Real Analysis II

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Lecture - 29.1
The Lebesgue Dominated Convergence Theorem

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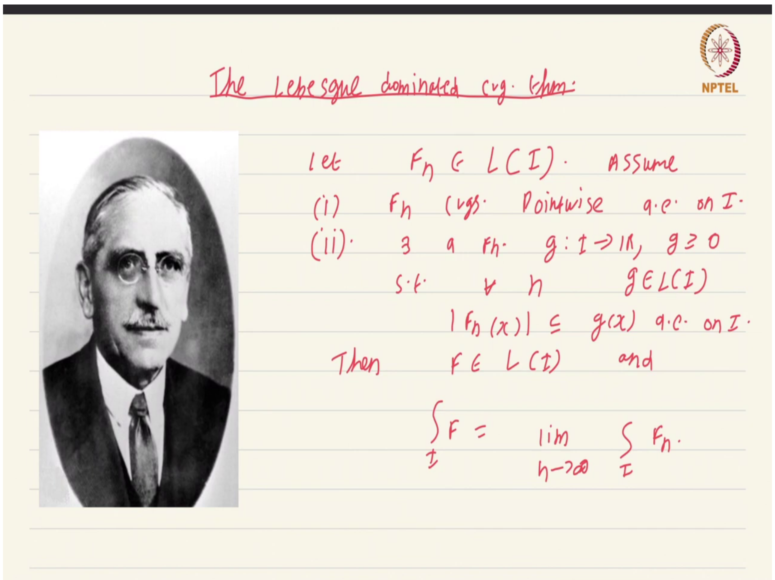

Here you see a picture of Lebesgue, the person who pioneered the theory of the Lebesgue Integral. In this video, we are going to study the most important theorem in Lebesgue Integration, Lebesgues Dominated Convergence Theorem. In fact, it is this theorem that makes this integral the canonical integral in analysis.

Apart from analysis, this theorem is extensively used in probability theory a usually to find out expectations. So, 25 years from now if somebody asks you what did you learn in your course on real analysis? The one thing you should never forget is Lebesgues Dominated

Convergence Theorem. Its going to be as relevant 25 years later as it is now. Its that important.

Lets state the theorem this is going to be an extension of the monotone convergence theorem that we have seen except now we are not going to assume any good behavior about the sequences of function in terms of monotonicity, but rather we are going to assume that the point wise limit of these functions exist. So, that is one crucial difference.

So, let us state the theorem let F n be Lebesgue integral over the interval I. Assume, assume first of all that F n converges point wise almost everywhere on I. We did not have this first requirement in the monotone convergence theorem because we got the existence of a point wise limit from monotonicity, rather from the fact that if you have an increasing monotone sequence of real numbers that is bounded above, its automatically going to be convergent.

So, that sort of helped us get a limit function for free, but because we are not going to assume anything about monotonicity of the functions F n, we have to assume that the pointwise limit exists. Number 2, this is the crucial hypothesis and the hypothesis that sort of lends the name dominated.

There exists a function g from I to R, g is greater than or equal to 0 such that for all n we have mod F n x is less than or equal to g of x almost everywhere on I. So, this function g which is a non negative of course, I must mention that, g is in l of I otherwise this whole thing is nonsensical. So, we have this non negative Lebesgue integrable function g which dominates the functions F n and this g is independent of n remember that ok.

What is the conclusion then? Then F is Lebesgue integrable and as you can expect the integral of F is nothing but the limit of the integrals of F n. So, the hypothesis are that you have a sequence of Lebesgue integrable functions that converge almost everywhere pointwise and these functions are dominated by a single non negative Lebesgue integrable function then the conclusion is that the limit function is Lebesgue integrable and the integral of the limit is the limit of the integrals ok.

