Real Analysis II
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Lecture - 29.2
Applications of the Convergence Theorems

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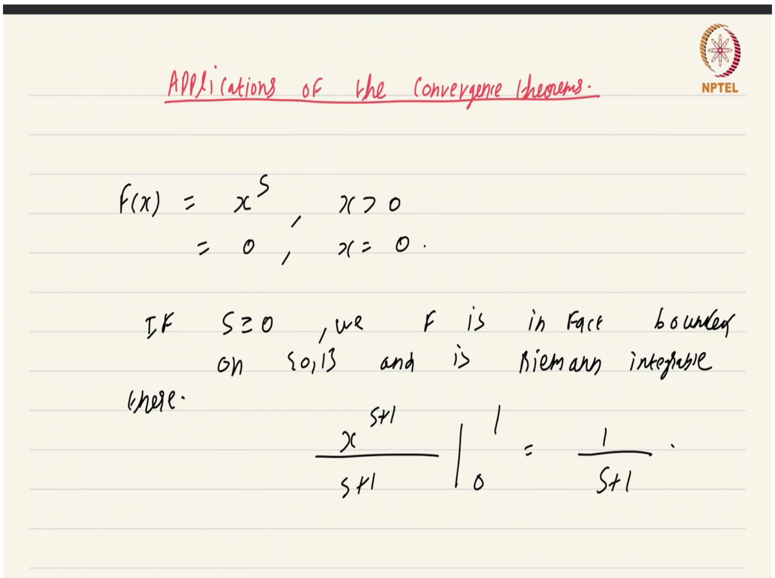

Let us see some Applications of the Convergence Theorems that we have studied. Let us begin with a simple application of the monotone convergence theorem. Consider the function F of x equal to x power S, where x is greater than 0 and 0 if x equal to 0. Look at this function. Now, if S is greater than or equal to 0, we know that F is in fact, bounded on close 0, 1 and is Riemann integrable there and it is Riemann integrable there.

And, what is the Riemann integral of this function going to be? Well it is just going to be x power S plus 1 by S plus 1 limits 0 to 1 which is just 1 by S plus 1, right.

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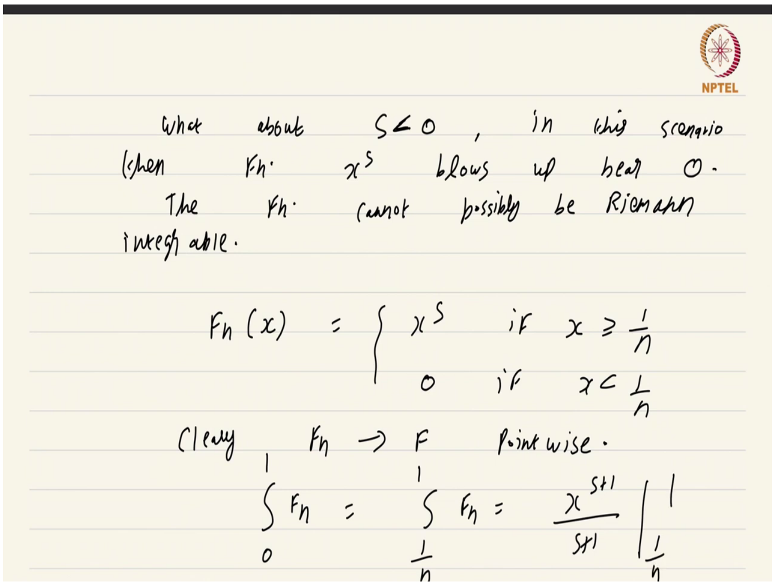

Now, what about S less than 0, ok? Well, then in this scenario in this scenario the function x power S blows up near 0. So, the function is cannot possibly be Riemann integrable cannot possibly be Riemann integrable, but is it Lebesgue integrable at least well we can see that.

How do we see that? Define the sequence of functions F n of x by x power S if x is greater than or equal to 1 by n and 0 if x is less than 1 by n. What we are essentially doing is since the function blows up near 0; we are truncating the function at the point 1 by n and setting it to be 0 when it is closer when x is closer to 0 than 1 by n, ok. So, essentially what we have done is we have sort of cut the function of before it blows up. Clearly F n converges to F point wise, right.

