Real Analysis II
Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 32
Solution to the Problem of Measure

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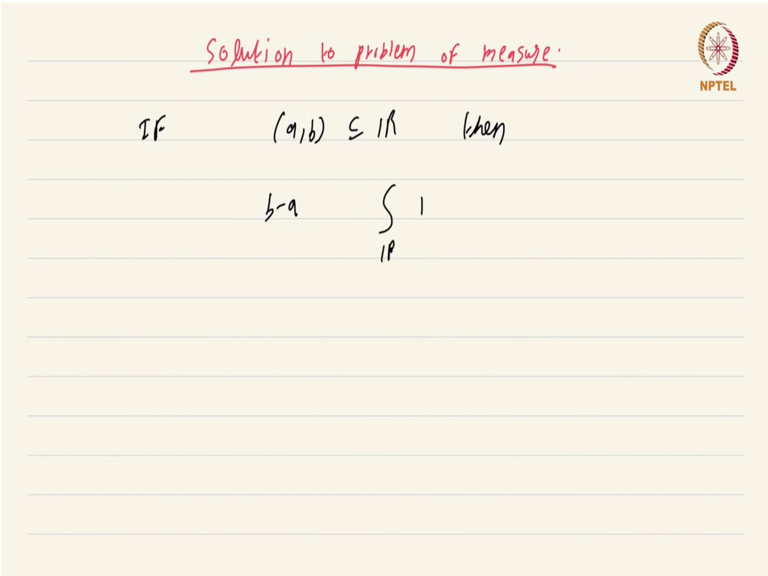

We have developed a powerful integral the lebesgue integral, one nice side effect of our construction is that we can get a solution to the problem of measure. Recall that the problem of measure asks us to assign for any subset of the real numbers a length.

This length should agree with the length on intervals the length of an interval a b is just b minus a irrespective of whether a or b is there in that set or not. We want to find a countable additive notion of length that agrees with the length on intervals. The solution is to observe that if a b is subset of R, let us say for concreteness sake I am taking the open interval a b then b minus a is just integral of 1 on R on a to b, b minus a is just the integral of the function 1 on from a to b.

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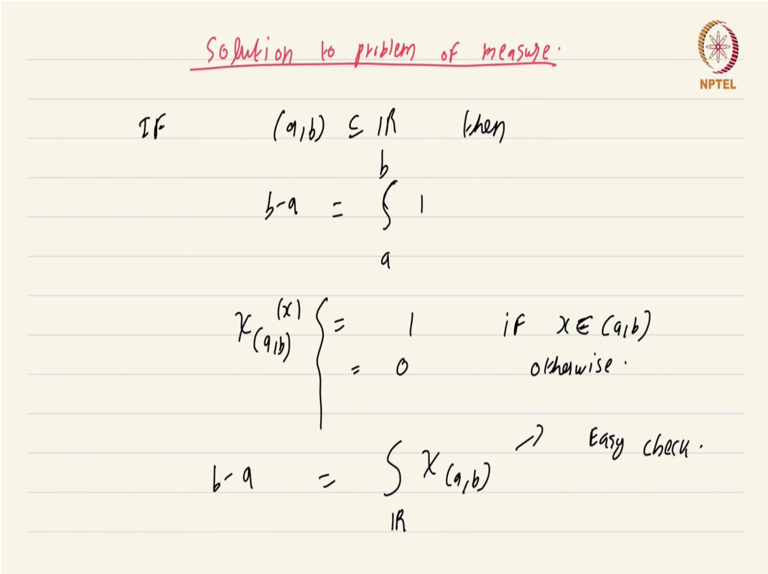

So, what we now do is you consider a new function which is we define it to be chi a b. This function is defined to be 1 if x is in a b 0 otherwise. Then it is clear that b minus a is nothing, but the integral of chi a b over R ok easy check. So, you can obtain the length of an interval by integrating this associated function.

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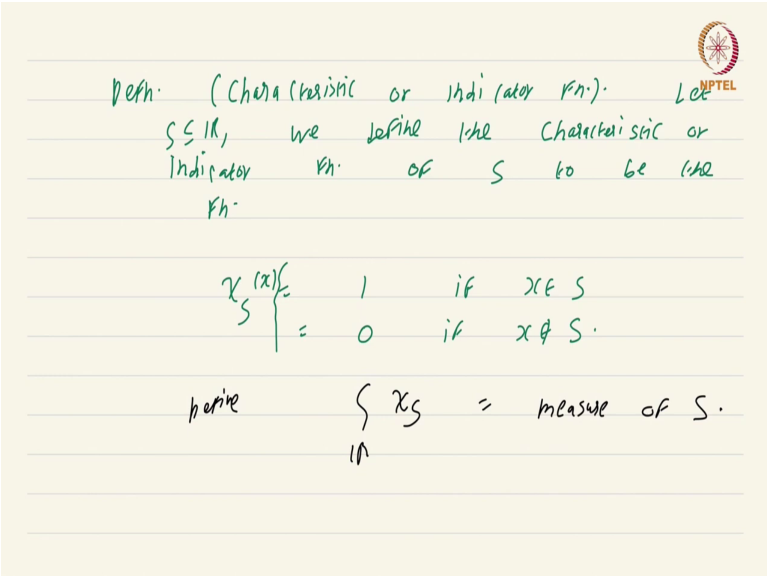

So, this prompts the following natural definition of that of a characteristic function. So, this is called characteristic or indicator function. Let us subset of R we define the characteristic or indicator function of S to be the function to be the function chi S.