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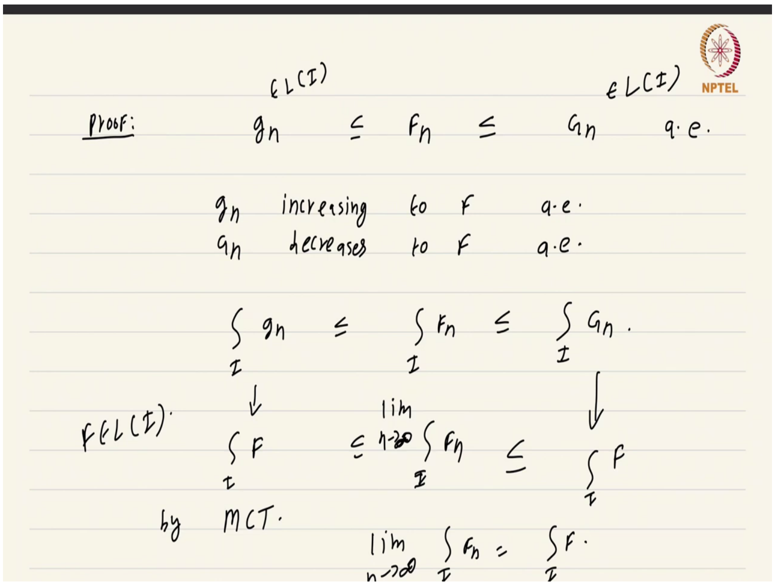

So, let us go on with the proof. Briefly speaking there is nothing deep going that is going to happen in this proof we are just going to manipulate the sequence F n, but the basic idea is to apply the monotone convergence theorem to get functions g n and capital G n both of these functions will be Lebesgue integrable we have to cook up these functions such that this little g n is always less than or equal to F n and this is less than or equal to G n of course, all this we need to be true only almost everywhere.

We want g n less than or equal to F n, less than or equal to capital G n and we want g n increasing to F, increasing to F almost everywhere whereas, capital G n decreases to F, decreases to F almost everywhere ok. So, let us assume for the time being that we have actually found out these functions g n.

So, little g n and capital G n with the with the property that little g n is less than or equal to F n less than or equal to capital G n almost everywhere and we have g n increasing to F almost everywhere and capital G n decreasing to F almost everywhere ok.

With this done, the result follows almost immediately because you can integrate this entire expression integrate this expression out. So, you will get integral of g n over I is less than or equal to integral over i F n is less than or equal to integral over I capital G n ok, but because this little g n, for instance this little g n increases to F this side sort of converges to integral over I of F by monotone convergence theorem.

Because g n increases almost everywhere to F it follows that integral of I, g n must converge to integral of I F. In fact, as a consequence we get F is in L of I. F is Lebesgue integrable. This much we get of course, the middle side remains the same. Now this capital G n this sequence decreases to F when we establish the monotone convergence theorem we established it only for increasing sequences, but there is nothing special about increasing sequences.

You can formulate and easily prove an analogous version of the monotone convergence theorem for decreasing sequences also. So, once having done that it will be easy to see that this converges to integral of I F as well ok and you will have this inequality. So, integral of I F is less than or equal to integral of I of course, ok I made a slight error. I said there is no change in the middle term, but there is a change there is a limit n going to infinity ok.

So, we get integral of I F is less than or equal to limit n going to infinity integral of I, F n is less than or equal to integral over I F which just shows that limit n going to infinity integral of I. F n is integral of I F ok.

So, this will finish the proof. So, all we have to do is manufacture these functions little g n and capital G n. I will manufacture the function capital G n with the required properties the little g n function will be can be easily produced by adopting the same procedure by replacing maximum by minimum in certain places.

