I mean since here I am not going to show that $f^{-1}(G)$ is open, I am going to show, given $G \subseteq Y$ open, and $f(x) \in G$, this is important. Sorry about that, I missed an important thing.

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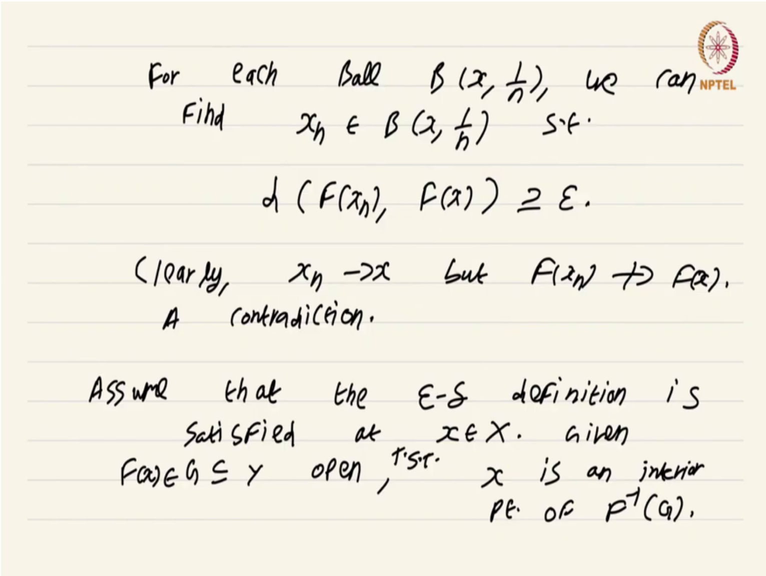

$G \subseteq Y$ open, $f(x) \in G$. I will show that x is an interior point of $f^{-1}(G)$. So, I will be very precise. The statement asks us to prove that whenever you take an open set $G \subseteq Y$, we have to show that $f^{-1}(G)$ is open in X.

I will prove the much stronger claim that if the epsilon-delta definition is satisfied at a particular point $x \in X$, then no matter what open set G you take in Y that contains the point $f(x)$. I will show that x is an interior point of $f^{-1}(G)$.

In fact, from this the fact that $f^{-1}(G)$ will be open if f were continuous through at all points or rather if f satisfies the epsilon-delta definition at all points is rather obvious, and I am going to leave it to you. So, coming back, I am we have this

$$B(f(x), r) \subseteq G.$$

So, I am going to show that $f^{-1}(\ B(f(x), r)$ for which we have this x is obviously, an element of this.

I am going to show that x is an interior point of $B(x, r)$. Now, this is rather easy, the epsilon-delta definition gives us a $\delta > \ 0$ such that

$$f\big(B(x, r)\big) \subseteq B(f(x), r).$$

Here r plays the role of epsilon that is all, that means,

$$B(x,r) \subseteq f^{-1}(B(f(x),r).$$

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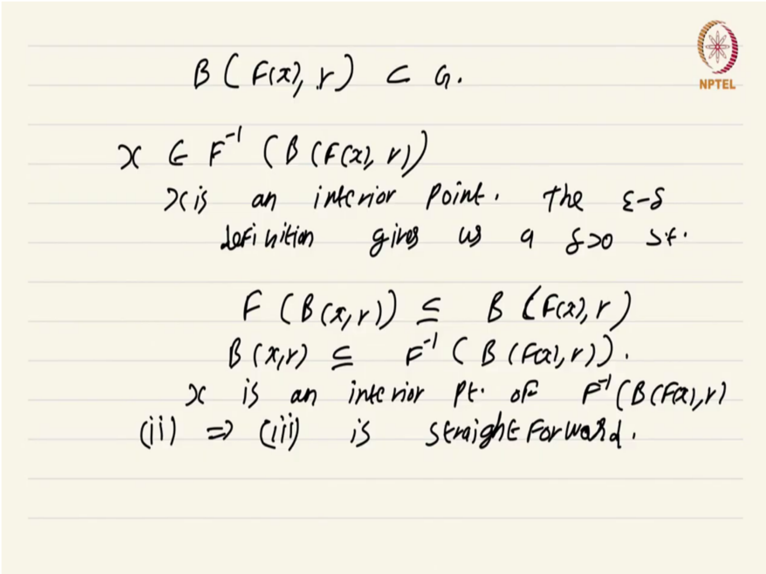

This immediately shows that x is an interior point of $f^{-1}(B(f(x),r)$. Now, it is the fact that (ii) implies (iii) is straightforward. I want you to think about how with this stronger fact that whenever you have continuity at a point, you automatically get that x will be an interior point of an inverse image of a ball that contains $f(x)$. So, I want you to think about it. How to prove the full statement (iii) from the stronger fact is rather easy.

