Real Analysis II
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Lecture - 33.2
Square - Integrable Functions

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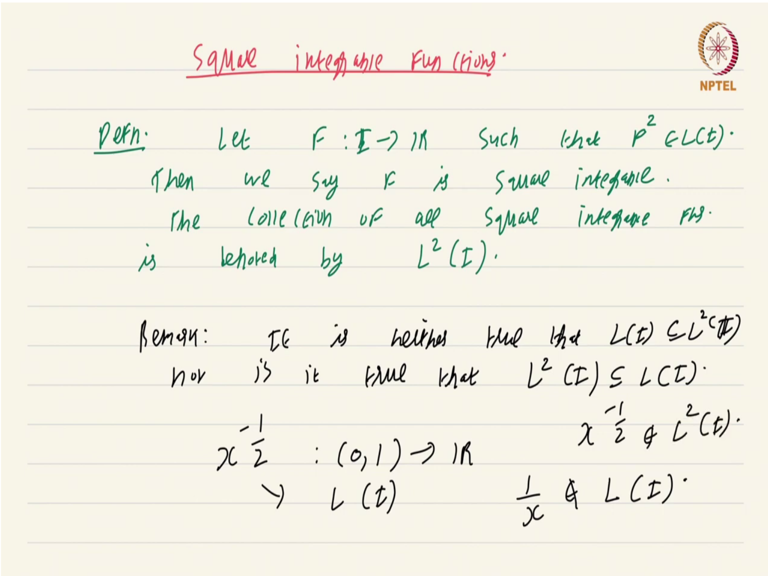

In this video we are going to see the definition of the space of Square Integrable Functions. This space is all important in the study of Fourier analysis. We will not study much about Fourier analysis in this course, but what we are about to do in this video will definitely serve as good motivation for the study of Fourier analysis in a future course.

So, the definition of square integrable function is sort of an illustration of a statement that I keep making that mathematicians are not very creative when it comes to naming stuff. Let F be a function from I to R such that F squared is in L of I, then we say F is square integrable.

So, those functions whose square are integrable are called square integrable functions. The collection of all square integrable functions square integrable functions is denoted by L squared I or L 2 of I more commonly. Now, first of all let me make a remark it is neither true that L of I is a subset of L 2 of I nor is it true is it true that L 2 of I is a subset of L of I.

Neither of these statements are true. So, to see this consider the function x power minus half on the interval 0 1 ok. You can check that this function is going to be in L of I, but the square of this function which is just 1 by x this is not in L of I ok. So, this function x bar minus half is not an element of L 2 of I its square is not integrable.

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$\frac{1}{x}$ on $[1, +\infty)$ then this fn. is not Lebesgue integrable but

$$\frac{1}{x^2} \in L([1, +\infty)].$$

Proposition: Suppose $f: I \to \mathbb{R}$ is Lebesgue integrable and bounded a.e. Then $f^2 \in L(I)$ or in other words $f \in L^2(I)$.

On the other hand, if you consider the function 1 by x on plus 1 infinity ok, then this function is not this function is not Lebesgue integrable as you can check is not Lebesgue integrable, but 1 by x squared is in L of 1 infinity. So, neither containment that you expect is not going to hold true. However, we can write a sort of a trivial proposition which sort of gives one common criteria for checking whether a function is square integrable.

Suppose, F from I to R is Lebesgue integrable is Lebesgue integrable and bounded almost everywhere, then F squared is Lebesgue integrable or in other words in other words F is an element of L 2 of I. So, if you have a function that is both Lebesgue integrable and almost everywhere bounded above then it is going to be Lebesgue integrable the square is going to be Lebesgue integrable.

