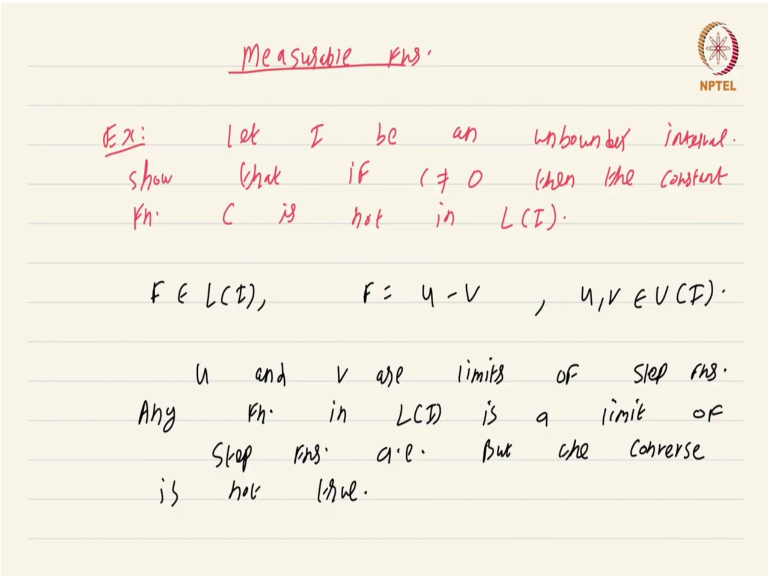
Real Analysis II

Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 31.2
Measurable Functions

(Refer Slide Time: 00:23)



The class of Lebesgue integrable functions is quite large, but nevertheless there are very very very simple looking functions that do not belong to this class. So, we begin with an exercise. Let I be an unbounded interval an unbounded interval show that if C is not equal to 0 then the constant function C the constant function C is not in L of I.

So, even constant functions need not necessarily be Lebesgue integrable if the interval is unbounded. So, if you start with the Lebesgue integrable function you can always write it as the difference of two functions u and v, where u comma v are both upper functions. This is the very definition of a Lebesgue integrable function. But u and v are limits of step functions this is also by definition. So, the net upshot is any function in L of I is a limit of step functions almost everywhere.

So, any Lebesgue integrable function can be expressed as a pointwise almost everywhere limit of step functions, but the converse is not true, as this exercise above will immediately tell you, but the converse is not true. The limit of step functions need not always be Lebesgue integrable so, this prompts the following definition.

(Refer Slide Time: 02:33)

Defn: A fn. $f: I \to \mathbb{R}$ is said to be measurable if it is a.e. a limit of step fns. The collection of measurable fns on I is denoted $M(I)$.

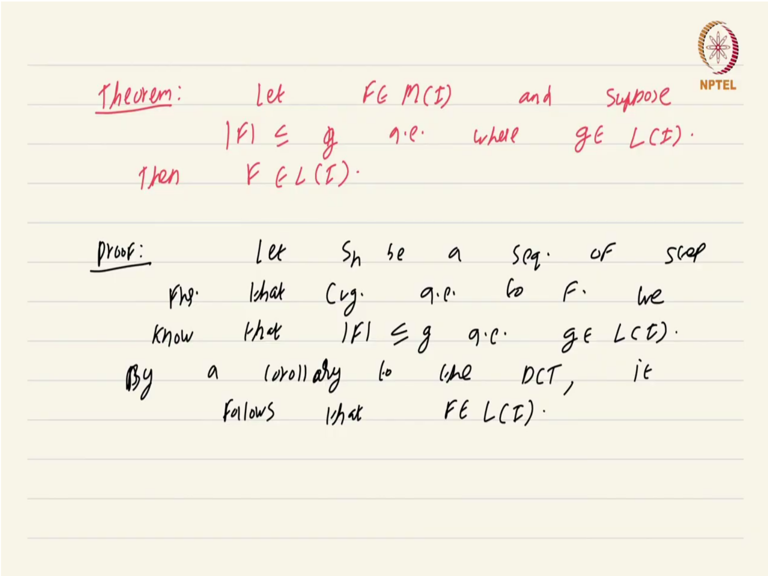
Observation: If $f \in M(I)$, and $J \subseteq I$ is a subinterval then $f|_J \in M(J)$.