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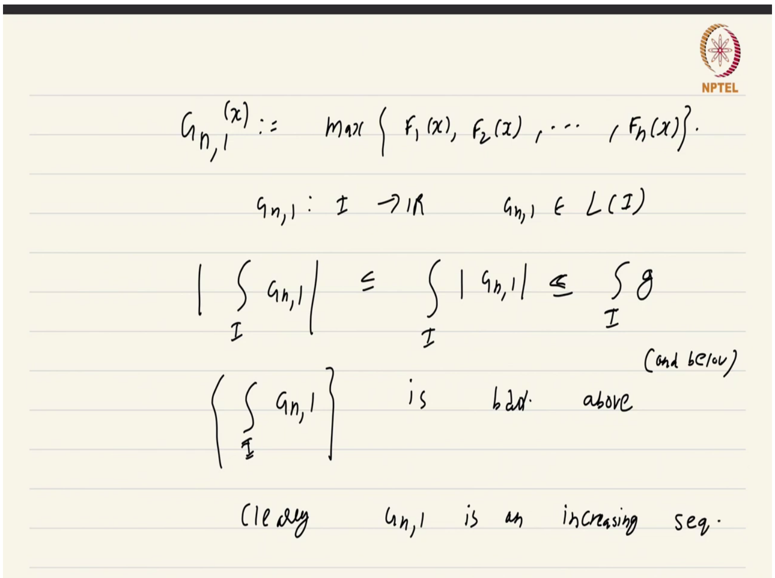

So, let us see how to do this. Let us see how to do this, what we do is we first define capital G n 1 to be maximum of F 1 x F 2 x comma dot dot dot F n x ok. So, G n 1 is a function from I to R, its obtained by taking the maximum of the first n of the functions, F 1 of the sequence F n you just take the first n of them and call that the call the maximum G n 1.

Now G n 1 is obviously, going to be Lebesgue integrable because it is given by taking maximum of Lebesgue integrable functions. So, there is no issue there furthermore you have integral of I G n 1 this is less than or equal to integral of I, the absolute value of G n 1, this is just a basic property of Lebesgue integrable functions we have already seen.

And this is less than or equal to integral of I g where g is the function that dominates each one of the sequences each one of the I mean every element in the sequence F n. So, this g right. So, because this G n 1 is just maximum of F 1 to F n and the absolute value of each one of these is dominated by g, we get this inequality, from the basic properties of Lebesgue integrals under ordering ok. So, in particular what we have is integral of I G n 1 this set you look at the is bounded above ok.

In fact, its bounded below also, bounded above and below also because there is an absolute value here. Its going to be bounded below also that will play a role a little bit into the future ok. Anyway this is bounded above and clearly G n 1 is an increasing sequence.

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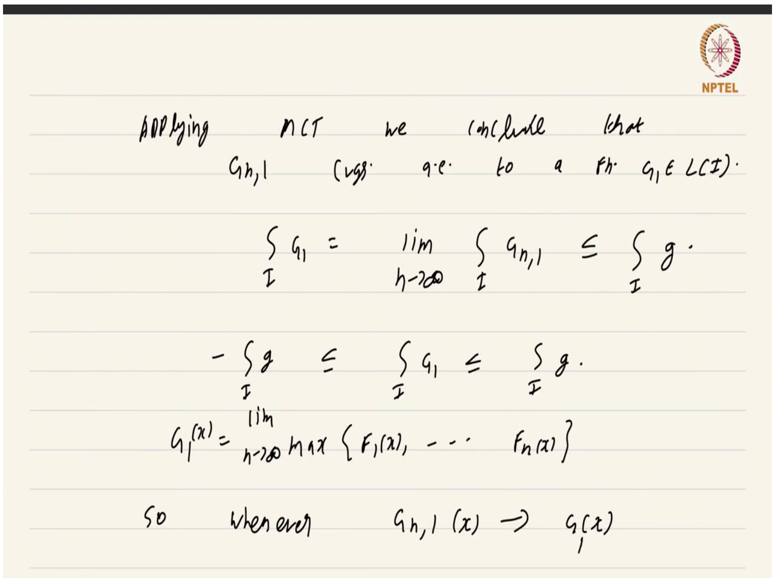

So, everything is set at this stage. We can apply the monotone convergence theorem. Applying, applying NCT we conclude that G n 1 converges almost everywhere to a function G 1 which is Lebesgue integrable ok and not only that we have that the Lebesgue integral of G 1 is nothing but the limit n going to infinity of Lebesgue integral of G n 1 which is less than or equal to Lebesgue integral of G right. This follows from the inequality that we have here.

