Real Analysis II
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Lecture - 28.3
Monotone Convergence Theorem for Lebesgue Integrable Functions

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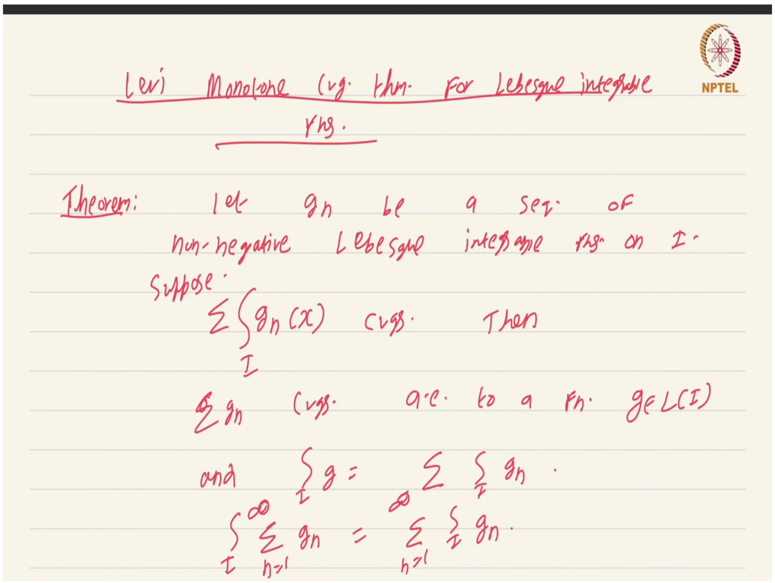

At last we come to Levi's Monotone Convergence Theorem for Lebesgue Integrable Functions. All the hard work has been done its just now going to be some manipulation with the results we already have to establish the result. What I am going to do is I am going to state and prove the version of Lebesgue's monotone convergence theorem for Lebesgue integrable function a version for interchanging sum and integral.

And I am going to leave it to you to formulate and prove a version for just sequences and not sums. If you are having any difficulty you can consult Apostles textbook. So, without further I do, here is the theorem. Let g n be a sequence of non negative Lebesgue integrable functions Lebesgue integrable functions on I.

Suppose, number 1, summation there is only one hypothesis suppose summation g n of x converges almost everywhere on I ok, sorry summation integral over I g n x converges, no almost everywhere of course. Then summation g n converges almost everywhere to a function to a function g which is Lebesgue integrable and integral over I g is just summation integral over I g n ok.

So, what you expect what you expect is actually holding true. To highlight the importance of this result we can write it like this, integral of I summation over n equals one to infinity g n is equal to summation n equals 1 to infinity integral of I g n. We have interchanged the summation and the integral, and such results are going to be very powerful.

Once we prove the dominated convergence theorem in the next video, we are going to see several applications of both this monotone convergence theorem as well as the dominated convergence theorem. In particular, if you recall in a Real Analysis-I, we had studied something about uniform convergence and something that is not true for point wise convergence, we are going to see that in more depth in the applications ok.

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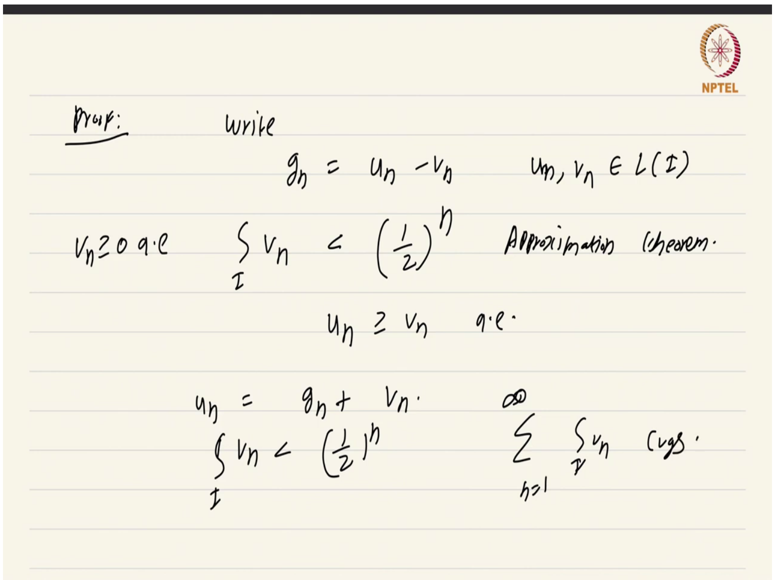

