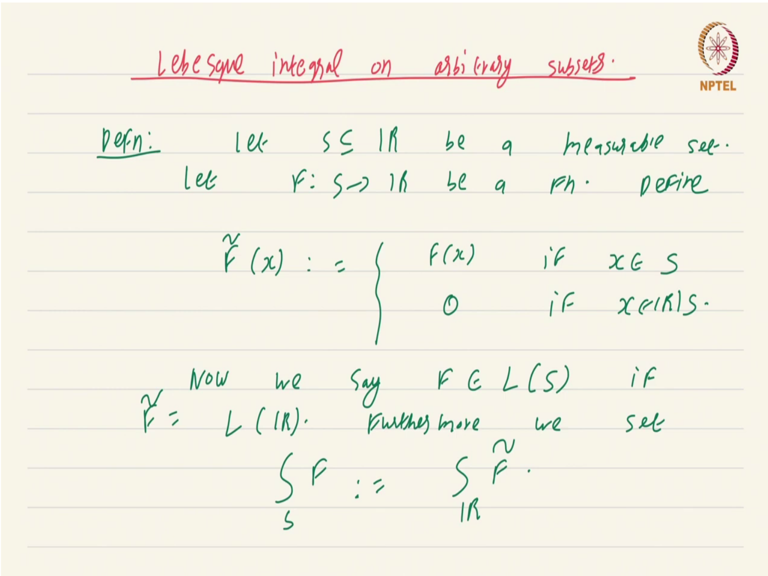
Real Analysis II
Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 33.1
The Lebesgue Integral on Arbitrary Subsets

(Refer Slide Time: 00:23)

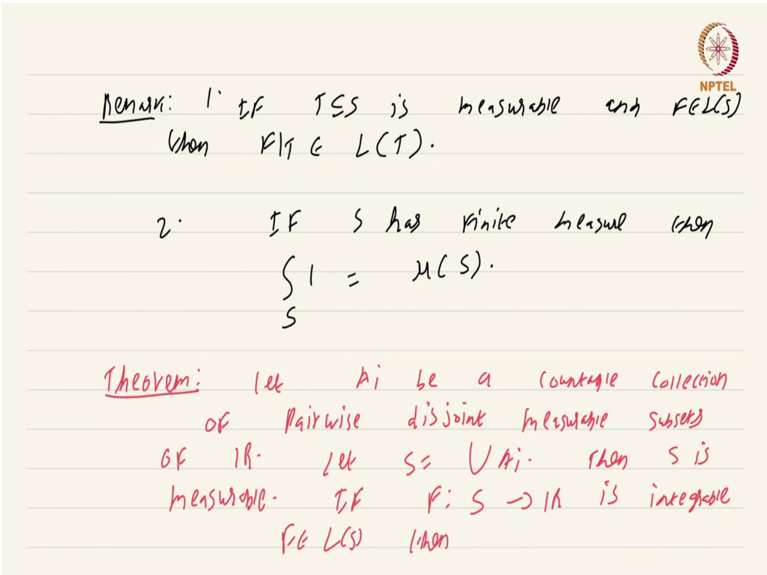


In this short video, let us see how we can define the Lebesgue integral of a function not necessarily defined on an interval, but an arbitrary measurable subset of the real line. The definition is very straightforward. And in various properties of the Lebesgue integral will transfer to these general Lebesgue integrals on arbitrary sets quite easily.

Let S subset of R be a measurable subset be a measurable set. Let F from S to R be a function define the new function F tilde of x which is by definition going to be equal to F of x, if x is in S, and 0 if x is in R minus S. So, we are essentially extending the function F by setting it to be 0 outside of its domain of definition which is the set S.

Now, we say F is Lebesgue integrable on S if F tilde is Lebesgue integrable on R. Furthermore, furthermore, we set integral of F over S to be by definition the integral over R of F tilde. So, the definition is rather straightforward. We just extend the value of the function F by 0 outside the set S, and then we define the integral of the function F on the set S to be that of the integral F tilde on the whole of the real line.

(Refer Slide Time: 02:34)



Some immediate consequences which I am going to leave it to you to show they just follow in one line from the definitions are the following. Remark, if T subset of S is measurable and F is Lebesgue integrable on S, then F restricted to T is Lebesgue integrable on T, rather straightforward property.

And the second property is also equally expected and equally trivial to show if S has finite measure, then the integral of the function 1 on the set S gives you mu of S. The measure of S can be obtained by integrating the function 1 on the set S this is almost just by definition.