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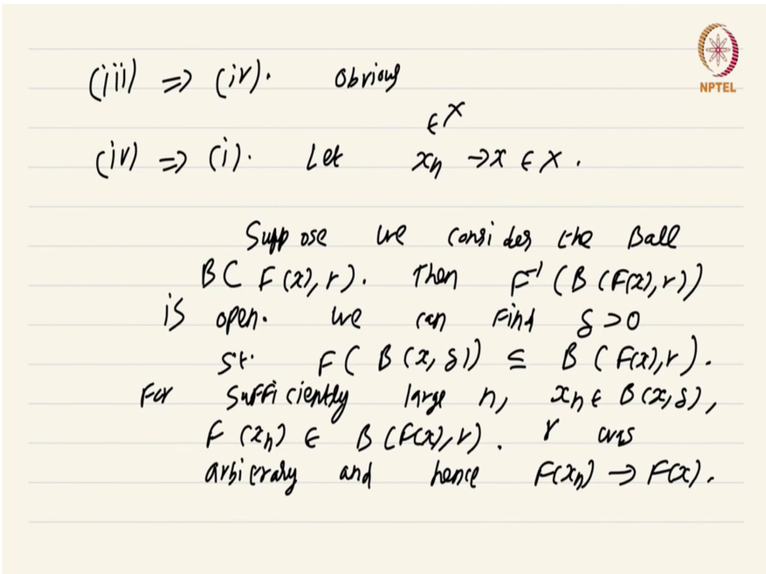

Now, (iii) implies (iv) is nothing. (iii) implies (iv) asks us to show that if whenever G is open, $f^{-1}(G)$ is open. We must have whenever B is open, $f^{-1}(B)$ i is open where B is a ball. So, finally, (iii) implies (iv) is obvious. Now, we have to show (iv) implies (i), that is, we have to show that whenever we have this condition that when you take a ball we have $f^{-1}(B)$ is open, we will have to show that if x_n converges to x, then $f(x_n)$ converges to $f(x)$.

So, let x_n converges to x. So, x_n is a sequence in X and x is a point in X. And you have a sequence converging to the point x; we have to show that $f(x_n)$converges to $f(x)$. Now, we already know that, so suppose we consider the ball $B(f(x), r)$.

Then $f^{-1}\big(B(f(x), r)\big)$ is open; that is the hypothesis. This means we can find $\delta > 0$ such that

$$f\big(B(x, \delta)\big) \subseteq B(f(x), r).$$

But because $f\big(B(x, \delta)\big) \subseteq B(f(x), r)$ and x_n converges to x for sufficiently large n, $x_n \in B(x, \delta)$. Consequently $f(x_n) \in B(f(x), r)$.

Because of the convergence of x_n to x for suitably large n, $x_n \in B(x, \delta)$, but $f(x_n) \in B(f(x), r)$. So, $f(x_n)$ is contained in $B(f(x), r)$. So, this r was arbitrary. And hence $f(x_n)$ converges to $f(x)$. So, the proof again was rather short.

So, we have shown that (i) implies, (ii) implies, (iii) implies, (iv) implies, and so all are equivalent. In the exercises, I am going to ask you to show some of these equivalences individually, directly showing that (iii) is equivalent to (iv), and (ii) is equivalent to (iv), and whatnot. I am going to leave a few exercises for you.

Another exercise that is again rather easy is to show that when set f is closed in Y, then $f^{-1}f$ is closed in X. So, you can characterize continuity completely using closed sets also. A mapping is continuous if and only if it pulls back open sets to open sets; equivalently, it only pulls back closed sets to closed sets.

So, this theorem sort of characterizes continuity completely in terms of open sets. Now, one important class of continuous functions that will be very, very useful in our study of metric spaces, and more importantly, in the future study of functional analysis, are linear mappings. And it is very easy to characterize linear mappings whether they are continuous or not.