Let us prove this result. And since all the technology has been done before, the proof is not very hard ok. Now, what we need to do is we need to express each of these g ns as a difference of upper functions that will come by definition, but we have to be a bit more finesse we have to do it with finesse. So, what we are going to do is we are going to use the approximation theorems for Lebesgue integrable functions we have shown.

What we are going to do is we are going to write g n as u n minus v n, where u n and v n are Lebesgue integrable functions with the additional property that integral over I v n is less than half power n you will understand in a moment why we are requiring this strange requirement. It is essentially to make sure a particular series is actually convergent ok. So, this follows from approximation theorems. Please revisit the video if you are not able to recall this from approximation theorem.

Of course since g n is greater than or equal to 0 almost everywhere needless to say u n is greater than or equal to v n almost everywhere. Of course, I forgot one thing v n is greater than or equal to 0 almost everywhere ok. So, the fact that you can choose v n greater than or equal to 0 almost everywhere with integral of I v n less than half power n follows from one of the approximation theorems that we have shown last week ok.

Now, that you got u n is greater than or equal to v n almost everywhere and the fact that integral of I v n is really small, we can start doing some manipulations. What we do is we write u n as g n plus v n ok which means let we know that integral of I v n is less than half power n which means the series integral of I v n n running from 1 to infinity converges ok.

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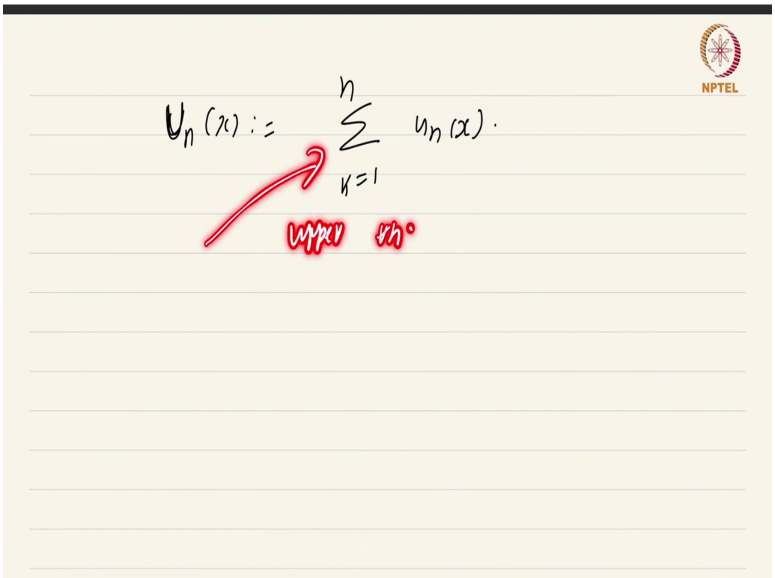

Now, what you do is you consider the partial sums which you denote by capital U n of x capital U n of x is just the partial terms sums k running from 1 to n u n of x ok.