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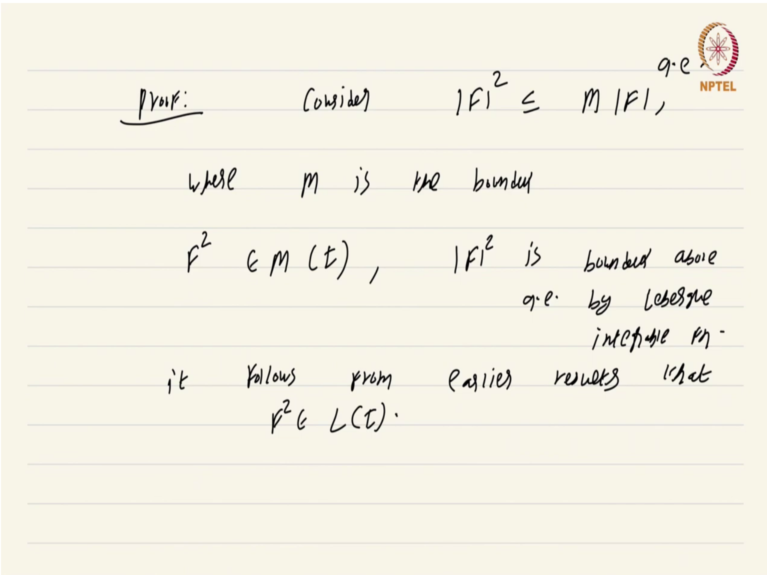

Well, the proof is really nothing proof is really nothing. Well consider mod F squared; well this mod F squared is going to be less than or equal to m times mod F where m is the bound of course, this is true almost everywhere ok.

So, this means that the function F squared which is of course, measurable because F is a measurable function being Lebesgue integrable and modulus of F squared is bounded above is bounded above almost everywhere by a Lebesgue integrable function by a Lebesgue integrable function namely M times mod F by a Lebesgue integral function it follows from earlier results that F squared is Lebesgue integrable as claimed.

So, this is one really simple and stupid criteria to check whether a function is Lebesgue integrable sorry a square integrable. Now, another natural question that crops up is what type of space is the space of square integrable functions. Well thankfully it is a vector space which is the essential content of the next theorem.

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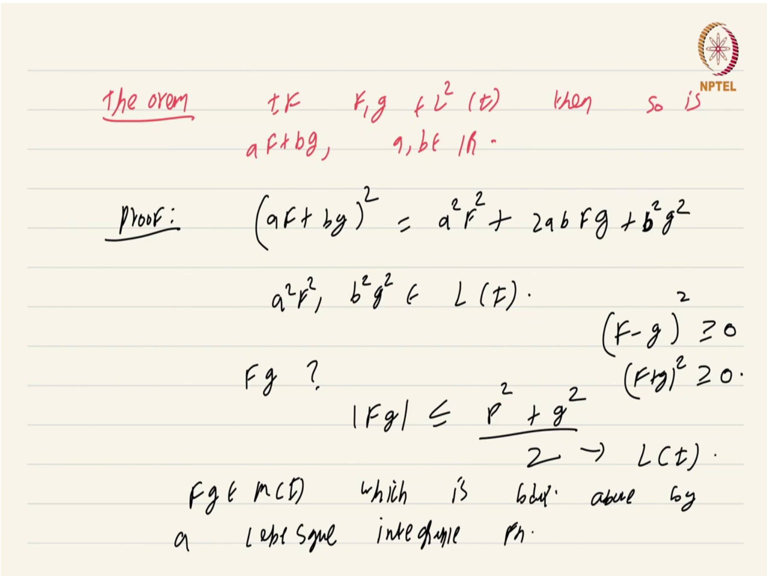

If F comma g are in L 2 of I then so, is a F plus bg where a and b are any real numbers. This is essentially showing that the space of Lebesgue square integrable functions is actually going to be a vector space. Proof; well what is a F plus bg the whole squared. Well this is nothing, but a squared F squared plus 2 a b F g plus b squared g squared ok.

Now, we already know that a squared F squared and b squared g squared are Lebesgue integrable simply because F and g are assumed to be square integrable. What about this middle term F g? Is this also going to be Lebesgue integrable? Is that clear, well observe that mod F g is going to be less than or equal to F squared plus g squared by 2 ok. This just follows from observing that F minus g the whole squared is greater than or equal to 0 and F plus g the whole squared is also greater than or equal to 0.

From that it follows immediately that mod F g is less than or equal to F squared plus g squared by 2, but this is Lebesgue integrable simply because F squared and g squared are therefore, F squared plus g squared is and F squared plus g squared by 2 is also a Lebesgue integrable. So, F g is a measurable function which is bounded above which is bounded above by a Lebesgue integrable function therefore, by an earlier result F g will be Lebesgue integrable.

