Real Analysis II
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Lecture - 31.1
The Lebesgue Integral on Unbounded Intervals

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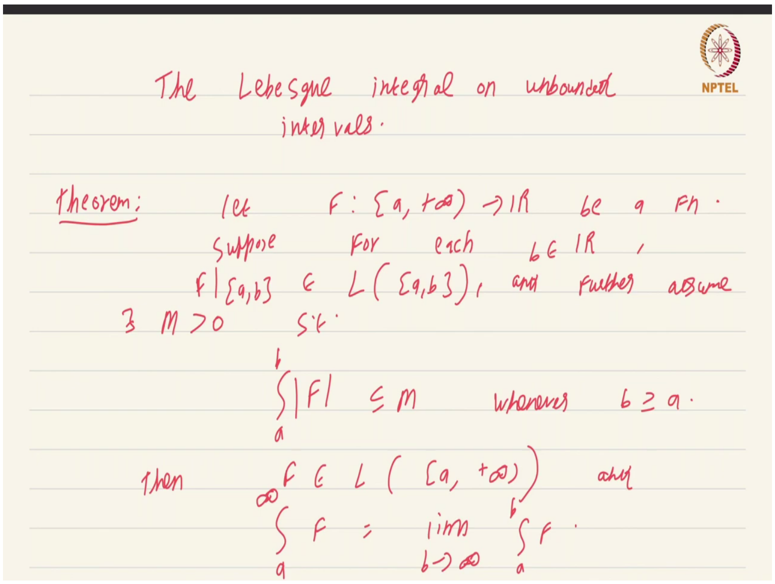

Unlike the Riemann integral we do not require the domain of definition of a function that is Lebesgue integrable to be a finite interval. It could very well be the whole of R itself. How do we check whether a function defined on an unbounded interval is Lebesgue integrable? Well, the following theorem gives one way of doing it and it will prove to be quite useful theorem. Let F from closed a open infinity to R be a function.

Suppose for each b in R F restricted to this closed a b is an element of L of a b, that is on any closed interval bounded closed and bounded interval when you restrict F to that F is Lebesgue integrable. And further assume there exists a constant M greater than 0 such that integral a to b of F a to b of the absolute value of F is less than or equal to M whenever b is greater than or equal to a.

So, independent of the choice of b suppose the integral of the absolute value of F is always bounded above by M. Then F is actually an element of L a plus infinity its Lebesgue integrable and as you can expect the integral of F a to infinity is nothing but limit b going to infinity of integral a to b F ok. So, you can obtain the Lebesgue integral of the function F on the whole interval all the way up to infinity. As a limit of its integral values on finite portions this is sort of intuitive, but it still requires a proof.

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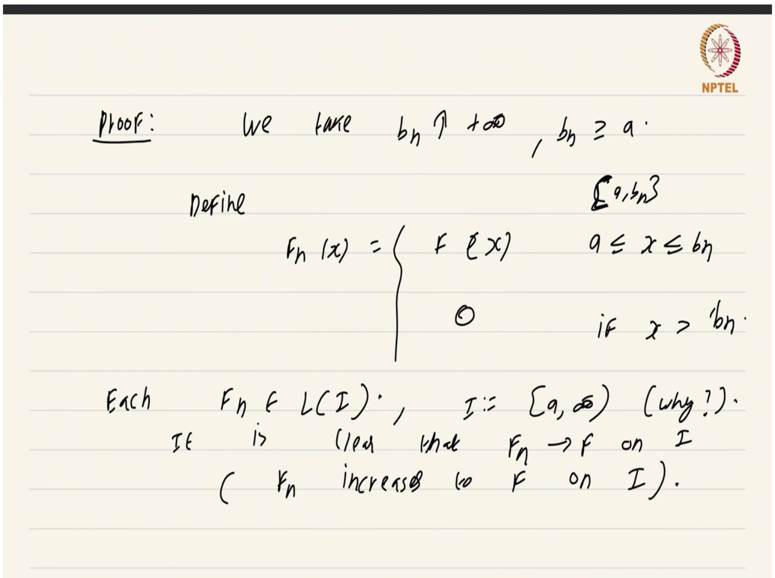

Let us move on to the proof this is going to be yet another application of Levi's monotone convergence theorem. So, what we do is we take b n increasing to plus infinity b n greater than or equal to a, take some sequence of real numbers greater than or equal to a that increases to plus infinity ok. Now, what we do is we restrict F to these intervals a b n and consider them as a sequence of functions F n, that is define F n of x by definition to be F of x if a is less than or equal to x is less than or equal to b n and 0 if x is greater than b n ok.

Now each F n is Lebesgue integrable on the interval I ok, where I is a infinity why is this the case well think about it. We already know that when you restrict F to this interval a b n it is going to be Lebesgue integrable, outside that interval it is 0. Therefore, it is got to be Lebesgue integrable on the whole of I will leave it to you to work out the details ok and it is clear that Fn let me say I defined to be a infinity, it is clear that F n converges to F on I. In fact, F n increases to F on I ok.