$L(I) \subseteq M(I)$.

A function F from I to R is said to be measurable if it is almost everywhere a limit of step functions ok. So, the collection of measurable functions on I is denoted M of I ok. As a trivial observation as a trivial observation observe that if f is a measurable function on I and J subset of I is a sub interval then f restricted to J is measurable on J. This is just an obvious consequence of the definition.

So, this class of Lebesgue integrable functions are a subset of the class of measurable functions. Obviously, we would like to have a nice criteria that will guarantee that a measurable function is Lebesgue integrable. This arises in practice because many times we can exhibit functions as a limit of step functions. But the limit of step functions need not in general be Lebesgue integrable. So, it would be good to have some sort of criteria.

(Refer Slide Time: 04:21)



And the next simple theorem which is actually just a trivial corollary of the dominated convergence theorem gives us simple criteria. Let f be in M of I and suppose mod f is less than or equal to g almost everywhere where g is Lebesgue integrable.

Then f is also Lebesgue integrable. So, we have this straightforward criteria to check whether a measurable function is Lebesgue integrable. The criteria is that its absolute value must be dominated by a Lebesgue integrable function let us see a proof of this it is very straightforward.

So, we have a sequence let s n be a sequence of step functions that converge almost everywhere to F. Now we know that we know that mod F is less than or equal to g almost everywhere and g is Lebesgue integrable. So, by a corollary to the dominated convergence theorem. So, if you want to know exactly which corollary this is watch that video on the applications of the convergence theorems by a corollary to the dominated convergence theorem it follows that F is Lebesgue integrable.

So, in the original statement of the dominated convergence theorem, you require each function in the sequence to be dominated by g in the corollary which was an application of the dominated convergence theorem we saw that it in it is enough if the limit function is dominated the in absolute value by a Lebesgue integrable function you can still apply the dominated convergence theorem ok.

(Refer Slide Time: 06:42)

Corollary 1: If $f \in M(I)$ and $|f| \in L(I)$, then f is $L(I)$.

Corollary 2: If $f \in M(I)$, bounded a.e. and I is bounded then $f \in L(I)$.

Theorem: Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a continuous fn. Let $f, g : I \to \mathbb{R}$ be measurable. Then $\phi(f, g) : I \to \mathbb{R}$ is also measurable. In particular, $f+g$, $f \cdot g$, $|f|$, $\max(f, g)$ and $\min(f, g)$ are all measurable.

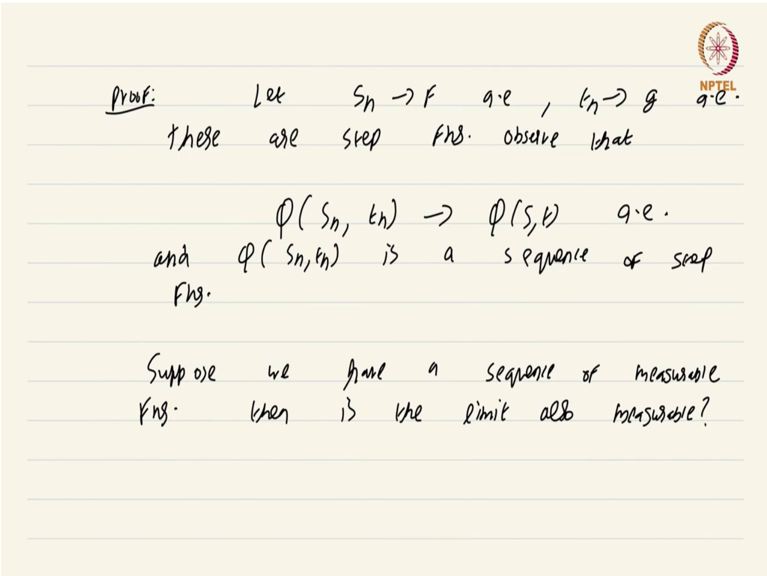
So, we have this simple criteria which immediately gives us several nice corollaries, corollary 1 if F is measurable and mod F is Lebesgue integrable, then f is Lebesgue integrable this follows immediately by applying the previous theorem.