In fact, from that same inequality we can In fact, conclude that integral of I G, the negative is less than or equal to integral of I G 1 is less than or equal to integral of I G because we had an absolute value here. So, what we have got is, we have got this function G 1 by monotone convergence theorem. We have got this function G 1 and this G 1 is Lebesgue integrable and its dominated by integral of I G and a I mean in absolute value its dominated by the integral of G in fact, ok.

Now, let us proceed with the proof. Let us see analyze this function a bit more. Notice that G 1 is how do you determine, how do you determine G 1 of x. Well what you are doing is, you are taking maximum of F 1 of x dot dot dot F n of x and then taking limit n going to infinity, right that is essentially what G 1 of x is going to be.

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$g_1(x) = \sup\{f_1(x), f_2(x), \dots\}$.

$G_{n,r}(x) = \max\{f_r(x), f_{r+1}(x), \dots f_n(x)\}$.
$\forall\ n \geq r$.

$G_{n,r}$ increases. By MCT
$G_{n,r}$ must cvg. a.e. to a fn.
$G_r \in L(I)$

$$-\int_I g \leq \int_I g_r \leq \int_I g.$$

So, whenever G n 1 x converges to G of x, remember this convergence is only almost everywhere on I. Whenever this convergence happens sorry G 1 of x, whenever G N 1 of x converges to G 1 of x we conclude that G 1 of x is nothing but supremum of F 1 x, F 2 x comma dot dot dot excellent ok.

Now, what we are going to do is we are going to manufacture G 2, G 3, G 4 in an exactly similar way we are going to define G n r x to be the maximum, but starting from the r th sequence F r x, F r plus 1 x comma dot dot dot F n x ok and this is defined for all n greater than or equal to r ok.

So, we define this new sequence J n r x defined for n greater than or equal to r by taking maximum of F r x F r plus 1 x dot dot dot F n x ok. Clearly this sequence G n r also increases exactly the same reason why G n 1 increases you are taking maximum of more and more function. So, thus the value can only increase.

And again by monotone convergence theorem. G n r x G n r must converge almost everywhere to a function, to a function g r which is Lebesgue integrable and we also have these inequalities minus integral over I G is less than or equal to integral over I G r is less than integral over I G, exactly like before excellent.

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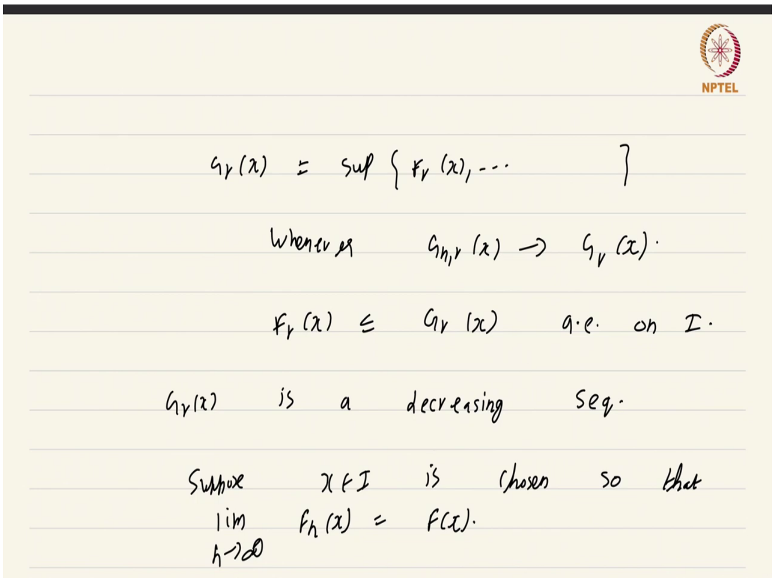

Now, observe exactly the same observation that we have here, exactly the same observation we have here we can write down that G r of x is nothing but the supremum of F r of x comma dot dot dot, all the way up to infinity whenever G n r x converges to G r x, at those points where you have convergence this G r x the value of that is nothing but the supremum of F r x and you F r x F r plus 1 x dot dot the supremum of the set ok.