Which is defined to be chi S of x is 1 if x is in S equal to 0 if x is not in S ok. Now, this the name characteristic or indicator function of S is self explanatory at any point of the set it gives the value 1 otherwise it gives the value 0.

Let me make a side remark here in the measure theoretic approach to integral you already have a notion of measure for sets. Once you have the notion of measure for sets you can define the integral you can define the integral of this chi S to be just the measure of S.

So, essentially what will happen in the classical treatment of measure theory first and integral second is the reverse of what is going to happen now. What we are going to now do is define the measure of a set to be the integral of the characteristic function.

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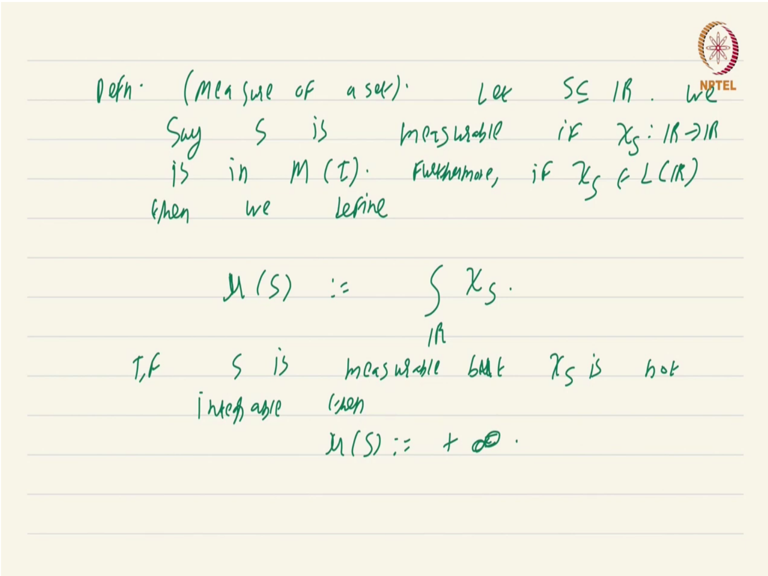

So, let us make that formal definition this is measure of a set let S subset of R we say S is measurable if chi S from R to R is in M of I. So, if the characteristic function of the set is measurable then we say the set itself is measurable.

Furthermore if chi S is lebesgue integrable on R then we define we define the measure which we call traditionally mu; mu of S is by definition just the integral over R of chi S. If S is measurable, but not integrable, but chi S is not integrable then mu of S is by definition plus infinity ok.

So, the only way by which the characteristic function which is a non negative function can fail to be integrable on R is if it is sort of going to take an infinite value that is the motivation behind defining mu of S to be plus infinity, if the characteristic function is measurable, but not integrable.

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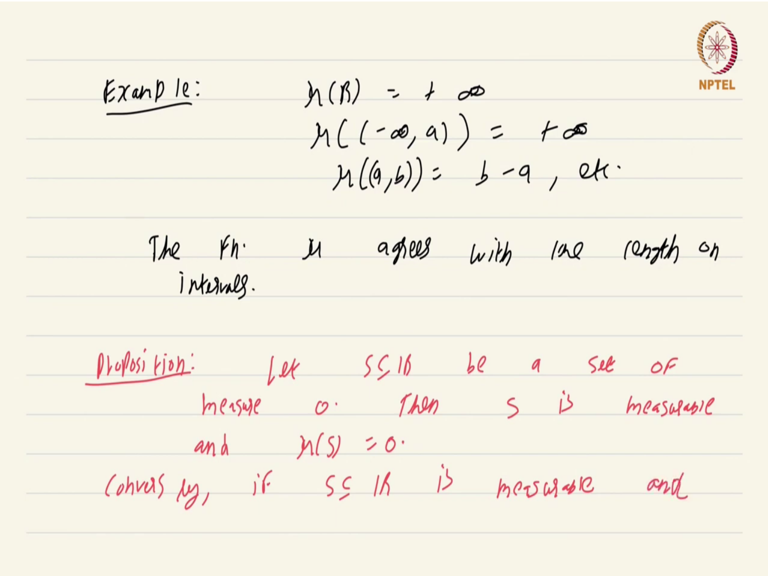

Now, immediate example you can see that you can see that mu of R is plus infinity then mu of any interval of the type minus infinity a is also plus infinity mu of a, b is going to be b minus a etcetera ok. So, in other words the measure the function mu agrees with the length on intervals and this is rather straightforward to see.

Now, is there any other larger class of measurable sets, and what is the measure of such sets going to be? Well the next simple proposition a sort of is very basic and tells us that sets of measures 0 are a measurable and b their measure is 0. If that were not the case the nomenclature given by mathematicians would be extraordinarily stupid. So, let S subset of R be a set of measure 0, then S is measurable and mu of S is 0. Conversely if S subset of R is measurable.