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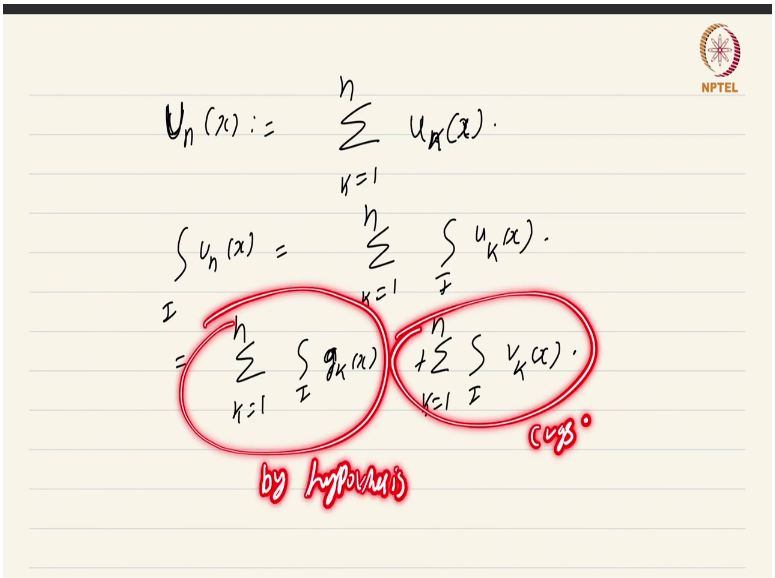

This being a sum of upper functions is also an upper function. And by the various properties of upper functions, you have integral of U n x is exactly equal to summation k equals 1 to n integral of I small u n x ok. But small u n x you can write this as summation k equals 1 to n integral of I g n x plus integral of I v n x.

You can break it up into these two terms. Well, let me just add the summation here k equals 1 to n. So, I made a mistake with the index. So, it should all be k, this should be u k of x and. So, are these places u k, g k of x, v k of x, excellent. Now, this term converges. And this term converges by hypothesis.

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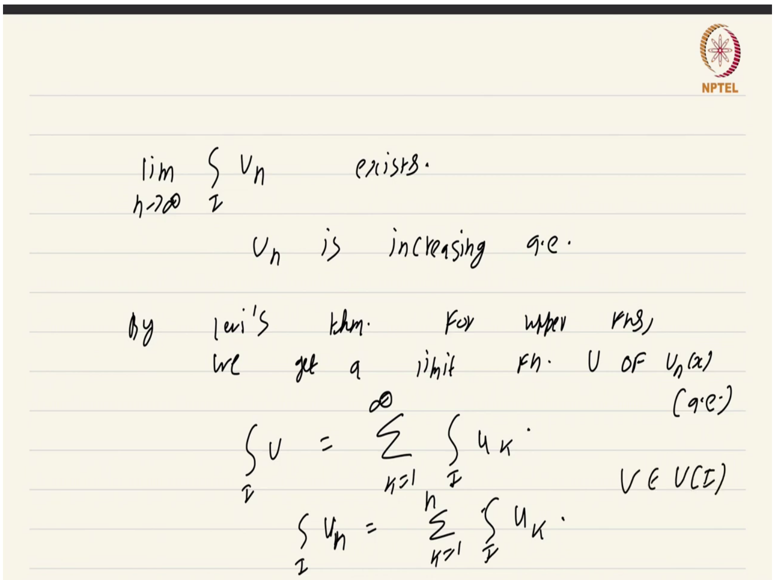

So, the net upshot of all this is that this integral of I U n limit n going to infinity exists. Note that U n is increasing U n is increasing almost everywhere. This follows because we assumed that u n is greater than or equal to v n almost everywhere which is greater than or equal to 0 almost everywhere. So, this is these partial sums will consist of non-negative terms almost everywhere. So, this will be almost everywhere increasing.

By Levi's theorem for upper functions by Levi's theorem for upper functions we get a limit function we get a limit function U of U n of x of course this is an almost everywhere limit. And this U as the property that this integral of I u is nothing but summation k equals 1 to infinity integral of I U k. Why does this follow, because you have integral of I U k U n is nothing but summation k equals 1 to infinity integral of I small u k this is 1 to n, of course this is small u k. Excellent, we are now in good shape.

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Similarly, we can find a fn. V

s.t.

$$V_n := \sum_{k=1}^{n} v_k(x) \quad \text{cvg. a.e.}$$

and $\int_I V = \sum_{k=1}^{\infty} \int_I v_k(x)$. $\quad V \in U(I)$.

Consider $U - V \in L(I)$.

$$\sum_{k=1}^{n} g_k = U_n - V_n \quad \text{this cvg. a.e. on } I \text{ to } U - V.$$

Similarly, we can find we can find a function V such that the partial sums V n defined to be summation k equal to 1 to n, so little v k of x converges almost everywhere, and the integral of I V n is nothing but summation sorry integral of I V nothing but summation k equals 1 to infinity integral over I little v k of x ok. Exactly are given as follows.