Furthermore, what is integral of F n 0 to 1? Well, it is just integral 1 by n to 1 of F n which is just going to be give which it is going to give us x power S plus 1 by S plus 1 with the limits 1 by n to 1.

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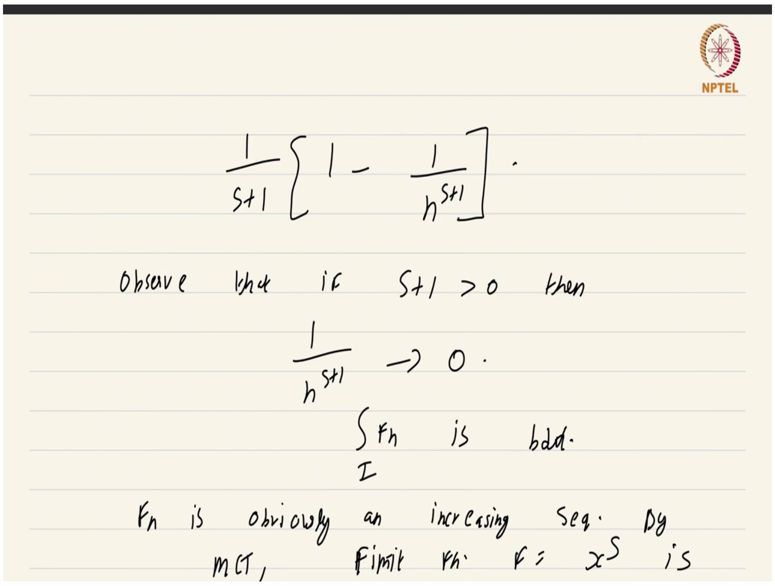

If you substitute this you will get 1 by S plus 1 into 1 minus 1 by n power S plus 1, ok. Now, observe that if S plus 1 is greater than 0, then 1 by n power S plus 1 converges to 0 ok which means that the sequence of integrals F n is bounded. Furthermore, F n is obviously an increasing sequence. obviously an increasing sequence.

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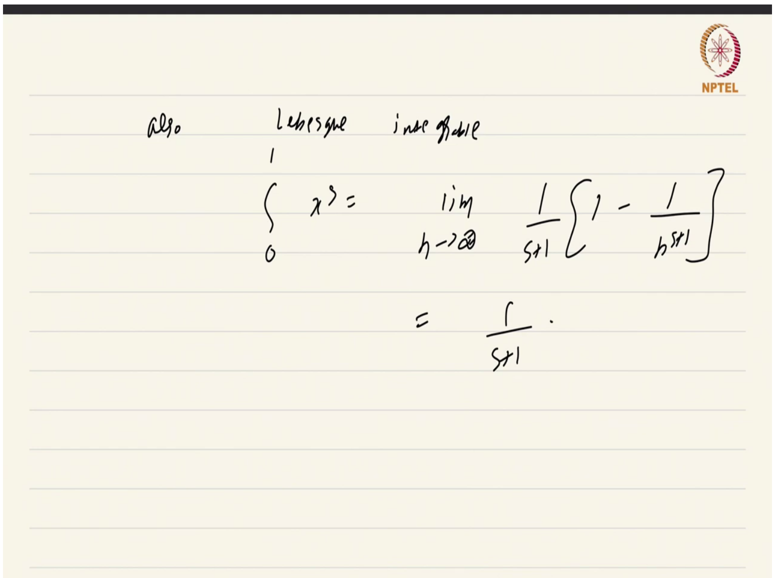

Therefore, by the monotone convergence theorem by the monotone convergence theorem F the limit of that which is just one by not one by the limit function F the limit function F given by x power S is also Lebesgue integrable. And, its integral is limit n going to infinity 1 by S plus 1 into 1 minus 1 by n power S plus 1 which is just 1 by S plus 1.

So, the functions x power S are all Lebesgue integrable provided S is greater than minus 1 and its integral is what you would expect 1 by S plus 1 excellent. So, this was one simple application of the monotone convergence theorem.