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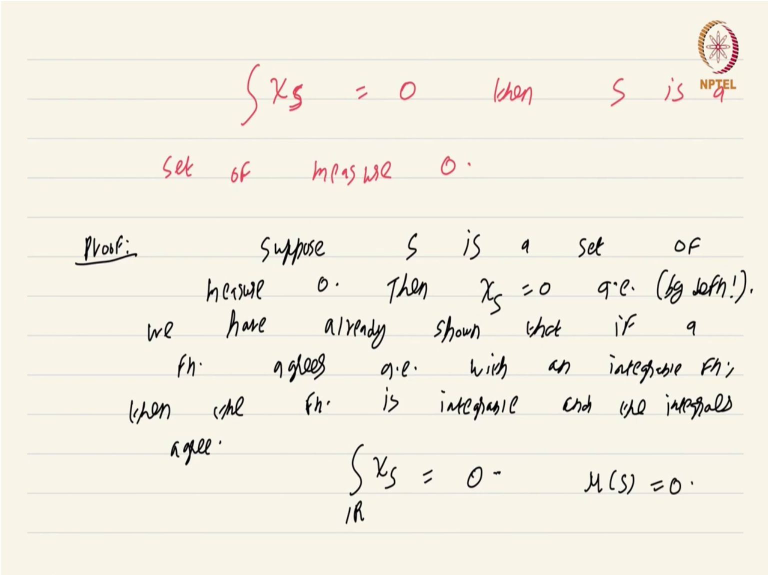

And integral of chi S is 0 then S is a set of measure 0 ok. Let us prove both parts proof suppose S is a set of measure 0 ok, then chi S is 0 almost everywhere this is just by definition this is just by definition the function chi S is 0 almost everywhere.

We have already shown we have already shown that if a function agrees almost everywhere with an integrable function, then the function is integrable function is integrable and the integral values agree.

We had shown this one once I mean as part of when we developed the theory of the lebesgue integral and convergence theorem and the integrals agree. So, therefore, we immediately get that integral of chi S is 0 as claimed. So, a set of measure 0 will be measurable and in fact, mu of S would be 0. Thank god ok.

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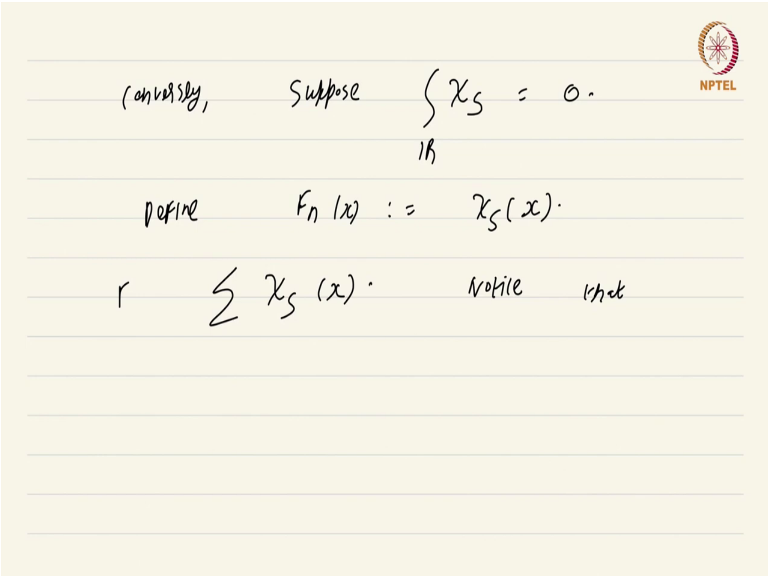

Now for the converse conversely suppose chi S integral over R is 0 ok. Now ,what you do is you define this function F n of x to be just equal to chi S of x, that is F n is just chi S ok and look at summation chi S of x look at this summation. Now, notice that if you call this so, what you do is instead of defining F n to be chi S.

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Conversely, Suppose $\int_{\mathbb{R}} \chi_S = 0$.

Define

$$F_n(x) = \sum_{k=1}^{n} \chi_S(x).$$ Notice that F_n's are non-negative and

$$\int_{\mathbb{R}} F_n = \int_{\mathbb{R}} \sum_{k=1}^{n} \chi_S = \sum_{k=1}^{n} \int_{\mathbb{R}} \chi_S = 0.$$

Define F n of x to be summation n equals or rather summation K equals 1 to n F n ok. Now, notice that Fn’S are non negative are non negative and integral of F n is just integral summation first of all integral K over R K equals 1 to n chi S which is just integral over R, but summation is now outside K equals 1 to n chi S which is 0 ok.

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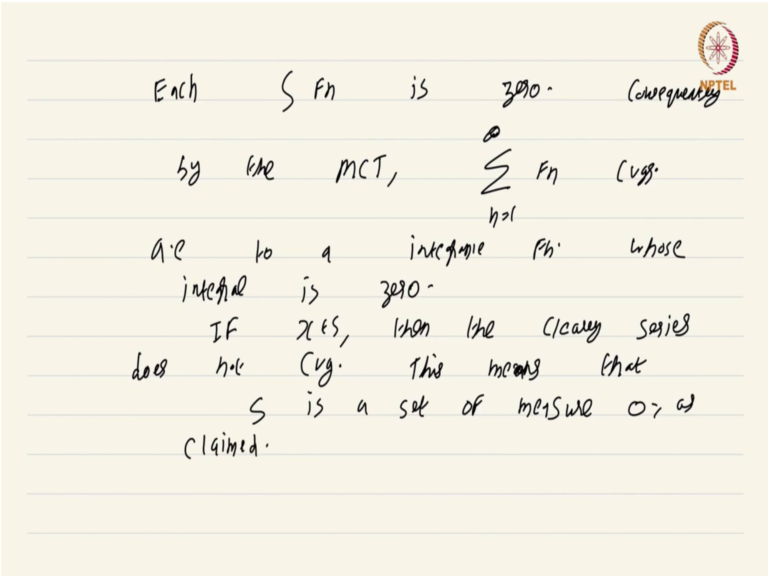

So, each integral F n is 0, consequently by the monotone convergence theorem the monotone convergence theorem this summation F n this function n equals 1 to infinity this converges almost everywhere. To a integrable function whose integral is 0 ok, this is just the monotone convergence theorem.