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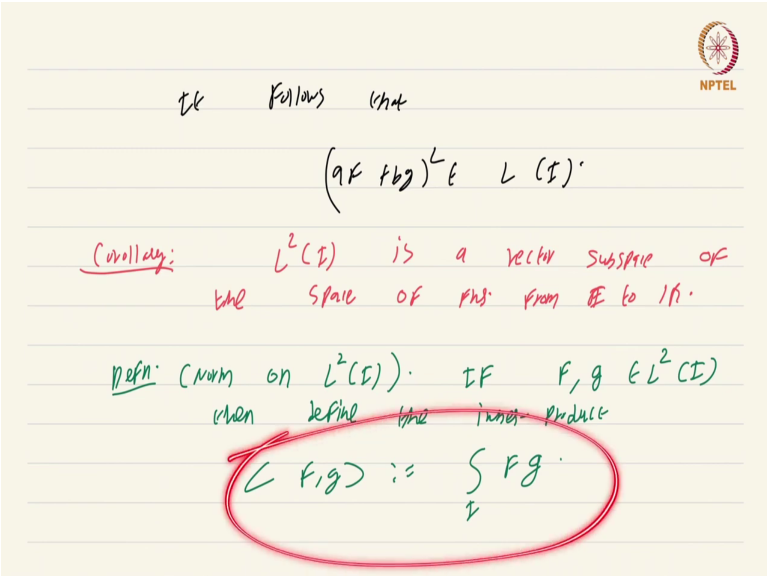

It follows that it follows that a F plus bg the whole squared is also a Lebesgue integrable. So, as a corollary of this simple observation we get that the space L 2 I is a vector subspace of the space of functions space of functions from I to R. So, the space of square integrable functions is going to form a vector subspace.

Now, we can define a norm or on this space L 2 I. Well what is this norm? Definition norm on L 2 I if F comma g are in L 2 I, then define the inner product the inner product of F and g to be just by definition integral over I of F g. This makes sense because we just saw in the previous proposition that if F and g are square integrable, then the product is going to be Lebesgue integrable therefore, this definition makes sense. Now, that you have an inner product the norm is automatic.

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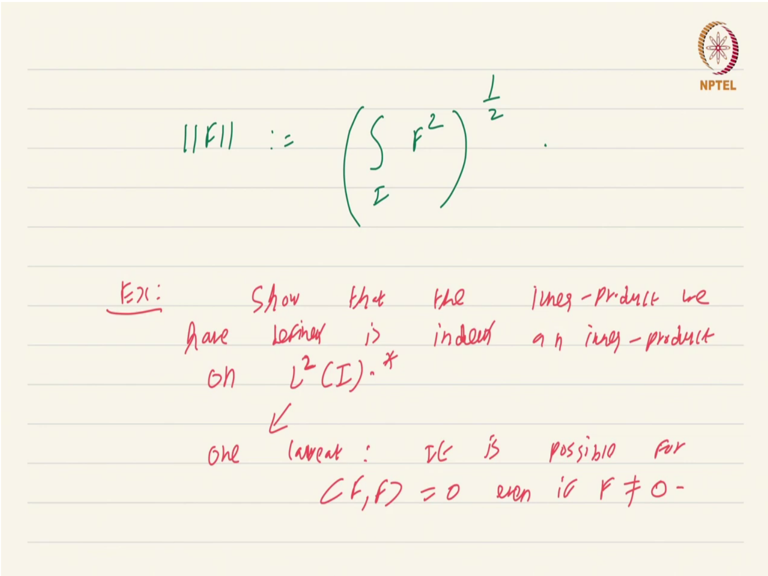

So, norm of a function F would be by definition integral of F squared I whole power half ok. Now, that this thing is an inner product is left as an exercise for you; there is really nothing much to prove. Show that the inner product we have defined we have defined is indeed an inner product on L 2 of I.

Only one property will fail so, let me put a star let me put a star because this is not going to be entirely satisfying all the properties of an inner product. One caveat which you will have to think over it is possible it is possible for inner product F comma F to be equal to 0 even if F is not equal to 0.

The reason is if you have a function which is 0 almost everywhere then that function is automatically going to be in L of I as we have remarked in earlier videos and the integral is going to be 0. So, functions that are not identically 0 could have 0 norm under this inner product and norm.