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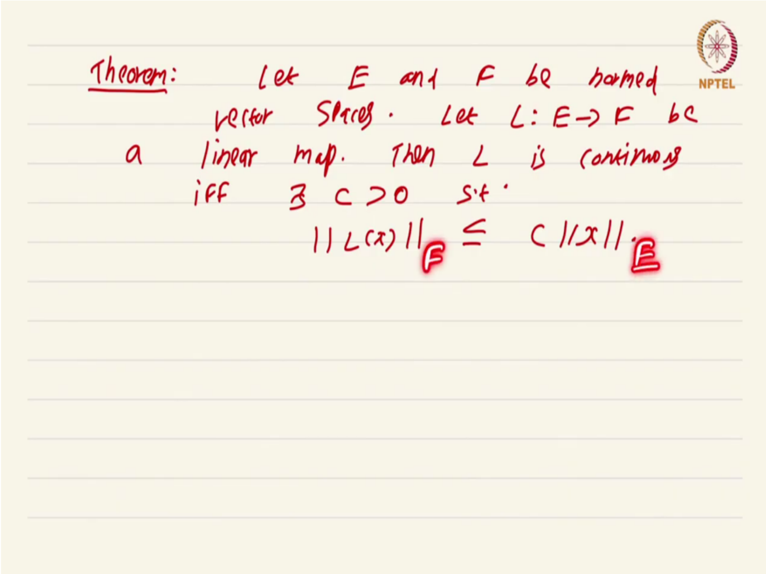

Theorem: Let E and F be normed vector spaces. Let $L : E \to F$ be a linear map.

Now, the temptation is to conclude that L will be continuous automatically, but that does not truly solve the exercises to see an explicit example where you can have a linear mapping between normed vector spaces that are not continuous.

Now, thankfully, it is true that whenever E and F are both finite-dimensional. It is always true that L is going to be continuous just because you do not even require the finite dimensionality of F that is again going to be left as an exercise for you to analyze what happens in the finite-dimensional case.

But, in the infinite-dimensional case, just linearity does not guarantee that L is going to be continuous. However, the condition for L being continuous is rather simple.

Theorem: Let E and F be normed vector spaces. Let $L : E \to F$ be a linear map. Then L is continuous if and only if there exist $c > 0$ such that

$$\|L(x)\| \leq c\,\|x\|.$$

Again I am committing abuse of notation by using the same notation for the norm in both the domain and the codomain. This is a common abuse of notation. If otherwise, if you are not satisfied with this, you can put the subscript if you want; you can put a subscript E here, and sorry the subscript F here and a subscript E here if you want, but that is just too much notational overburden.

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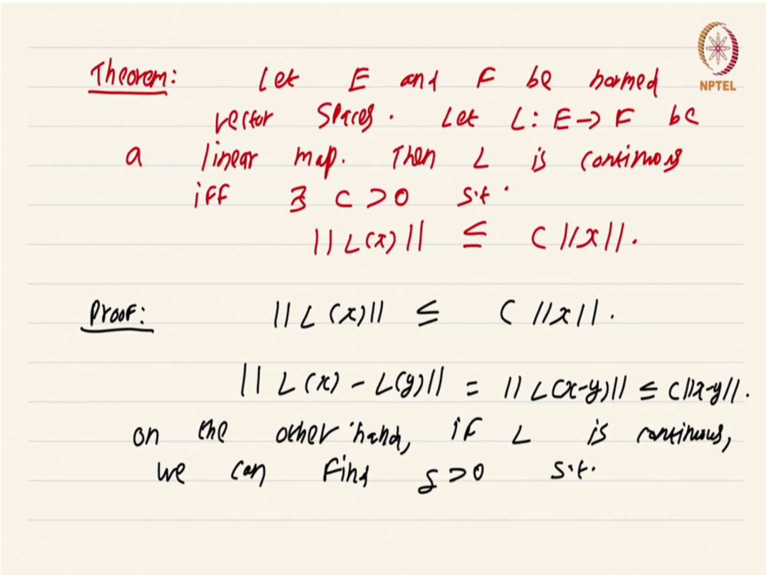

So proof, the proof is not at all difficult.

Proof: Suppose you have

$$\|L(x)\| \leq c \, \|x\|.$$

Then L is, in fact, Lipschitz, not just continuous; it will be Lipschitz. How do you see that? Because

$$\|L(x) - L(y)\| = \|L(x - y)\| \leq c \, \|x - y\|.$$

In some earlier chapter, we had briefly studied Lipschitz functions, which shows that L is Lipschitz, and therefore, it is continuous. On the other hand, if L is continuous, we can find, we can find; we can find $\delta > 0$.