One slightly little bit more involved theorem which again I am going to leave it to you because the hard work has been done when we prove the corresponding property for measures is the countable additivity property of the integral. So, let A i be a countable collection of pairwise disjoint subsets of pairwise disjoint measurable subsets disjoint measurable subsets of R. Let S be the union of A i, then of course S is measurable which is just a consequence of the material we did in the last video. Then S is measurable, if F from S to R is integrable that is F is in L of S.

(Refer Slide Time: 04:44)

$F|_{A_i} \in L(A_i)$ and we have:

$$\int_S F = \sum_{i=1}^{\infty} \int_{A_i} F|_{A_i}.$$

Conversely, suppose each $F|_{A_i} \in L(A_i)$ and the RHS above cvgs. Then $F \in L(S)$ and the equality above holds.

Then F restricted to A i is there in L of A i. So, each restriction is Lebesgue integrable and we have integral of F S is just summation integral of F over this A i, i running from 1 to infinity of F restricted to A i ok. So, the obvious thing that you expect holds provided the sets A i are all pairwise disjoint.

There is a converse also this. For this conversely suppose each F restricted to A i is in L of A i, and the RHS above converges, RHS above converges, then F is in L of S and the equality above holds above holds. So, this theorem is more or less just a trivial consequence of the countable additivity of the Lebesgue measure. It is straightforward to prove. And it is an instructive exercise for you to check whether you have understood the various definitions to prove them.

(Refer Slide Time: 06:23)

Defn: Let $F : I \to \mathbb{C}$ be a fn. We say F is Lebesgue integrable on I, if writing $F = u + iv$ we have $u, v \in L(I)$. We define

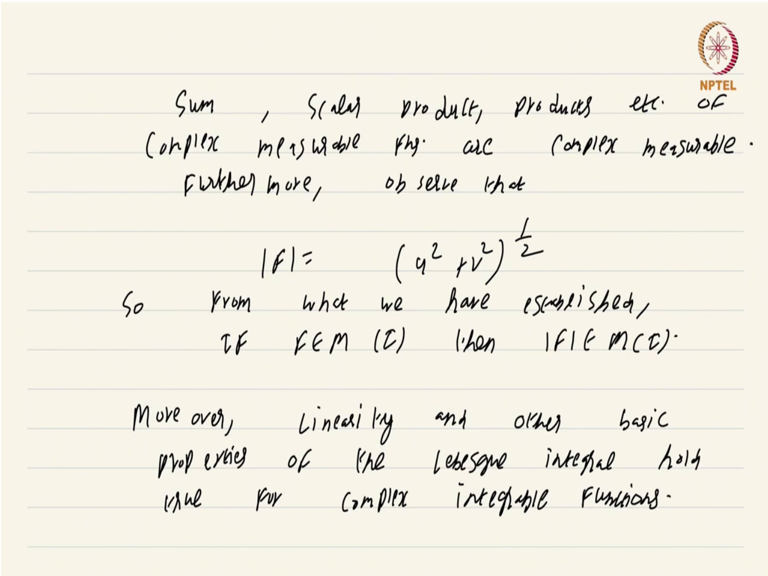
$$\int_I F = \int_I u + i \int_I v.$$

In a similar way, we say $F \in M(I)$ if $u, v \in M(I)$.

Now, let us move on and also define the Lebesgue integral on complex valued functions. And there is really nothing going to happen there the definition is utterly straightforward and exactly what you would expect. Let F from I to C be a function we say F is Lebesgue integrable, on I, if writing F equal to u plus i v we have u comma v are both Lebesgue integrable on I.

We define integral of F I to be nothing, but the integral over I of u plus i times integral over I of v. So, we just generalize in the most obvious way. In a similar way, in a similar way, we say F is measurable on the interval I, if u comma v are both measurable on the interval I ok.

(Refer Slide Time: 07:43)