So, from that we see that F r x is less than or equal to G r x almost everywhere on I because the sequence G n r x converges to G r x almost everywhere. At those points of convergence we have this expression and therefore, we have this inequality. Now this G r x is going to as the notation suggest this is going to be our required thing, we have to show that G r x converges to the function F pointwise almost everywhere.

First of all G r x is certainly is a decreasing sequence is a decreasing sequence why is G r x a decreasing sequence. Well by definition you notice that the supremum that you are taking is over less and less number of functions as r increases therefore, its a decreasing sequence.

Now suppose x in I is chosen so that limit n going to infinity of F n of x is equal to F of x. Remember the sequence F n itself does not converge everywhere to the function F. It converges only almost everywhere. What I am going to show is that at those points of convergence, we actually have that this sequence G n x also converges to F of x ok. Now what is the meaning of limit n going to infinity F n x equal to F x.

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Fix $\varepsilon > 0$ then $\exists N$ if $n > N$

$$F(x) - \varepsilon < F_n(x) < F(x) + \varepsilon.$$

$$F(x) + \varepsilon \leq \sup\{ F_n(x), F_{n+1}(x), \ldots \}$$
$$\leq F(x) + \varepsilon.$$

$$F(x) + \varepsilon \leq G_n(x) \leq F(x) + \varepsilon \quad \forall n > N.$$
$$G_n(x) \to F(x).$$

Well fix Epsilon greater than 0, then there exists n such that if small n is greater than N then F of x minus Epsilon is less than F n x is less than F of x plus Epsilon. This is just the very definition of limit n going to infinity F n x equal to F x. Now, that we have this, its clear that if you take supremum of F n x, F n plus 1 x comma dot dot dot, you will have F of x plus Epsilon is less than or equal to this is less than or equal to F of x plus Epsilon.

In other words we have F of x plus Epsilon is less than or equal to G n x is less than or equal to F of x plus Epsilon for all n greater than capital N ok and this just shows that G n x converges to F of x. So, what we have managed to show is that this sequence of functions G n x converges to F of x precisely at those points where F n x converges to F of x ok. What next.

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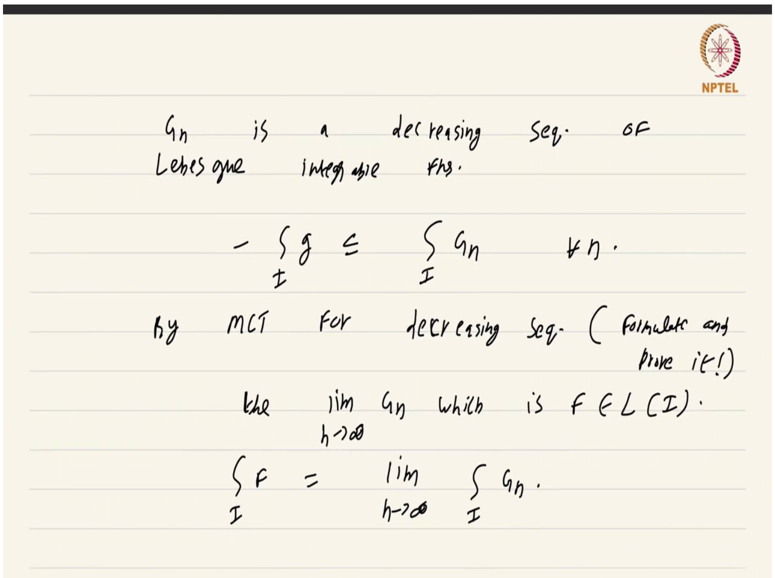

Well we already observed that G n is a decreasing sequence. G n is a decreasing sequence of Lebesgue integrable functions ok. Furthermore we also observed that integral of I g is less than or equal to integral of I g n for all n ok. Now by monotone convergence theorem for decreasing sequences; for decreasing sequences, you have to formulate and prove this formulate such a theorem and prove it also.

Its not that hard you can just adapt the proof of the increasing case or just once you state the theorem, you can see how you can use the theorem for the increasing case to prove the decreasing case. Anyway by the monotone convergence theorem for decreasing sequences G n converges I mean not we already know G n converges we know that the limit of G n which we know to be F, which we know to be F.