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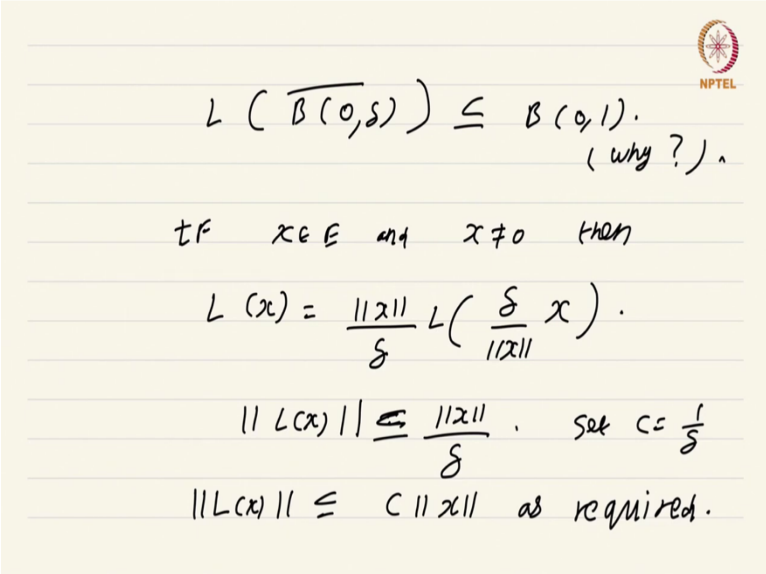

Such that

$$L\left(\overline{B(0,\delta)}\right) \subseteq B(0,1).$$

That is, you take the open unit ball in the codomain Y, you can find a δ such that L maps the close unit ball $\overline{B(0,\delta)}$ inside $B(0,1)$. I want you to think about why this is true. I want you to think about why this is true.

Again let me tell you, this is not at all hard. If you have understood the equivalent characterizations of continuity, this should be less than 30 seconds argument why this is true. That means, if $x \in E$, and $x \neq 0$, then

$$L(x) = \frac{\|x\|}{\delta} L\left(\frac{\delta}{\|x\|} x\right).$$

Again I am just using linearity. And, $\frac{\delta}{\|x\|} x$ is going to be an element of $\overline{B(0,\delta)}$. Because of that, this is going to be contained in $B(0,1)$. In other words,

$$\|L(x)\| = \frac{\|x\|}{\delta} \|L\left(\frac{\delta}{\|x\|} x\right)\| \leq \frac{\|x\|}{\delta} . 1.$$

This just follows because this quantity $L\left(\frac{\delta}{\|x\|}x\right)$ is here. This whole thing, this image is there in this. So, the norm of that is less than or equal to 1. In other words, if we set $c = \frac{1}{\delta}$, we get

$$\|L(x)\| \leq c\|x\|$$

Again this proof was not complicated at all. So, there is an easy way to characterize linear mappings that are continuous between normed vector spaces. Now, I will make a remark that would, at the outset, seem rather silly, but it needs to be made, especially when you are going to study functional analysis in the future.

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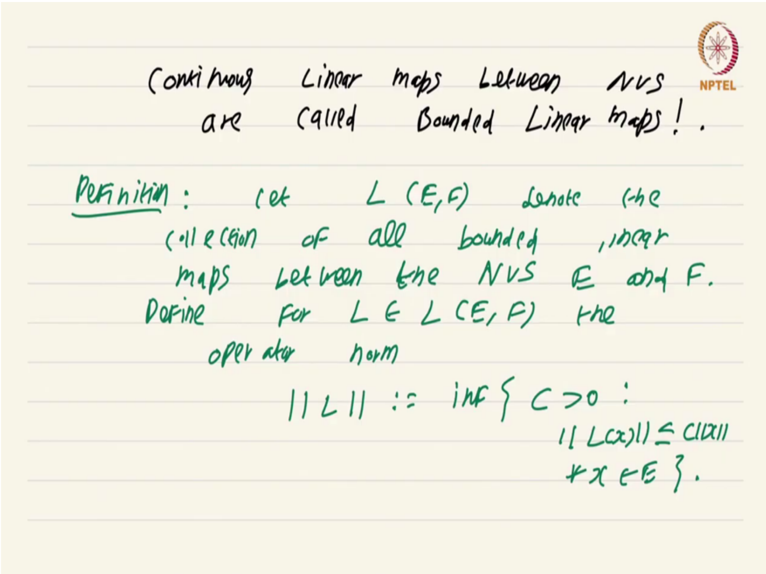

Normed vector spaces when you have linear mappings that are continuous between normed vector spaces, so continuous linear maps between normed vector spaces are called bounded linear maps, bounded linear maps. Now, they are called bounded because we have this inequality ok.

