Real Analysis II

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Lecture – 2.3
Basic Properties of Open and Closed Sets

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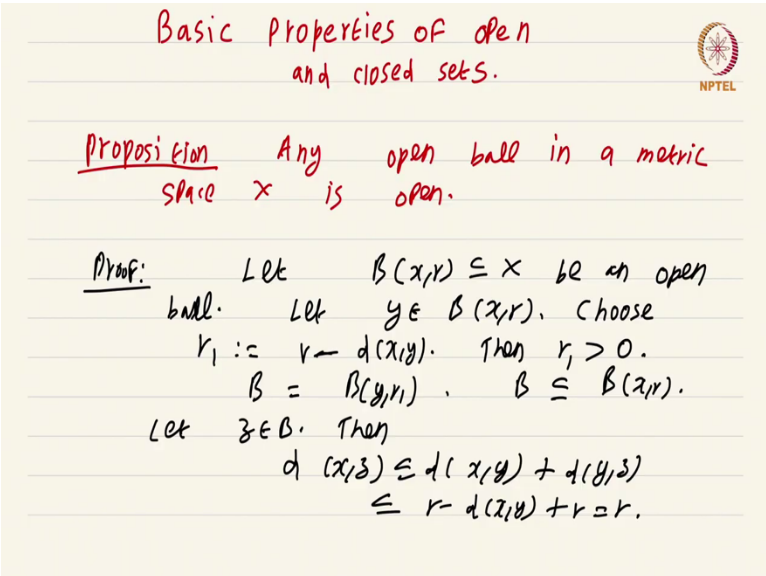

In this short module, we will prove the basic properties of open and closed sets. The proofs are more or less the same as what we have seen for the real numbers. So, we shall be very brief in the proofs.

Proposition: Any open ball in a metric space X is open.

Proof. The proof is rather trivial. Let $B(x,r) \subseteq X$ be an open ball. Let $y \in B(x,r)$. We have to show that this point y is an interior point of $B(x,r)$. So, we choose

$$r_1 = r - d(x,y).$$

Then, clearly $r_1 > 0$. Consider $B = B(y,r_1)$. The triangle inequality will immediately tell you that $B \subseteq B(x,r)$. To see this more precisely let $z \in B$, then

$$d(x,z) \leq d(x,y) + d(y,z) \leq r - d(x,y) + r = r.$$

So, this proof was rather trivial.

Now, we are going to do a different characterization of open and close sets. We are going to show that open and close sets are dual notions.

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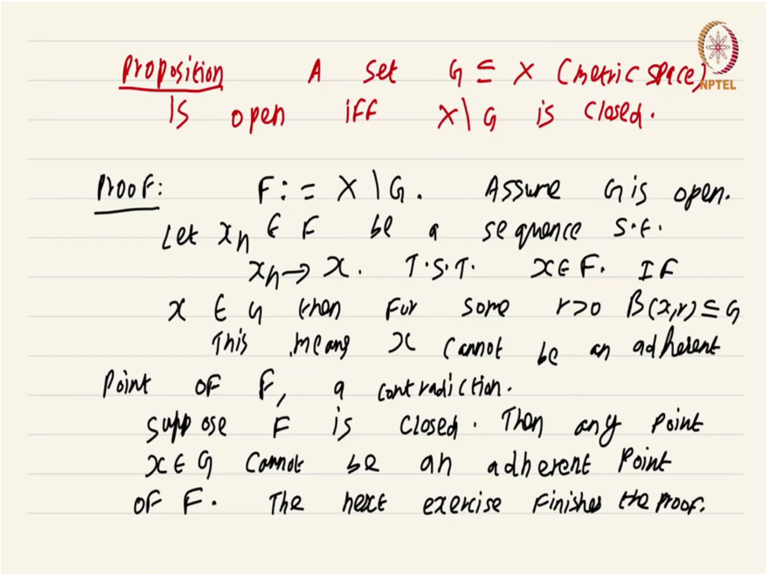

So, again the proof of this is very similar to what we have already seen for the real numbers.

Proposition: A set $G \subseteq X$ (metric space) is open if and only if $X \setminus G$ is closed.

Proof. Define $F = X \setminus G$. Assume G is open. Our goal is to show that F is closed. Let $x_n \in F$ be a sequence such that $x_n \to x$.

We have to show that $x \in F$.

If $x \in G$, then for some $r > 0$, $B(x,r) \subseteq G$. That is just because G is an open set. Therefore, x must be an interior point. Therefore, we can find a ball, such that $B(x,r)$ is fully contained in G. This means x cannot be an adherent point of F, a contradiction. What will happen is after a particular stage, we must have $x_n \in B(x,r)$. That is simply not possible because $B(x,r)$ is now a subset of G.

Now, suppose F is closed, then any point $x \in G$ cannot be an adherent point of F because F and G are complements of each other.

Now, the next exercise finishes the proof. I will leave a very simple exercise for you guys to do, which is exactly similar to what you have done previously for adherent and limit points in the case of the real numbers.