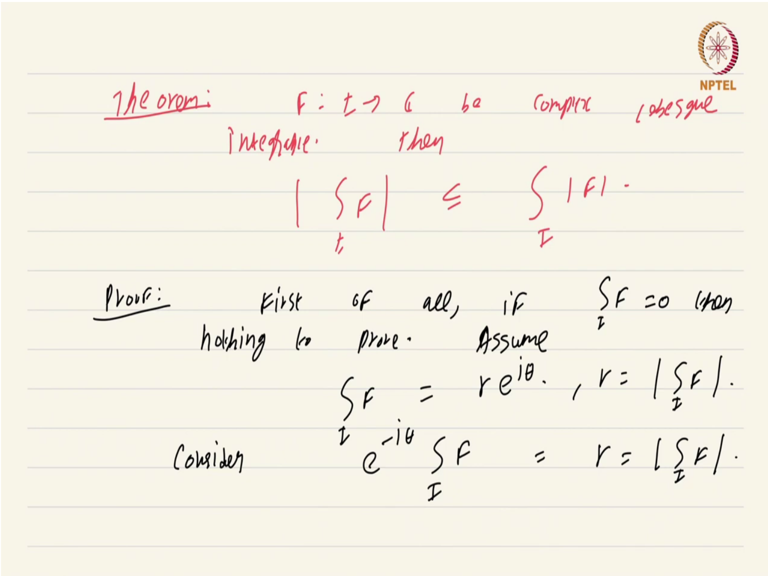
Now, one easily checks that some scalar a product products etcetera of complex measurable functions, complex measurable functions are complex measurable. Furthermore, observe that mod F is u squared plus v squared the whole power half. So, from what we have established, from what we have established if F is measurable on I as a complex valued function, then mod F is measurable on I as a real valued function.

This is because the function whenever you compose a continuous function with two measurable functions, then what you end up with still is still a measurable function. Because of that and because of this expression for mod F, we immediately get that mod F is also measurable. So, these are all straightforward to show.

Moreover, linearity and other basic properties other basic properties of the Lebesgue integral of the Lebesgue integral hold true for complex hold true for complex integrable functions also. And the proofs are nothing more than just writing F equal to u plus i v.

And applying the same I mean results that hold in the real case for both the integral over u part and the integral over v part. Since the integral of F is just defined to be integral of u i plus integral of i times integral over capital I v you just apply that corresponding results for both this u and v part, and you are done. There is really nothing much to do.

(Refer Slide Time: 10:20)



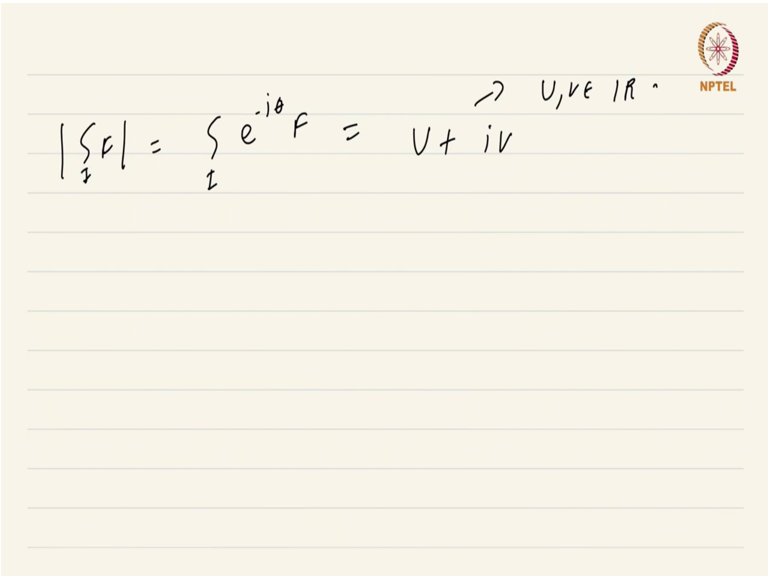
Only one new result we have I mean it is not a new result you cannot just get it straight forward from the real case there is a little bit of work to be done. So, let F from I to C be complex integrable complex Lebesgue integrable, then the absolute value of the integral is less than or equal to the integral of the absolute value.

We have already established this for real valued functions. Now, we are going to establish for complex valued functions also. Here a little bit of work is involved. So, first of all first of all if integral of I F is just 0, then nothing to prove, then nothing to prove.

So, assume, integral of I F is sum r e power i theta. So, here I am going to borrow a basic fact from the theory of complex numbers which you would have definitely learnt in your high school. Any nonzero complex number can be written as r e power i theta this is the polar representation of the complex number. Here r would be the absolute value of that complex number, and theta is the argument. So, this is just a basic fact that you would have no doubt learnt when you studied the basics of complex numbers ok.