So, if you take an element x in S then clearly the series does not converge, clearly series does not converge. This means that S is a set of measure 0 because this series summation F n must converge outside a set of measure 0 it does not converge on S, consequently S must be a set of measure 0 as claimed, might look like a roundabout way of showing it, but nothing really deep is happening excellent.

Now, we have got a nice collection of sets that are definitely going to be measurable they are all the sets which are sets of measure 0. We want to enlarge and see that this collection of measurable sets is in fact, large. In fact, it will be so large that the only way to construct a non measurable set will be to appeal to the axiom of choice.

Please revisit some remarks I made when we showed that the set of measurable functions is in fact, closed under taking point wise limit. It is very hard in practice to land up with a non measurable function, it is equally hard to land up with a non measurable set for all practical purposes you can assume that all sets are measurable. Anyway we still need to see some examples some concrete examples of measurable sets which are not just intervals or sets of measure 0.

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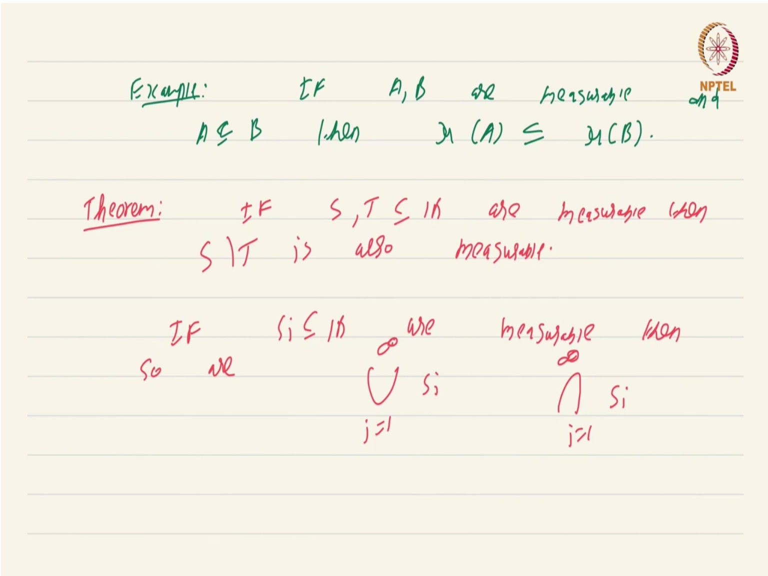

Also just one example if A and B are both are measurable and A subset of B A subset of B then you can check easily that mu of A is less than mu of B this just sort of follows from the monotonicity property of the lebesgue integral ok. Keep this in mind because this will be repeatedly used without mention.

Theorem, we are going to show that some natural set theoretic operations do not take us outside of the class of measurable sets if S comma T subset of R are measurable, then S set minus T is also measurable. And if Si and if Si subset of R are measurable then so are union i equals 1 to infinity Si and intersection Si i equals 1 to infinity.

Arbitrary not arbitrary countable union and countable intersection of measurable sets are measurable. For computing the measure of this union just wait a moment the next theorem that we study will sort of tell you that there is some sort of additivity which is very nice.

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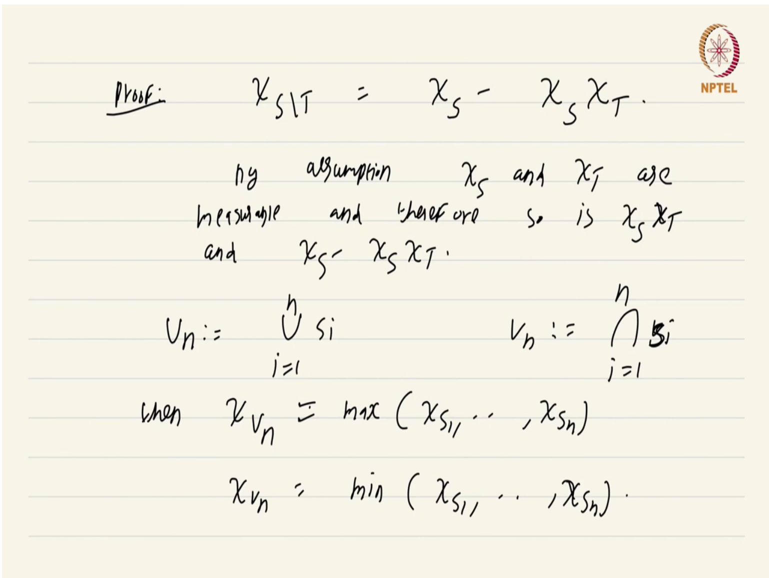

Proof, for the first part it is just a simple observation observe that chi of S minus T is just chi of S minus chi of S times chi of T. To see why this is true note that chi of S minus T is non zero precisely if you have a point of S which is not also a point of T that is why we are subtracting chi of S into chi of T; chi of S into chi of T will be non zero precisely on S intersection T. So, chi of S minus T is chi of S minus chi of S into chi of T. .

Now, by assumption chi S and chi T are measurable they are measurable and therefore so is and therefore so is the product chi S chi T. Because we have proved a proposition that says that any continuous combination of measurable functions is measurable and so, is and chi of S minus chi of S chi of T excellent.

So, this shows that the difference of two measurable sets is measurable. Now, coming to the union and intersection that is actually easier than this. Notice that if you define U n to be just union i equals 1 to n S i. And V n to be by definition intersection i equals 1 to n V i sorry Si, then chi of U n is nothing but max of chi S 1 to chi S n and chi of V n is nothing, but the minimum of chi of S 1 to chi of S n.

Well the easiest way to see this is to close your eyes and contemplate it for a few seconds and you will get it there is no explanation I could give that it will convince you of this it is so trivial ok.