Corollary 2 if F is measurable bounded almost everywhere and I is also bounded, then F is Lebesgue integrable, this is also immediate by observing that a bounded function there is a supremum and if you take that maximum absolute value of F that constant will be integrable on I simply because I is a bounded interval.

So, we have several simple criteria for checking whether a function is measurable. Let us see yet another criteria theorem. This theorem will tell us how we can combine measurable functions to get measurable functions. Let phi from R 2 to R be a continuous function be a continuous function.

Let f comma g from I to R be measurable. Then phi of F comma g from I to R is also measurable. In particular F plus g, F times g, mod F maximum of F g and minimum of F g these are all measurable functions. Simply by varying the choice of phi we can get each one of these functions.

(Refer Slide Time: 09:12)



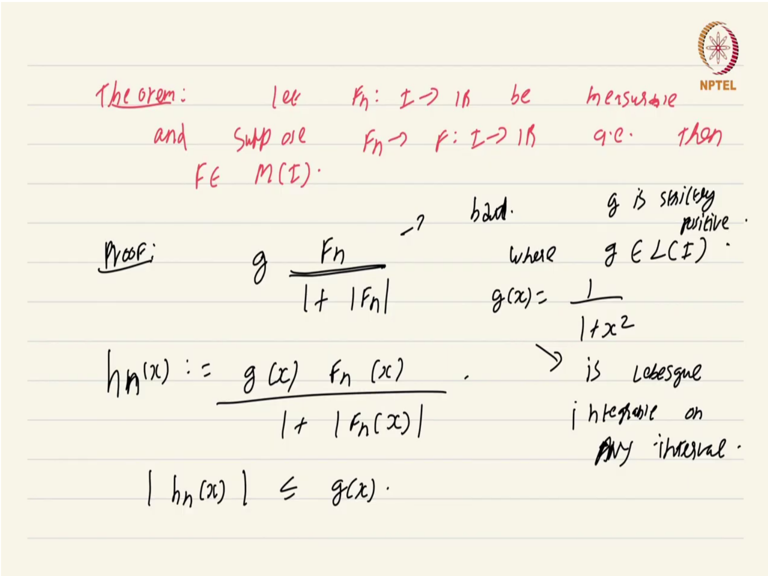
Let us see a proof of this again there is nothing really much to show. Let S n converge to F almost everywhere and t n converge to g almost everywhere these are step functions because F and g are measurable such step functions exist. Now observe that observe that phi of S n comma t n; obviously, converges to phi of S comma t almost everywhere by continuity and phi of S n, t n is a step function or rather a sequence of step functions.

The fact that this is going to be a sequence of step functions can be easily derived by looking at the partition which makes S n step function and another partition that makes t n a step function and taking a common refinement and observing that on that common refinement in each sub interval the value is just given by phi of s n, t n. So, this allows us to combine measurable functions to get more measurable functions.

Now, finally, suppose we have a sequence of measurable functions sequence of measurable functions, then is the limit also measurable. This is an interesting question to ask simply because a limit of Lebesgue integrable functions is not Lebesgue integrable you can check that. So, this property of being closed under point wise limits is not enjoyed by the class of Lebesgue integrable functions. That is why we are interested in this class of measurable functions which cannot be enlarged in any easy manner.