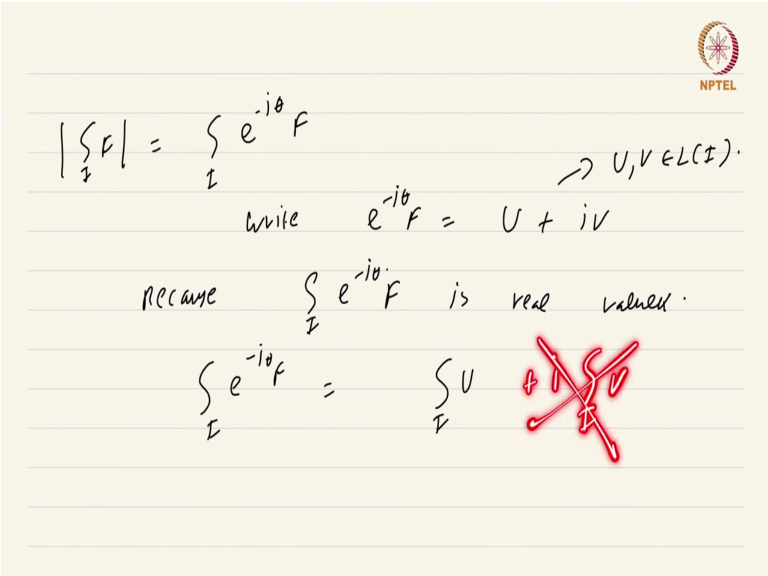
Now, what you do is, consider, e power minus i theta integral I F ok this is just r which is integral over I F the absolute value ok. Now, by the linearity of the Lebesgue integral which we have not shown for complex valued functions, I will invite you to provide a proof. As I said these follow immediately from the fact that the integral of F is defined in terms of the integral u and integral v. So, from that, you can get the linearity.

(Refer Slide Time: 12:35)



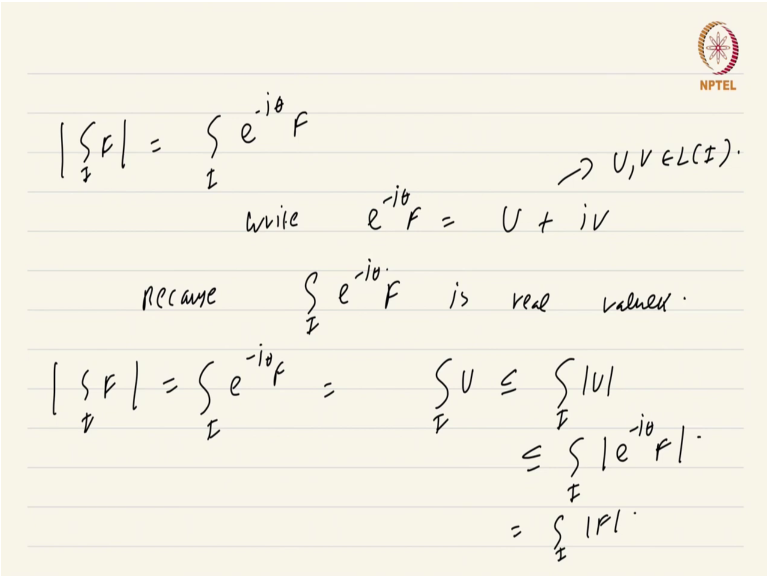
So, you can write this as integral over I e power minus i theta F ok. Now, this is going to be the absolute value of integral of I F from what we have just from the definition of what this theta is. Now if you write this as if you write this as capital U plus i V these are U and V are real numbers ok.

(Refer Slide Time: 13:13)



If you or rather do not do that if you write e power minus i theta F as capital U plus i V where U comma V are in L of I, real valued integrable functions write it like that. Because, integral of I e power minus i theta F is real valued is real valued, we must have integral of e power minus i theta F I to be nothing but the real part of the integral which is just integral over I U that is in other words what I am trying to say is this other term which is there plus i times integral I of V, this is actually 0, this term will not play a role in what is to come. Just that integral I U that will be the nonzero term ok.

(Refer Slide Time: 14:07)



Now, this is less than integral over I mod U simply because of the ordering property of the Lebesgue integral. And this is going to be less than integral over I of modulus of e power minus i theta F. This is simply because e power minus i theta F is U plus i V ok. And by the basic property that the real part the absolute value of the real part of a complex number is less than or equal to the complex the absolute value of the complex number itself ok.

Excellent, we have this. Well, e power minus i theta this is a unimodular complex number. So, this is equal to integral over I of mod F ok. And now we immediately get what we wanted because the left hand side is just integral over I mod F. So, this proves this inequality that the absolute value of the integral is less than or equal to the integral of the absolute value.

So, this concludes this short video on Lebesgue integrals on arbitrary subsets and some properties of complex valued Lebesgue integrals. This is a course on Real Analysis. And you have just watched the video on Lebesgue Integral on Arbitrary Subsets.