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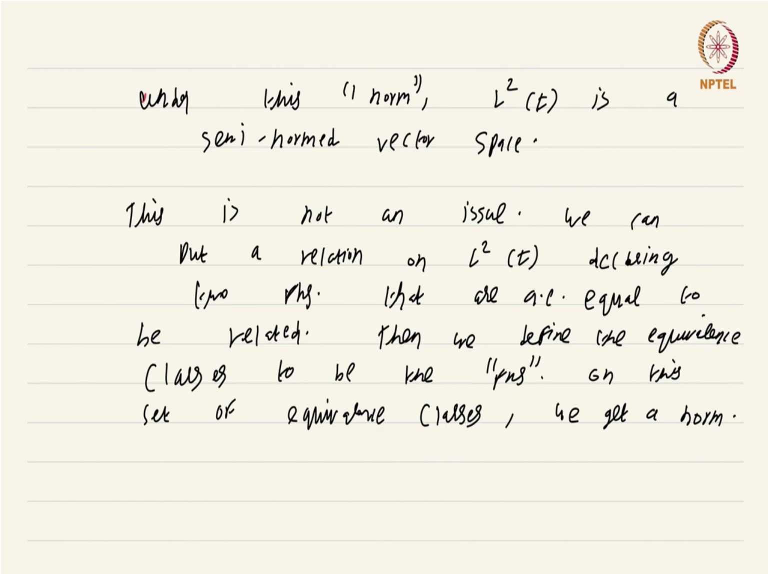

So, what is essentially going to happen is that; under this norm under this norm in quotes L 2 I is a semi normed vector semi normed vector space. So, something that satisfies all the properties of a norm except for the fact that if the norm is 0 the function is 0 that is known as a semi normed vector space.

Now, one thing that I must remark which is sort of a somewhat advanced remark at this stage is that this is not really an issue this is not an issue. After all from the perspective of integration theory two functions that are almost everywhere equal are indistinguishable because their integrals are also equal.

So, what we can do is we can put a relation put a relation on L 2 I declaring two functions that are almost everywhere equal to be related and then we can define the equivalence classes we can define the equivalence classes. So, equivalence classes will consist of functions which are pair wise almost everywhere equal.

We can define the equivalence classes to be the functions because from the perspective of the Lebesgue integral, functions that are almost everywhere defined and almost everywhere equal to other functions should be identified because the Lebesgue integrable is blind to almost everywhere differences, that is sorry the Lebesgue integral is blind to differences on sets of measure 0.

So, two functions that are almost everywhere equal will behave exactly the same from the perspective of the Lebesgue integral. So, what you do is you declare any two functions that are almost everywhere equal to be the same. That is you do that formally by setting up an equivalence relation.

On this set of equivalence classes on the set of equivalence classes we get a norm not just a semi norm ok. So, for many many technical purposes many authors identify functions that are almost everywhere equal and study that space instead. Now, there are some annoying technical issues that will crop up if you go this route. For instance, if you just take a continuous function that makes no sense anymore because this function is part of an equivalence class and that equivalence class has functions that are completely discontinuous everywhere ok.

So, recall that the indicator function of the rationals is not continuous it is discontinuous everywhere, but it is almost everywhere equal to 0. Therefore, the 0 function which is as nice and as continuous as possible is in this perspective equal to the characteristic function of the rational numbers which is everywhere discontinuous. So, this sort of annoying technicalities can prop up, but that is part of life ok. So, one final remark if you do not go to this route.

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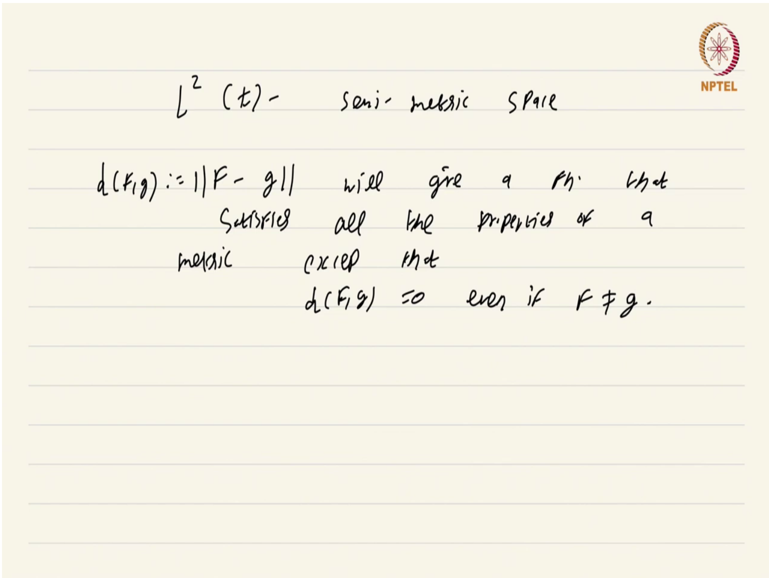