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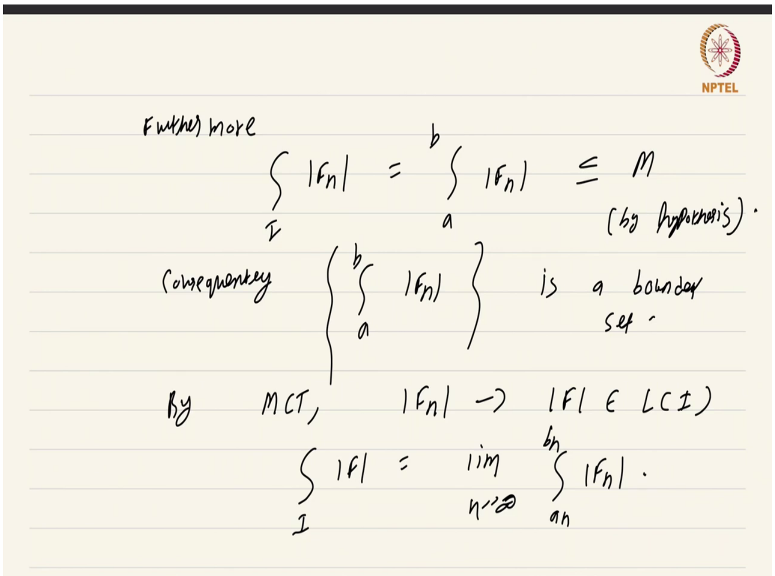

Furthermore integral over I of mod F n is equal to integral over a to b of mod F n which is going to be less than or equal to M by hypothesis ok. Consequently the sequence a to b mod F n this collection is a bounded set, is a bounded set ok. This means by monotone convergence theorem mod F n converges, of course it converges to mod F and this mod F is going to be Lebesgue integrable. And integral of mod F is nothing but limit b n or n going to infinity, n going to infinity a n to b n mod F n ok excellent.

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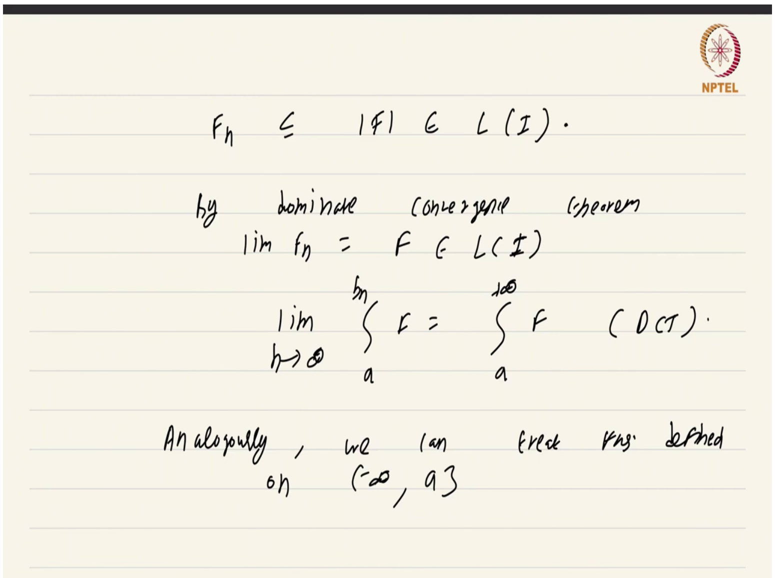

Now, what we essentially have is F n is less than or equal to mod F and mod F is an element of L of I right. Which means that, by dominated convergence theorem by dominated convergence theorem the limit function of F which is nothing but a limit function of F n which is nothing but F is also an element of L of I ok. And furthermore we have limit n going to infinity of integral a to b n F is nothing but integral a to plus infinity of F.

Again this is also by dominated convergence theorem. So, this shows that you can obtain the Lebesgue integral of F on that half infinite interval by just taking some sequence. It does not matter what sequence b n you take and taking the integrals and taking the limit. As a corollary we also get that irrespective of what sequence you choose these integral values will converge to the same value ok. Analogously we can treat functions defined on minus infinity a, exactly in the same way you just consider a sequence b n that now decreases to minus infinity.