Now, you are all familiar with a bounded function; that just means that the range of that function has a positive but finite diameter right that is the usual notion of boundedness. Now, you can check that if this linear map between any two vector spaces, any two norm vector spaces that are not identically zero can never be bounded if you want the diameter of the range to be a positive finite quantity that can never happen at all.

So, this notion of bounded when you say when you use bounded terminology applies only to linear maps between normed vector spaces. So, whenever you see this, do not make the mistake of thinking it is bounded in the usual sense. So, this is bad terminology. We could have just used continuous linear maps and be done with it.

But for whatever reason, these maps are also called bounded linear maps, and we have to live with this terminology. If there was no existing notion of bounded, then this property is a nice way to say that the map is bounded. Unfortunately, there are two different notions of bounded now.

And you have to make sure that you understand what notion of bounded is being meant in that given situation; it is not too hard actually to guess because whenever you are in the situation of linear maps between normed vector spaces, people usually mean the notion that

$$\|L(x)\| \leq c\|x\|$$

So, let me make a definition,

Definition: Let $L(E,F)$ denote the collection of all bounded linear maps between the normed vector spaces E and F.

I will violate my own thing that saying continuous is better, and I am going to follow the norm, no pun intended, of calling continuous linear maps bounded.

So, we consider the collection.

Definition: Let $L(E,F)$ denote the collection of all bounded linear maps between the normed vector spaces E and F. Define, for $L \in L(E,F)$ the operator norm

$$\|L\| = inf\{c > 0 : \|L(x)\| \leq c\|x\| \ \forall x \in E\}.$$

You look at the collection of all c's such that $\|L(x)\| \leq c\|x\| \ \forall x \in E$.

We know that there is at least one because this L is a bounded linear map. Look at the least possible one, the infimum, defined as the operator norm of L. Now that this is a norm is a check for you to do that is rather easy to do. So, let me leave it as an exercise.

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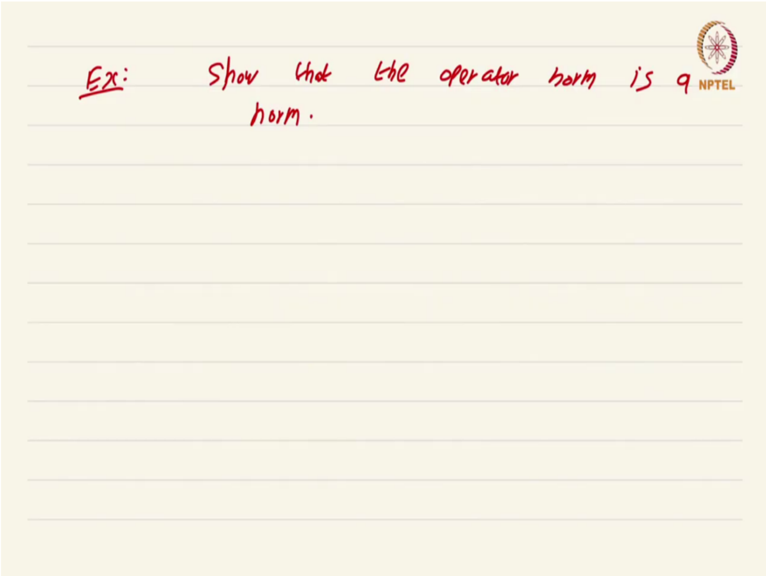

Exercise: Show that the operator norm is a norm.

It is rather easy to show. And in the exercises I am in the notes, there are several other formulas I am giving you to check for the operator norm. This is not the only way to find out what the operator norm is. There are several other ways. Please solve that exercise that will give you an idea about this operator norm.

Now, let me make one more remark on why this is called the operator norm. Usually, linear maps from a vector space to itself are often called an operator, especially in norm vector spaces. If you have a bounded linear self map, it is often called an operator. And this is the norm that you put on the space of operators.

Many authors call linear mappings between two different normed vector spaces also as operators. So, just for not having too many notations, this is just called the operator norm. So, this was a brief study of continuity in the context of metric spaces and normed vector spaces. Please do solve the exercises; there are some interesting ones. This is a course on Real Analysis, and you have just watched the video on continuity.