What you end up is the space L 2 I is a semi metric space is a semi metric space. What I mean by that is it the definition d of F g by definition equal to norm F minus g will give a function that satisfies all the properties of a metric space of a metric except that d F g could be equal to 0 even if F is not equal to g.

If F and g just agree almost everywhere still the distance between F and g will be 0. So, this is what is known as a semi metric space. Of course, to check that it is a semi metric space you can just appeal to the abstract results that we have shown. That once you have an inner product you have a norm and once you have a norm you have a metric. This was all done way back in the beginning of this course when we did a detailed study of metric spaces and norm vector spaces and inner product spaces ok.

So, all the properties of the normed spaces will continue to hold except for this one caveat. Let me just summarize the various properties that will hold and I am not going to provide a proof because we have already proved it once when we studied the theory of metric spaces and norm vector spaces.

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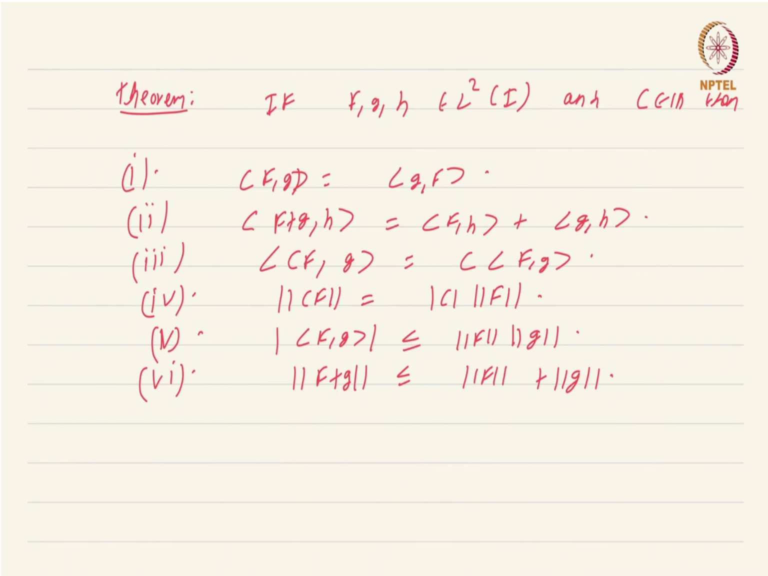

If F g and h are in L 2 of I and C is a real number, then we have these properties we have inner product F g is inner product g F this you have to prove when you show that F the inner product we defined is indeed an inner product you would show this as a part of that, but really there is nothing to show it is just because integral F g is same as integral g F there is really nothing to show.

And then you have integral of F plus g comma h is just integral F h plus integral g h. This is again part of the proof that this is going to be an inner product again there is really nothing to show this just says that integral of F into g plus h is integral F g plus integral F h which is just immediate from the properties of the Lebesgue integral.

And three is the scalar thing integral C F comma g is C times integral F comma g again straight forward trivial to show. And then you get the properties of the norm which I have which we have already shown once when we studied norm vector spaces. Norm C F is norm C sorry absolute value of C times norm of F.

And then we have the famous Cauchy Schwarz inequality which says that absolute value of inner product of F g is less than or equal to norm F norm g. This will follow from the abstract Cauchy Schwarz inequality we proved long back. And finally, a consequence of the Cauchy Schwarz inequality which is the triangle inequality norm of F plus g is less than or equal to norm F plus norm g. So, this is just a summary of the properties enjoyed by this norm on this semi normed vector space L 2 of I. In next week's videos we will see some convergence theorems on L 2 of I and also show that L 2 of I is a complete space.

This is a course on real analysis and you have just watched the video on square integrable functions.