Now, this means that of course, V is in U of I, it is an upper function ok. Needless to say U is also an upper function. These come from the conclusion of Levi's monotone convergence theorem for upper functions ok. Where does this leave us? Well, consider capital U minus V, this is a function that will be Lebesgue integrable simply by definition because it is a difference of two upper integrable functions ok.

And observe that this sequence summation k equals 1 to n g k is nothing but U n minus V n ok. And this converges with this converges almost everywhere on I that is what we have essentially shown to U minus V ok.

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set $g := U - V$ Then $g \in L(I)$

$$\int_I g = \int_I U - \int_I V = \sum_{k=1}^{\infty} \int_I (u_k - v_k)$$

$$= \sum_{k=1}^{\infty} \int_I g_k \quad \text{as claimed.}$$

Now, to finish the proof set g to be U minus V, then we have that g is Lebesgue integrable and integral of I g is nothing but integral of I U minus integral of I V which is nothing but summation k equals 1 to infinity integral over I u k minus v k which is nothing but summation k equals 1 to infinity integral over I g k as claimed.

So, once the hard work has been done for upper functions, there is really nothing much to do. You just use the approximation theorem to put yourself in a situation where you can repeatedly apply the theorem for upper functions to both the positive and the negative function that is the functions U n minus V n, and you are done.

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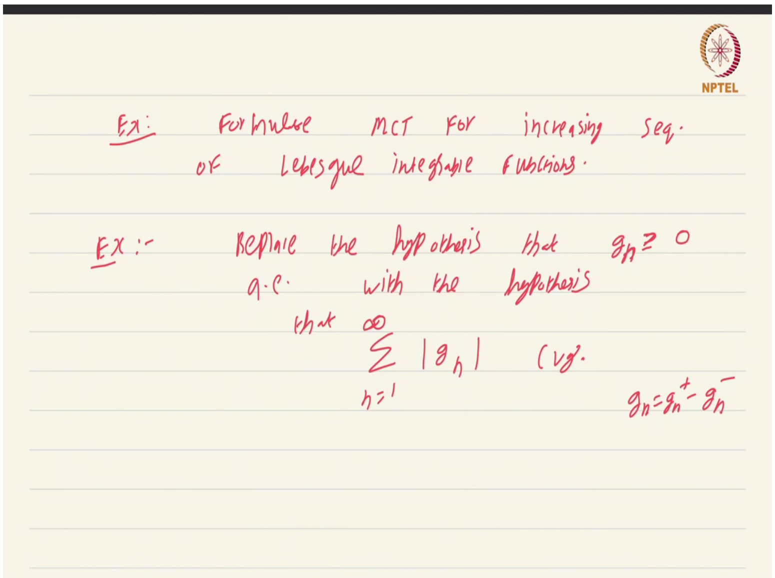

Now, I am going to give two exercises. And you can consult Apostles book if you get stuck. Formulate MCT that is Monotone Convergence Theorem for increasing sequence of Lebesgue integrable functions, I mean series is more complicated than sequences. We have done the more complicated case.

So, this should be rather easy. You just have to figure out how to convert the sequence to a series, so that everything works out and that is a nice trick that I want you to discover.

Exercise, another exercise, remove or replace the hypothesis replace the hypothesis that g n greater than or equal to 0 almost everywhere with the hypothesis with the hypothesis that the summation of the absolute values converge, it converge ok. Instead of assuming that summation g n converges assume that the summation absolute value converges and now remove the hypothesis g n greater than or equal to 0.

Hint; consider the positive and negative parts that is you can always write g n as g n plus minus g n minus ok. Recall what these g n plus and g n minus are, and apply the similar arguments, and you will get it easily.

So, this concludes this video. In the next video, we shall see the really powerful dominated convergence theorem then onwards to applications. This is a course on Real Analysis, and we have just watched the video on Levi monotone convergence theorem for Lebesgue integrable functions.