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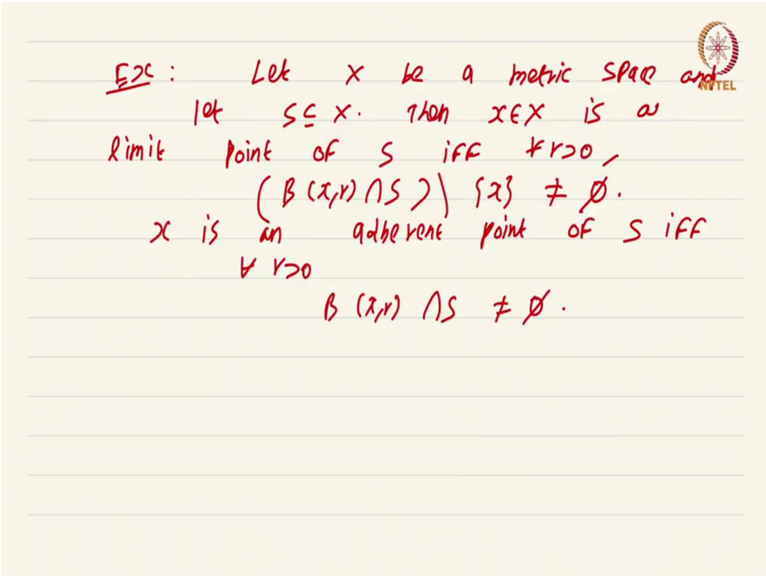

Exercise: Let X be a metric space and let $S \subseteq X$. Then $x \in X$ is a limit of S if and only if for each $r > 0$,

$$(B(x,r) \cap S) \setminus \{x\} \neq \phi.$$

Similarly, x is an adherent point of S if and only if for all $r > 0$,

$$B(x,r) \cap S \neq \phi.$$

These are exactly similar to the characterizations that you have previously seen for adherent and limit points in the context of real numbers.

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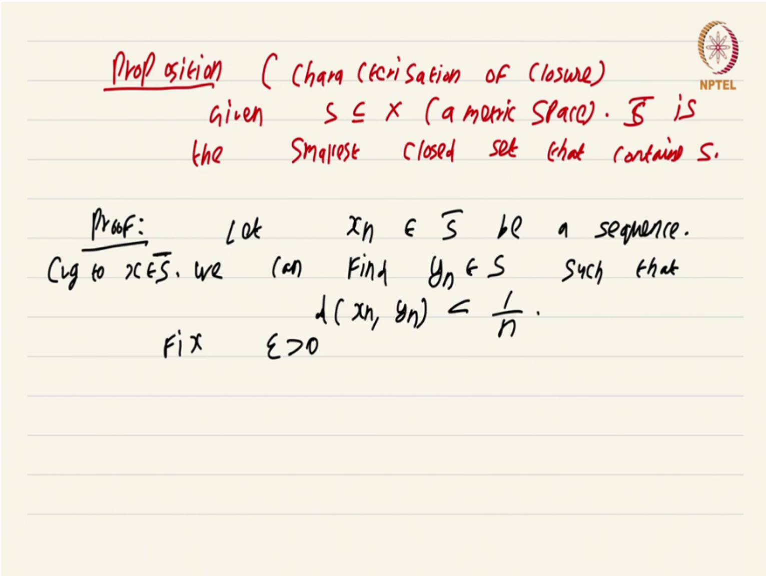

The final proposition of this short video is the characterization of closures.

Proposition: Given $S \subseteq X$ (metric space), then $\bar{S}$ is the smallest closed set that contains S.

Proof. There are two parts to this. The first is to show that $\bar{S}$ is a closed set. The second part is to show that $\bar{S}$ is the smallest closed set. So, let us go ahead and prove this.

Let $x_n \in \bar{S}$ be a sequence converging to $x \in X$. Now, since x_n's are elements of $\bar{S}$, we can find, $y_n \in S$ such that

$$d(x_n, y_n) < \frac{1}{n}.$$

We can do this because each x_n is an adherent point of S.

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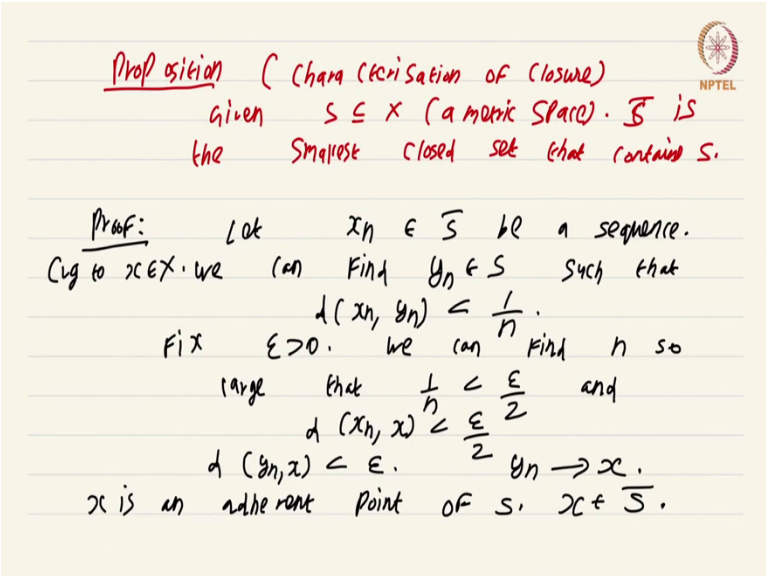

Our goal is to show that this x is an element of $\bar{S}$, which will prove that $\bar{S}$ is closed.

Fix $\epsilon > 0$. We can find n so large that simultaneously $\frac{1}{n} < \frac{\epsilon}{2}$ and

$$d(x_n, x) < \frac{\epsilon}{2}.$$

A trivial application of the triangle inequality would tell you that

$$d(y_n, x) < \epsilon.$$

Consequently, $y_n \to x$ and this means x is an adherent point of S, which means $x \in \bar{S}$. This shows that the closure of a set is always a closed set.

So, this concludes this video on Basic properties of open and closed sets, and you are watching this course on Real Analysis.