The previous theorem sort of says that any algebraic combination of measurable functions will be measurable. The next question is can you go outside the class of measurable functions by taking point wise limits well the next theorem will say this is not possible.

(Refer Slide Time: 11:41)



Theorem; let F n from I to R be measurable and suppose F n converges F f from I to R almost everywhere. Then the limit function f is measurable ok. Proof; what we are going to do is we are going to sort of modify this function F n to make it Lebesgue integrable ok. Now, the basic idea is that first we sort of make the function bounded by making F n by 1 plus mod F n, if you consider F n by 1 plus mod F n this this would be bounded, but a bounded function need not necessarily be Lebesgue integrable.

So, what we are going to do is we are going to sort of mollify this function by multiplying by a function g, where g is some function which is there in the class L of I. For instance one function which will always be there in L of I irrespective of what I s is the function 1 by 1 plus x square for instance you can take g of x to be this. This function is Lebesgue integrable for any interval which we have seen as an application of the convergence theorem is Lebesgue integrable on any interval it really does not matter.

So, once you consider this new function F n of x capital F n of x to be by definition g of x. So, let me not call it capital F n of x and confuse you because there is a small. So, we let us call it h n of x, h n of x is by definition g of x into f n of x divided by 1 plus mod f n of x ok.

Now, observe that mod h n of x is less than or equal to g of x oh of course, where g is in L of I and g is strictly positive this is also crucial we want a strictly positive function which is Lebesgue integrable.

(Refer Slide Time: 14:29)

$h_n \in M(I)$ by previous theorem.

$|h_n(x)| \le g(x)$, it follows that

$h_n \in L(I)$.

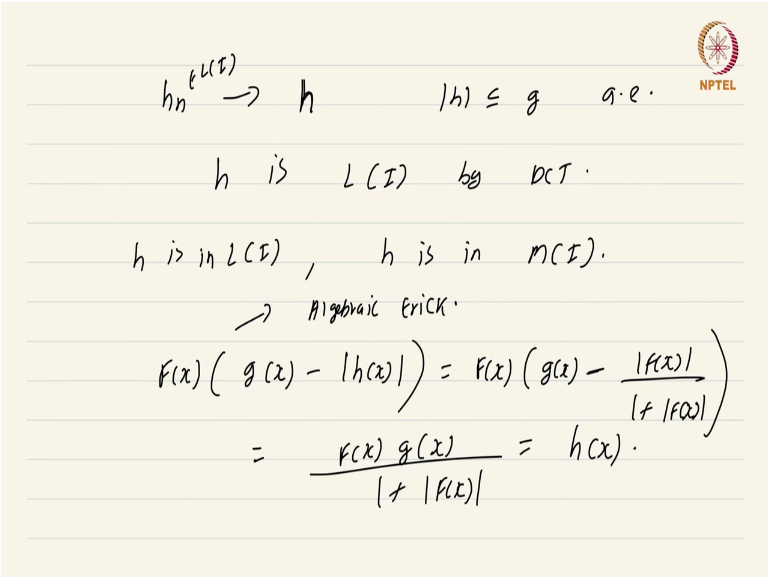
$$h(x) := \frac{f(x)\, g(x)}{1 + |f(x)|}.$$

$f_n \to f$ a.e.

$h_n(x) \to h(x)$ a.e.

So, mod h n of x is less than or equal to g of x, h n is measurable by previous theorem by previous theorem ok and because mod h n is less than or equal to g of x it follows it follows that h n is Lebesgue integrable ok. Now observe that if you define capital H of x or just small h of x by definition to be F of x times g of x divided by 1 plus mod F of x we have h n x converges to h of x almost everywhere because F n converges to F almost everywhere ok.

(Refer Slide Time: 15:38)



Which means we have the sequence of Lebesgue integrable functions that converge to this function h and mod h is also less than or equal to g almost everywhere. It follows that h is Lebesgue integrable by dominated convergence theorem ok. Now, because h is in because h is Lebesgue integrable h is in M of I as well h is measurable as well.