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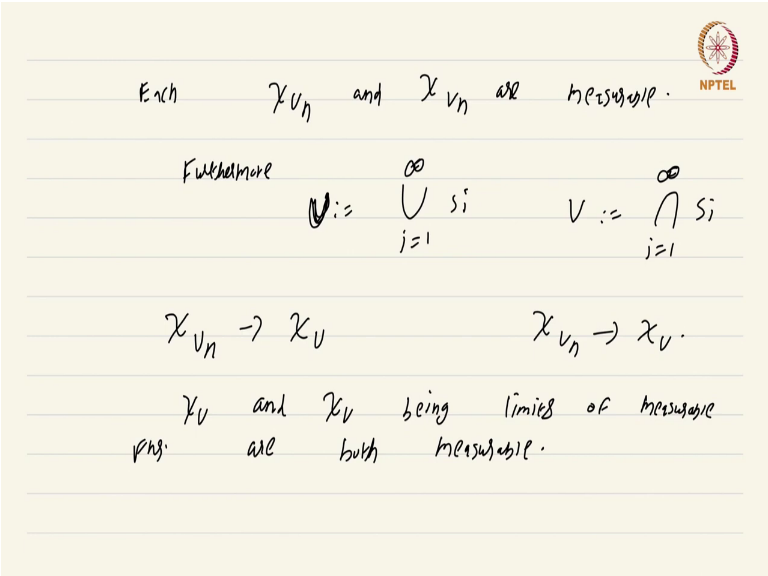

Now, that you have this each of each chi U n and chi V n are measurable being the maximum and minimum of measurable sets again recall that any continuous combination of measurable functions is going to be measurable ok..

Furthermore if you define U to be union i equals 1 to infinity Si and V to be intersection i equals 1 to infinity Si, then this chi U n; obviously, converges to chi U and chi V n converges to chi V right. Which shows that chi U n and sorry chi U and chi V being the limits being limits of measurable functions is measurable. This is one nice property of the class of measurable functions being limits of measurable functions are both measurable.

So, this concludes the proof that unions countable unions and countable intersections of measurable sets are continued to be measurable. Now, we want to deal with that main property of this lebesgue measure that is that of integrability sorry additivity.

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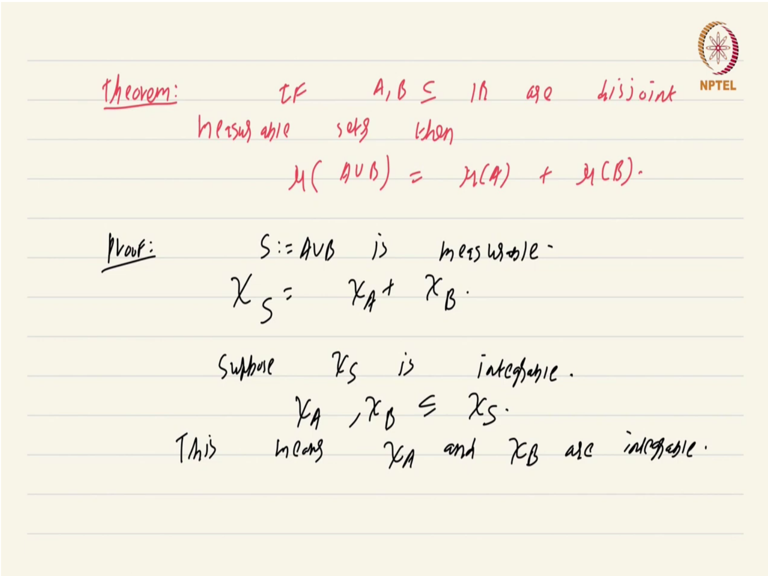

What this additivity says is that if you take two disjoint sets then the measure of the union is the sum of the measures which is natural, we will now first prove it for two sets then later extend it for a countable union. This is a powerful property and this is very useful in proving various stuff about measurable sets.

If A comma B subset of R are disjoint measurable sets then mu of A union B is mu of a plus mu of B ok, note that no assumption is made about whether these sets are going to have a finite measure or not no such assumption is made.

Proof well first of all A union B is measurable we have just shown that ok and because of the disjoint nature of A and B this union if you call it S chi of S is just chi of A plus chi of B ok. Now, suppose chi S is integrable then both chi A and chi B are less than or equal to chi S. And since these are already measurable this means chi A and chi B are integrable. We have already seen that any measurable function whose absolute value is dominated by an integrable function is integrable. So, this means chi A and chi B are integrable.

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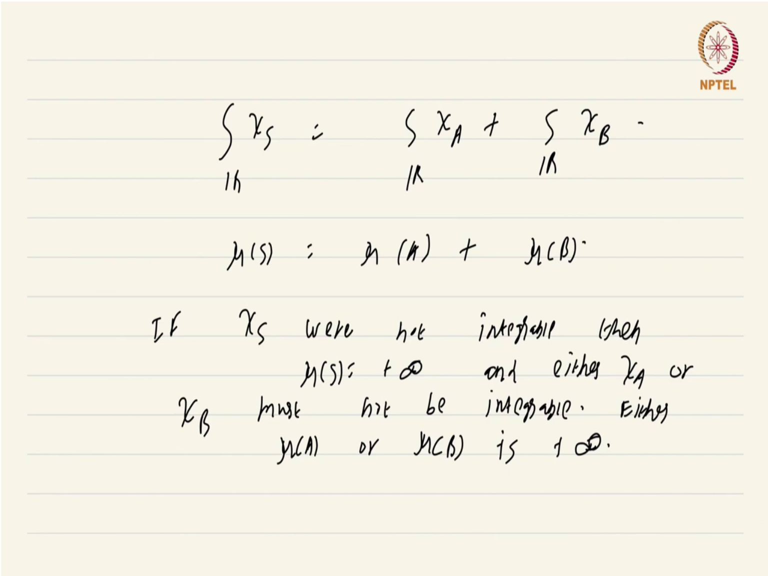

And it follows that integral of chi S over R is just integral over R chi A plus integral over R chi B. This is just by the additivity property of the lebesgue integral. So, in other words mu of S equal to mu of A plus mu of B.