This is Lebesgue integrable and we also have that Lebesgue integral of F is nothing but limit n going to infinity of integral of I G n ok.

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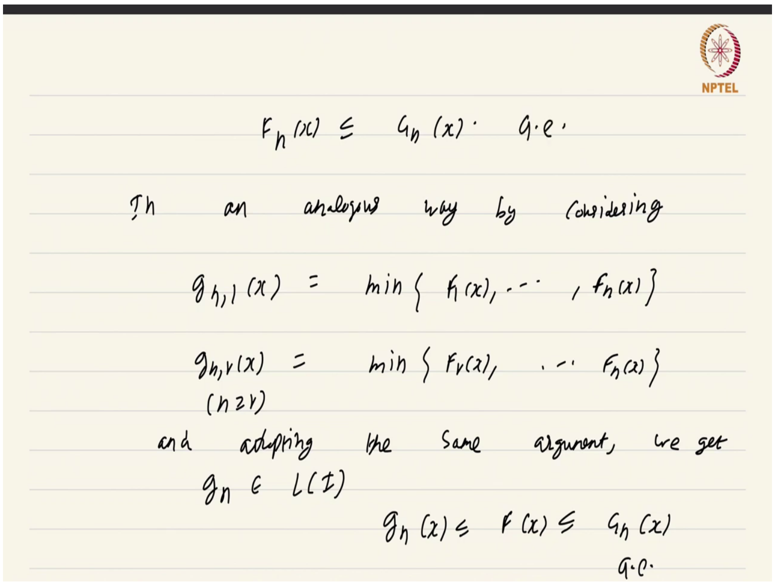

Now, we already know, we already know by a construction that F n of x is less than or equal to G n of x. In an analogous way, in an analogous way by considering g little one, little g n 1 x to be by definition the minimum of F 1 x dot dot dot F n x and G n r x for n greater than or equal to r to be minimum of F r x comma dot dot dot F n x.

And adapting the same argument, adapting the same argument, we get functions G n which are Lebesgue integrable and G n of x is less than or equal to F of x is less than or equal to capital G n of x almost everywhere even this is just almost everywhere.

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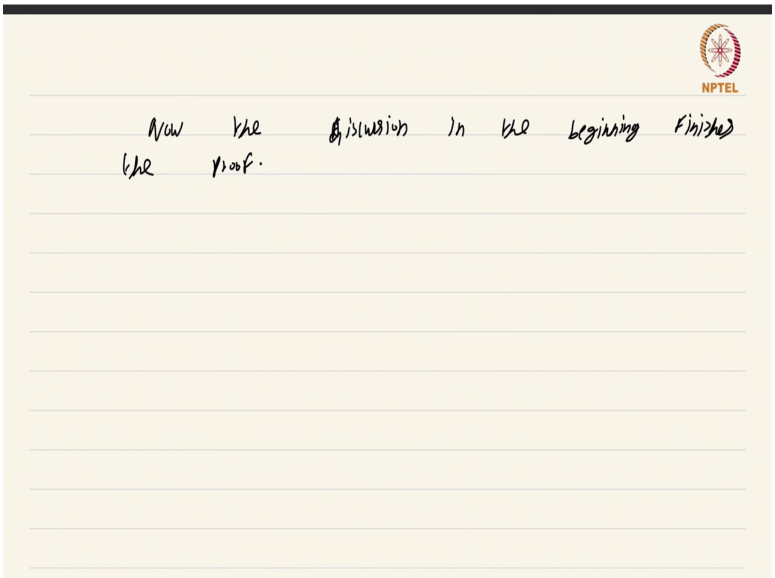

Now, the first part finishes the proof, the very first the discussion in the beginning. In the beginning, finishes the proof. So, there is nothing really much happening in this proof. You just have to construct these sequences and play around with them for some time and you will eventually get the result. In the next video we shall see applications of both the monotone convergence theorem and the dominated convergence theorem including a particularly interesting one about uniform convergence.

This is a course on real analysis and you have just watched the video on the Lebesgue Dominated Convergence Theorem.