So, all this trick to mollify this function f n by multiplying by g is just to get a function that involves this F which is measurable ok. So, we have got that this function h is measurable. Now, what we are going to do is, we are going to play this algebraic trick you look at F of x multiplied by g of x minus absolute value of mod h of x ok.

So, this is just an algebraic trick ok. What is this let us expand it out this is going to be F of x into g of x minus mod F of x divided by 1 plus mod f of x. And if you expand this out further you will get F of x g of x divided by 1 plus mod F of x which is just h of x ok.

(Refer Slide Time: 17:35)

$$F(x) = \frac{h(x)}{g(x) - |h(x)|}$$

By the previous thm, it follows that

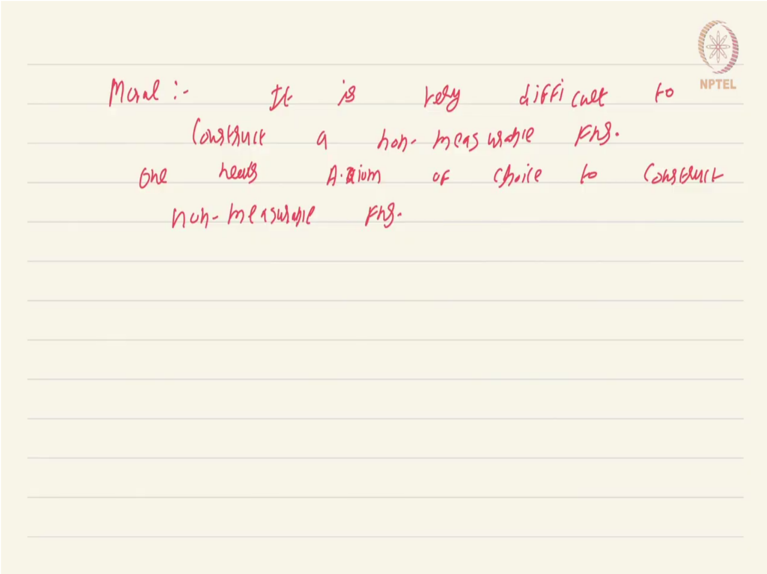
F is in $M(I)$.

(we have also used repeatedly the fact that if $F(x) \neq 0$ a.e. and $F \in M(I)$ then $\frac{1}{F}$ is measurable).

So, what we get is that F of x is nothing, but h of x divided by g of x minus mod h of x ok. And by the previous theorem by the previous theorem it follows that f is in M of I. Of course, I must mention we have also used the fact we have also used repeatedly the fact which was not state explicitly in the previous theorem, but I am stating it here we have also used repeatedly the fact that if f of x is not 0 almost everywhere and f is measurable then 1 by f is measurable.

Essentially, I stated things about product and absolute value the reciprocals also do not take you outside the class of measurable functions. If you have a function that is not 0 almost everywhere and measurable if you take its reciprocal it continues to be measurable ok. So, we have managed to show that F is a measurable function.

(Refer Slide Time: 19:07)



So, what is the moral of the story moral is it is very difficult it is very difficult to construct a non-measurable function. In fact, recall that when we had constructed this example or rather we had shown that it is not possible to define a measure on the whole of power set of R we had sort of come up with this weird set which has one point from each equivalence class well it turns out that you can use that to construct a non-measurable function.

So, to quantify you can say that one needs one needs axiom of choice axiom of choice to construct non measurable functions. Whatever you do in normal day to day mathematics that is algebraic manipulation of functions or taking limits of functions it is not going to take you outside of the class of measurable functions.

So, for all practical purposes you can assume that any function that you encounter in practice is just a measurable function. So, this class M of I is sort of like then optimal class that the analyst is interested in. This is a course on Real Analysis and you have just watched the video on Measurable Functions.