Now, if chi S were not integrable were not integrable then by definition mu of S is infinity mu of S is infinity and either chi A or chi B must not be integrable because if chi A and chi B were both integrable then chi S would be integrable because chi S is just chi A plus chi B ok. Which means either mu of A or mu of B or even both could is plus infinity.

So, with this additivity holds even in this scenario where one of the sets is not going to have a finite measure ok. So, immediately by induction, so we are done with that proof.

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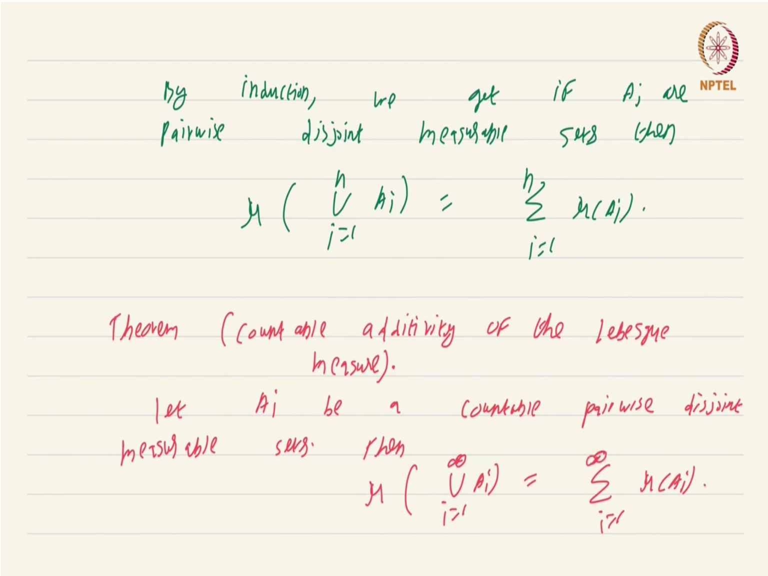

So, immediately by induction we get that if Ai are pairwise disjoint measurable sets, then you get that some mu of union i equals 1 to n Ai is just summation mu of Ai i equals 1 to n this is just obtained by the previous theorem just by applying induction ok.

Now, the question is what if you had i equals 1 to infinity if you had infinitely many countably infinitely many measurable disjoint sets, is it still true that the measure of the union is the sum of the measures of the individual sets. Well thankfully this is also true and this will be the major theorem of this section, this is the countable additivity of the lebesgue measure the lebesgue measure.

So, it states the following let Ai be a countable pairwise disjoint measurable sets. Then mu of union i equals 1 to infinity Ai is just summation mu of Ai i running from 1 to infinity. So, we have countable additivity of the lebesgue measure.

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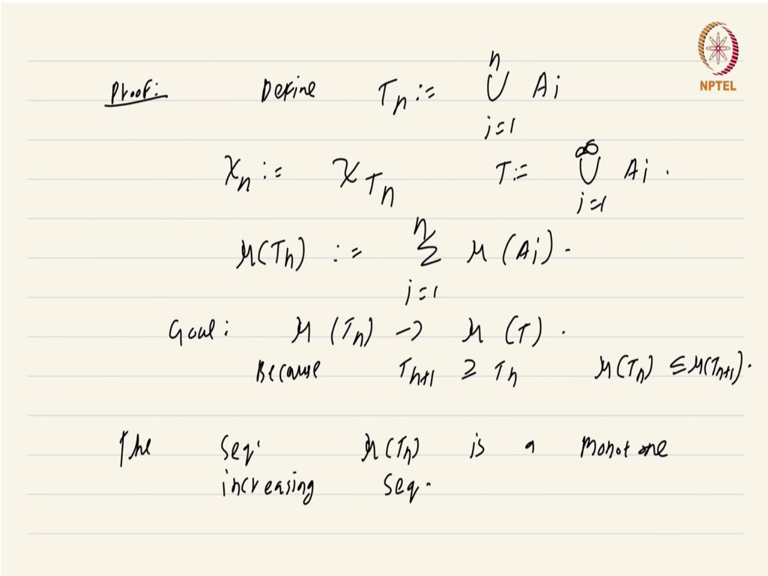

Proof, and all of these proofs are just standard somehow try to apply a convergence theorem. So, you define T n to be just union i equals 1 to n of Ti and you consider chi n to be just shortcut for chi of T n. And T you define it to be union i equals 1 to n one second I made a mistake here this is Ai ok. So, T you defined to be i equals 1 to infinity Ai ok.

Then by finite additivity which we have just established mu of T n is nothing, but summation i equals 1 to n mu of Ai, this is just by finite additivity ok. Now, the goal is to prove that mu of T n converges to mu of T right.

Now, because T n plus 1 contains T n we get mu of T n is less than or equal to mu of T n plus 1, this is just by the monotonicity property of the lebesgue integral which we have already mentioned. So, this collection mu of T n or the sequence mu of T n is a monotone increasing sequence monotone increasing sequence ok.