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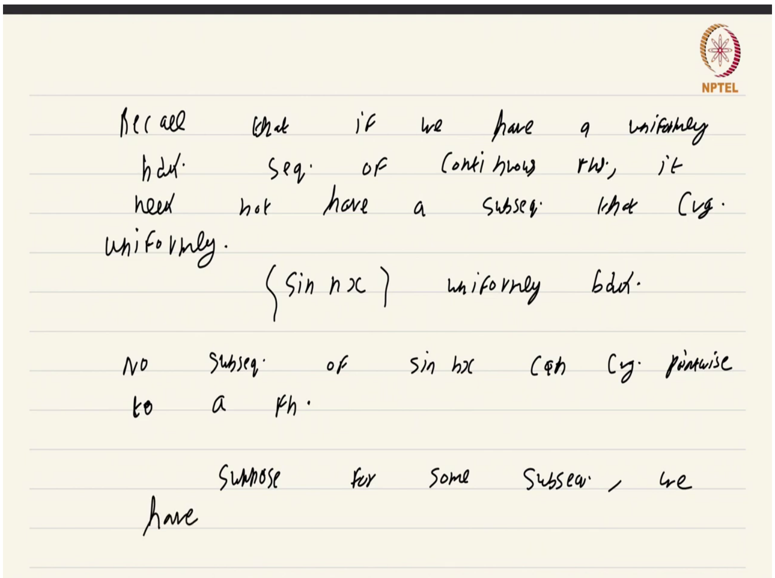

Now, let us see another application. Recall that if we have a uniformly bounded sequence of continuous functions it need not have it need not have a subsequence that converges uniformly. We have studied this extensively when we studied Ascoli Arzela to find out when you will have subsequences that are uniformly convergent.

In that context we have studied this example sin n x which is uniformly bounded, this is a uniformly bounded sequence of continuous functions and we had actually shown by an haddock argument that no subsequence of sin n x can converge point wise to a continuous function to a function forget continuity.

You cannot even have a point wise convergence subsequence of sin n x, forget about uniformly convergence subsequence of sin n x and we had constructed haddock clever argument. Well, now we can give a more conceptual proof. Suppose, for some subsequence for some subsequence we have sin n K x converges to some function F let us say ok.

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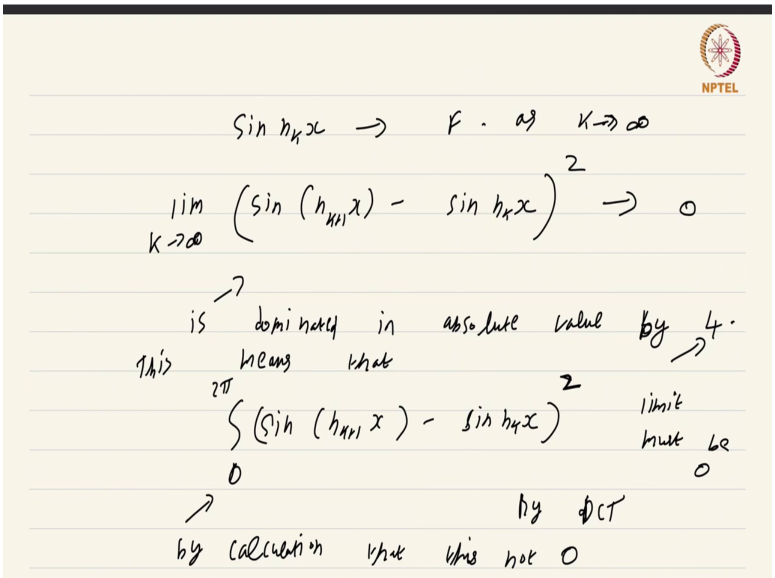

Now this just means that limit n going to infinity of sin n K plus 1 x minus sin n K x whole squared this goes to 0, right. But, what is inside is dominated in absolute value by 4 right that is the maximum possible value that you can get for sin n K plus 1 x minus sin n K x which means this means that integral over I or not I should mention what I is integral over 0 to 2 pi of sin n K plus 1 x minus sin n K x the whole squared this limit must be 0 as K.

One second I made a slight error this is K going to infinity ok here also it is as K goes to infinity ok. So, what was I saying yeah since we have this is dominated by 4 this means that sin n K plus 1 x minus sin n K x the whole squared limit must be 0 by dominated convergence theorem.

Here we are using the fact that the function 4 is Lebesgue integrable on the interval 0 to 2 pi it is obviously, because it is Riemann integrable there. But you can check by calculation check by calculation that this is not 0, this is not 0. I think it is pi or 2 pi or something like that.