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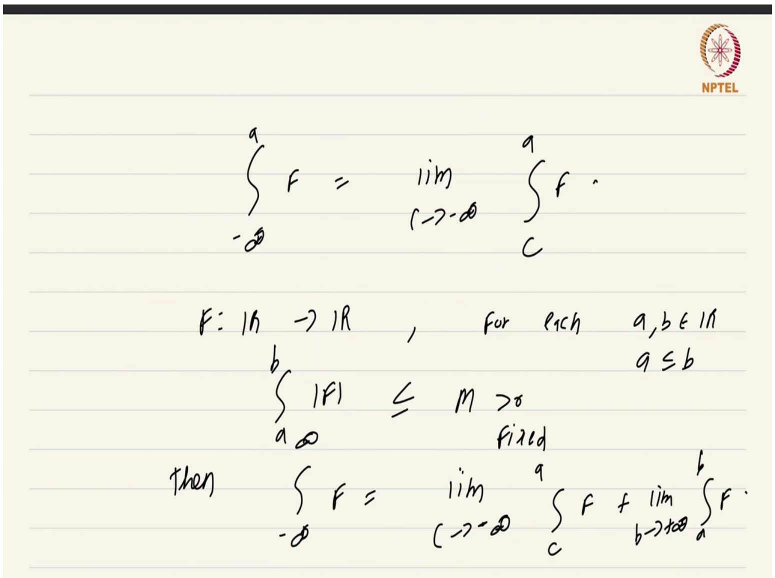

And you get almost the same result you will get integral minus infinity to a of F is limit C going to minus infinity integral C to a F ok. So of course, you have to assume that the integral of a mod F is less than or equal to m no matter what choice of C you choose. So, you have to put an analogous hypothesis on the boundedness of the absolute value integral ok.

Now, together if you have a function F from R to R then a with the hypothesis that for each a comma b in R a less than or equal to b. If we have integral a to b mod F is less than or equal to m greater than 0 fixed quantity, then we can combine the theorem that we have and the analogous version for the negative infinity interval we can combine both of them.

And write minus infinity to infinity of F is nothing but limit c going to minus infinity integral C to a F plus limit b going to plus infinity integral a to b F ok. And as a consequence we also get that it does not matter which point you break up, I mean typically what we will do is we will break up this integral into these 2 pieces by choosing this point a to be the problematic point that is usually how it is done.

Here as a consequence we get it really does not matter which point you choose we will get that integral minus infinity to infinity F can be broken down into 2 integrals and taking the limits ok.

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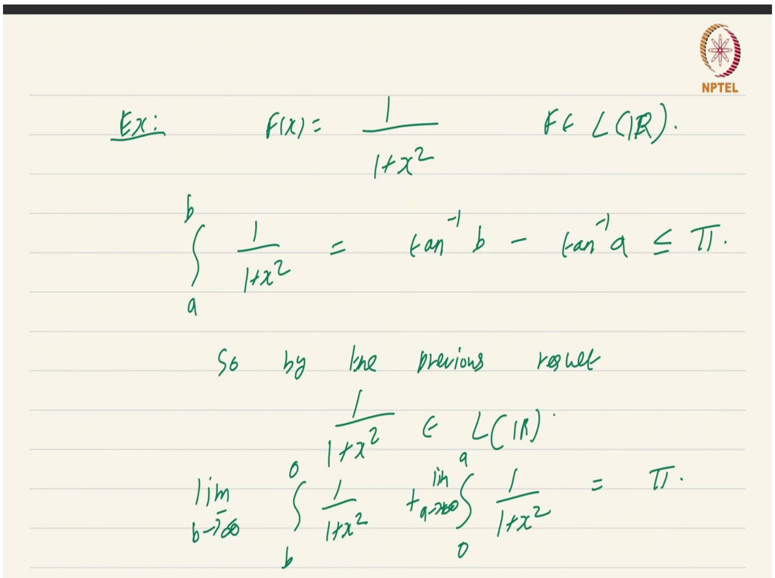

Let us work out an example, let us work out an example let us see an example of a function that is genuinely defined on the whole of R and is still Lebesgue integrable. Consider the function F of x equal to 1 by 1 plus x square ok. Claim is F is Lebesgue integrable on the whole of R. So, here is an example of a function that is nowhere 0, in fact, it is always going to take a positive value throughout the real line yet the integral is very much defined and it is a finite quantity.

Well, what do you do? When you fix a and b and consider integral a to b 1 by 1 plus x squared by high school integration you know that this is going to be just tan inverse b minus tan inverse a. This is from high school calculus and the range of the inverse trigonometric functions is from minus pi by 2 to pi by 2. So, the maximum value of this difference can be pi. So, the integral a to b 1 by 1 plus x squared is always going to be bounded by pi. So, by the previous result, by the previous result 1 by 1 plus x squared is in L of R ok.

Furthermore by that fact that you can break up the integrals you can write this as limit, let us say b going to infinity minus infinity integral b to 0 of 1 by 1 plus x squared plus integral 0 to a limit a going to infinity of 1 by 1 plus x square ok. And this would just give pi from the basic properties of the tan function. So, we have found that the function 1 by 1 plus x squared is Lebesgue integrable on R and it is integral value is exactly pi.

So, this is a sort of a stop cap video where we see how you can find out the Lebesgue integral on unbounded intervals. This is a course on Real Analysis and you have just watched the video on the Lebesgue integral on unbounded intervals.