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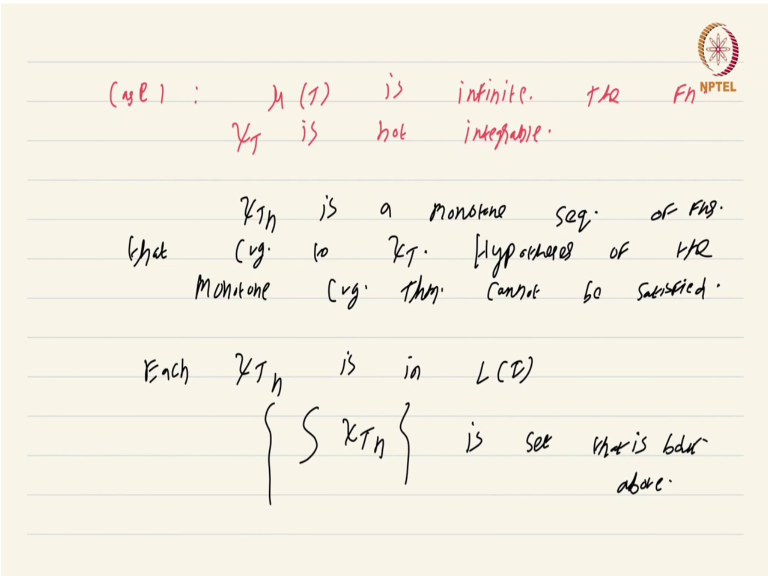

Now, again we will have to split up into two cases; case 1 mu of T is infinite. Now, this means that integral essentially what does this mean; this means that the function chi of T is not integrable ok.

Now, observe that chi of T n is a monotone sequence is a monotone sequence of functions that converge to chi T. But, since chi of T is assumed to be not lebesgue integrable that means, the hypothesis of the monotone convergence theorem cannot be satisfied.

Hypothesis of the monotone convergence theorem monotone convergence theorem cannot be satisfied because the conclusion of the monotone convergence theorem is that the limit function is integrable. Since, we are ending up with a non integrable limit function the hypothesis cannot be satisfied.

Now, what were the hypothesis of the monotone convergence theorem well the hypothesis were in translated to our particular situation each chi T n is in L of I, that was one of the hypothesis and this collection integral chi T n this is a set that is bounded above right. So, one of these will have to fail.

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χ_{T_n} is not in $L(I)$ for some n,

$$\mu(T_n) = +\infty$$

$$\mu(T) = \lim_{n \to \infty} \mu(T_n).$$

$$\mu(T) = \lim_{n \to \infty} \mu(T_n).$$

Well if chi of T n is not in L I for some n is not in L I for some n, then by definition mu of T n is plus infinity, we already have mu of T is plus infinity that is our global assumption in this case. So, put together we will get mu of T mu of T is limit and going to infinity of mu of T n ok in this scenario. On the other hand it can happen that this integral of chi T n is unbounded is not bounded above, which just means that mu of T is limit n going to infinity of mu of T n.

So, in either ways by which the hypothesis of the monotone convergence theorem can fail in both scenarios we end up with mu of T is limit n going to infinity of mu of T n as required.

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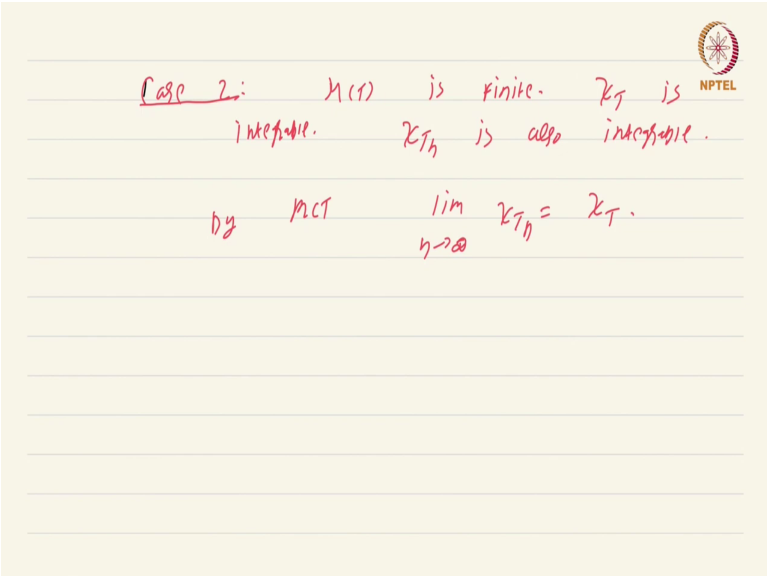

On the other hand if case 2 mu of T is finite ok this means that chi T is integrable consequently each chi T n is also integrable simply because chi T dominates chi T n and chi T n is measurable which just means that by monotone convergence theorem limit n going to infinity of chi T n is equal to chi T ok. In fact, we need we cannot use monotone convergence theorem for this we need the dominated convergence theorem.