Just evaluate this expression and do the integration and find out the value. You will notice that this limit is not going to be 0, it is going to be pi or 2 pi or something like that. So, here is a scenario where we have shown that sin n K x cannot possibly converge point wise to a function F that is not possible. It will violate it will violate the dominated convergence theorem, ok.

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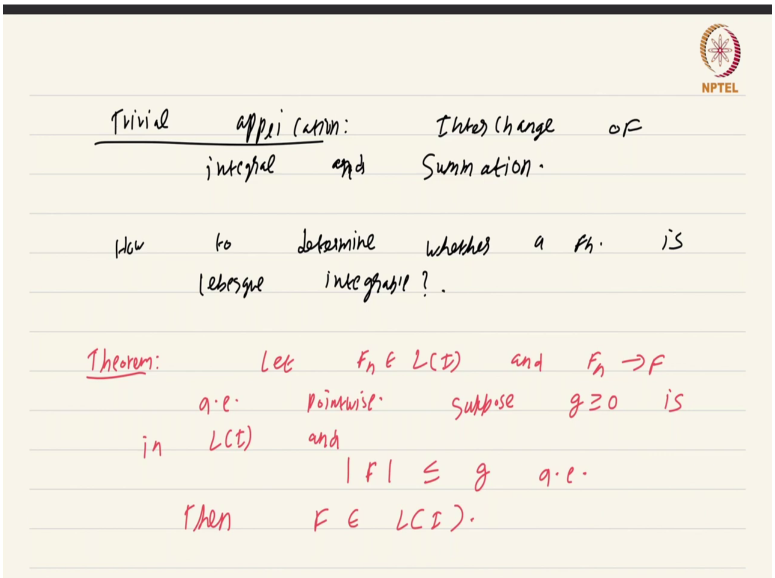

Yet another application is how to show ok of course, let me mention a trivial application interchange of integral and summation. So, I am not even going to elaborate this any further. So, let me just verbally say what I mean by this.

Suppose, if you have a sequence of Lebesgue integrable functions non negative and you look at their sums and let us say their sums converge to a function, then you can evaluate the integral of the sum as the sum of the integrals provided that the sums are all dominated by a single Lebesgue integrable function. So, I leave it to you to formulate a version of the Lebesgue's dominated convergence theorem for series it is not too hard and it is very useful also, ok.

Finally, another application is how to determine how to determine whether a function is Lebesgue integrable? So, what happens is often we do not have the full force of lebesgues dominated convergence theorem. We do not have mod F n's all dominated by g, we might have somewhat restrictive hypotheses. Let me just state that as a theorem.

Let F n be Lebesgue integrable and F n converge to F almost everywhere point wise. Suppose g greater than 0 is in L of I and mod F is less than or equal to g almost everywhere, then F is Lebesgue integrable. Notice the crucial difference between this theorem and Lebesgue's dominated convergence theorem. In Lebesgue's dominated convergence theorem we had that each F n is dominated in it is absolute value by g.

Here, only for the limit function we have that property, yet the function F will be Lebesgue integrable. This is very useful to show that certain functions are Lebesgue integrable. Let us see how to prove this.

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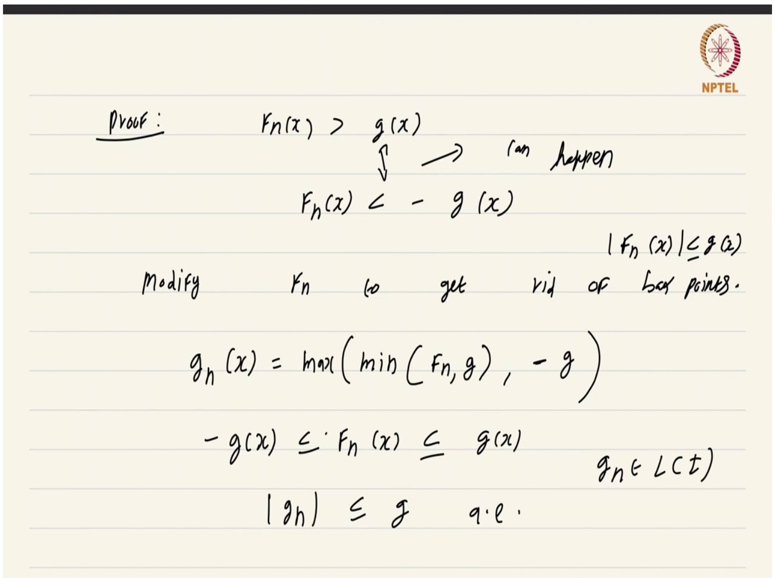

Proof: Now, what can happen? It can happen that F n of x is greater than or equal to or g of x or in fact, strictly greater than g of x or F n of x is less than minus g of x ok, both of these can happen. At those points where F n is less than g of x or F n is greater than minus g of x, then we have that mod F n of x is less than g less than or equal to g of x at those points.