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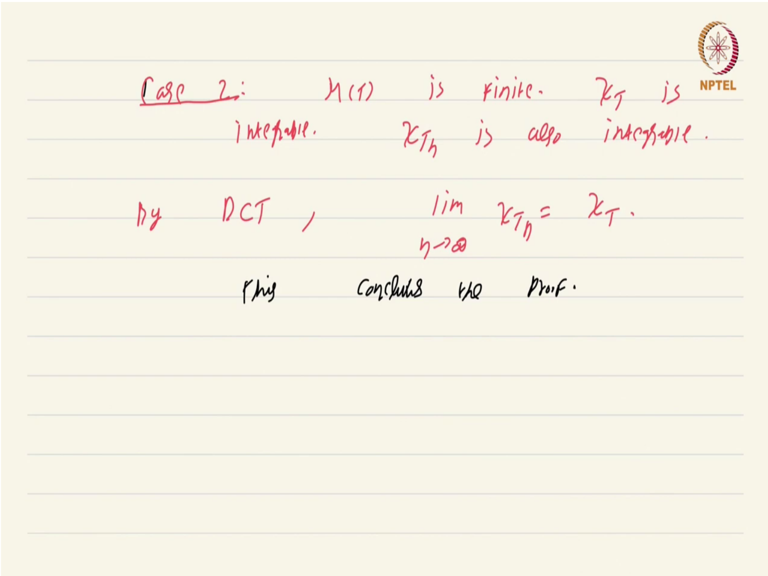

By dominated convergence theorem limit n going to infinity chi T n equal to chi T excellent. So, this concludes the proof. So, we have countable additivity of the lebesgue measure. So, let me just summarize a bit of useful information by giving a definition.

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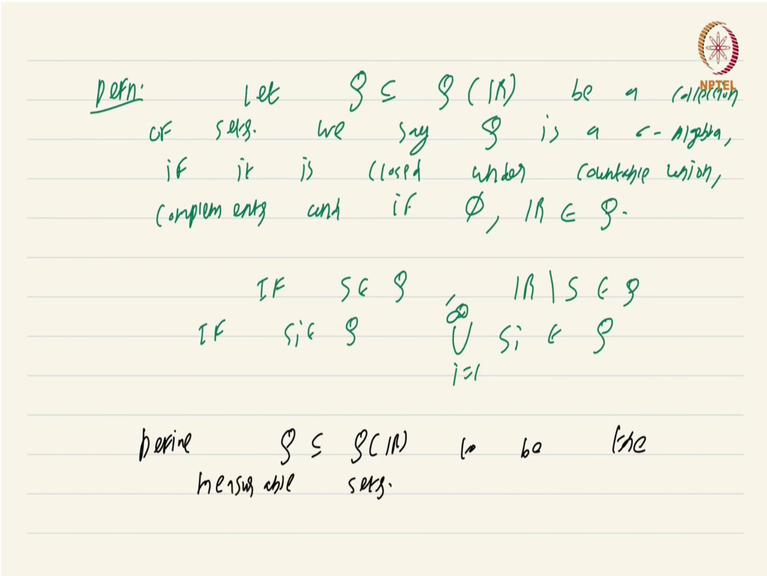

Let p subset of power set of R be a collection of sets. We say p is a sigma algebra if it is closed under countable union complements and if empty set comma R are both elements of p.

A collection of sets such that the empty set the whole real numbers and this collection should be closed under countable union and complements, closed under countable union and complement means if S is an element of rho then R minus S or let us call this p naught rho.

If S is an element of p R minus S is also an element of p. If Si are in p then union i equals 1 to infinity Si is also in p ok. So, why am I making this definition well it can summarize all the properties that we have the function. So, define p subset of power set of R to be the measurable sets to be the measurable sets.

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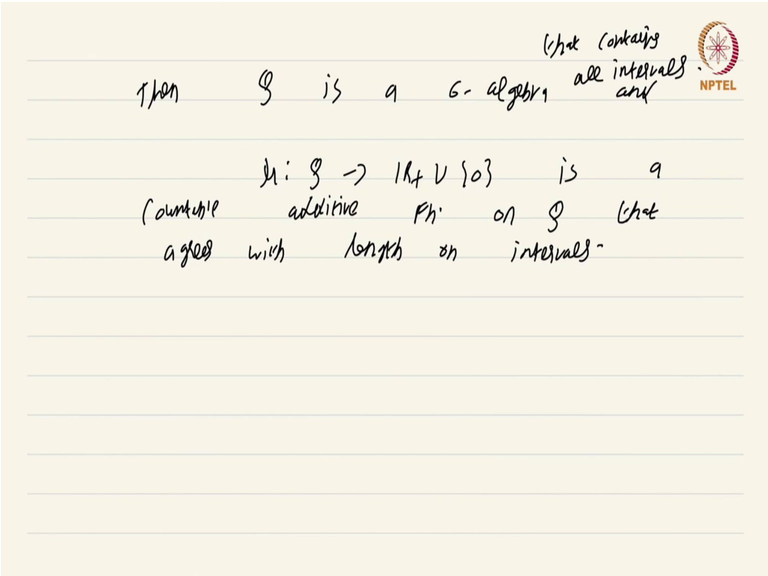

Then we can summarize everything that we have done in this video by saying this p is a sigma algebra and mu from this p to R plus union 0 is a countable additive function on p that agrees with length on intervals ok. And p is a sigma algebra that contains all intervals contains all intervals.

We can summarize the entire discussion here. So, of course, the fact that the empty set and R are the measurable is rather obvious and the fact that we have countable additivity we have just shown. The fact that this is the sigma algebra is just a consequence of the fact that R is there in this collection p and it is closed under complementation which we have already shown.

So, in the measure theoretic approach to constructing the lebesgue integral we start with the sigma algebra and try to construct a measure on that sigma algebra and then from that measure you try to go to the integral by defining the integral of a characteristic function to be just the measure of that set.

And then we proceed to define what are known as simple functions which are just linear combinations of characteristic functions and then you take the limits of such functions and then define the integral in more or less a similar way with to what we have done.

So, this is an alternative approach to the theory of measure going via integrals. This is a course on real analysis and you have just watched the video on solution to the problem of measure.