But, there could be many such problematic points where either F n is greater than g or F n is less than minus g. So, what we are going to do is modify F n to get rid of bad points to get rid of bad points. Well, how do we do that? Well, at those points where F n is greater than g we are just going to truncate it to g. At those points where F n is also less than minus g we are going to set the function to be minus g.

Well, there is a fancy formula you can write down that does the job for us. You define g n of x to be the first you take the minimum you take the minimum of F n and g, what this would do is if it so happens that F n exceeds g, then it will just truncate it to g at that point; wherever F n is less than g this will continue the function g n will continue to be F n.

Now, this does not take care of those points where F n is less than minus g. So, to take care of that what you do is you put a minus g here and take max. So, let us think what happens. Let us take the three possibilities. Suppose, it happens that F n x is less than or equal to g of x and minus g of x is less than or equal to this that is we are in the good point x is a good point where everything is maintained.

Well, what will happen is when you take minimum of F n comma g you will end up with F n and when you take maximum of F n comma minus g you will end up with F n again. So, this complicated looking function does nothing if x is a good point. Now, if F n x happens to be greater than g of x then this minimum of F n comma g will be just g, then when you take maximum of g and minus g because g is a non negative function you will end up with g.

So, at a bad point where F n exceeds g you still end up with g. Similarly, if F n is less than minus g, then what will happen is minimum of F n comma g would actually end up with the g again and when you take maximum of g comma minus g. So, sorry about that. Let me clarify again if F n is less than minus g, then minimum of F n comma g would actually be F n right because F n is in fact, less than minus g therefore, it is less than g.

Now, when you take maximum of F n comma minus g because we have F n is less than minus g you will you will get minus g. So, at those points where F n dips below minus g you will get minus g ok. So, the net upshot of all this is mod g n is less than or equal to g almost everywhere and also we have g n is Lebesgue integrable because it is a maximum and minimum of Lebesgue integrable functions.

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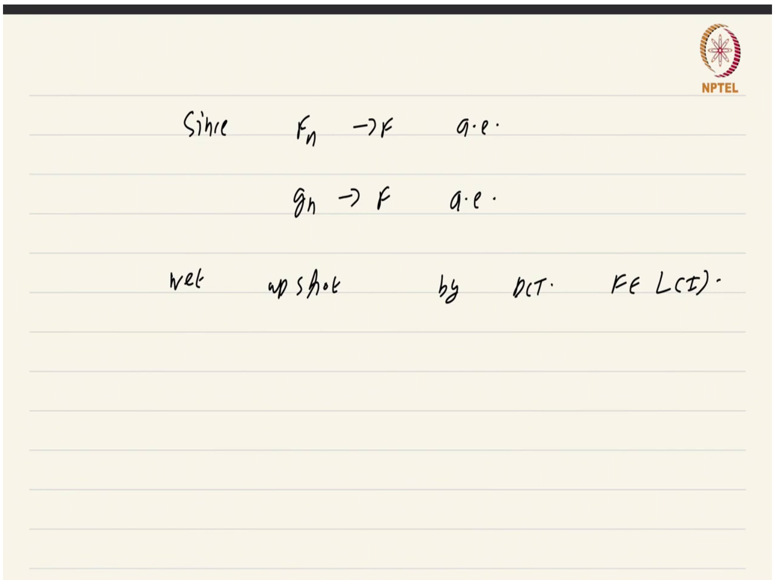

And, it is also easy to see that since F n converges to F almost everywhere g n converges to F almost everywhere ok. Net up shot is net up short is by dominated convergence theorem F is Lebesgue integrable, ok. This is another useful thing to remember about showing how to show a function is Lebesgue integrable.

This is a course on real analysis and you have just watched the video on applications of the convergence theorems.