Real Analysis II
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Lecture - 28.1
Levi Monotone Convergence Theorem for Step Functions

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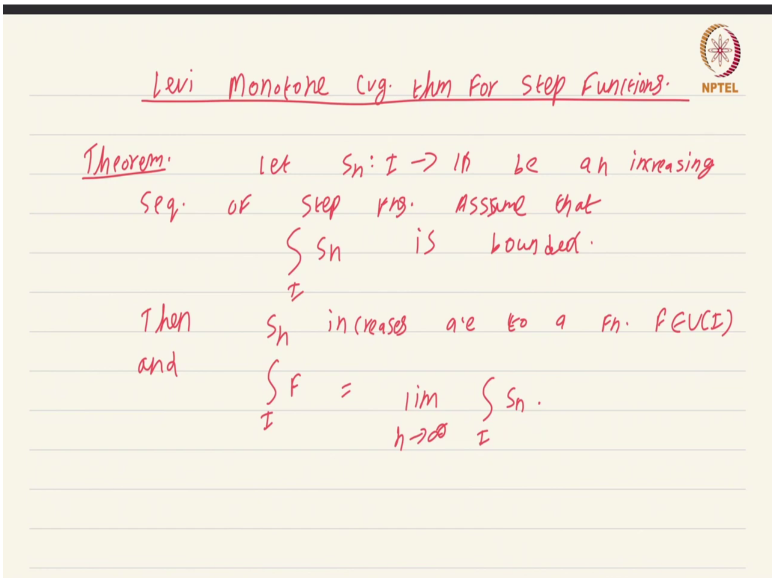

Last week, we did the hard work of setting up the theory of the Lebesgue integral. This week we reap the rewards. We shall prove various convergence theorems that make the Lebesgue integral far more suited for analysis compared to the Riemann integral. The first theorem that we are going to establish is the Levi's Monotone Convergence Theorem.

Recall that we had first defined the integral for step functions then we considered increasing sequences of step functions that converge to a function such that the integrals are bounded and the limits of such functions we called upper functions and then we define the integral for upper functions.

Now, what we are going to show is the moment that the integrals of step functions are bounded, then automatically the step functions converge and the resultant function is an upper function. So, essentially we are trying to see that the process by which we created the Lebesgue integral sort of is closed; in a some sense we are trying to prove a completeness result.

Without further ado we are going to first prove Lebesgue Levi's monotone convergence theorem for step functions, then we will prove the same theorem for upper functions and finally, we will deal with the general class of Lebesgue integrable functions. We are essentially going to show that this class is closed under increasing functions. So, you will understand better once I state the theorem.

So, the setup is as follows let S n from I to R be a be an increasing sequence increasing sequence of step functions. Assume that integral over I S n is bounded. Then the conclusion is then S n increases almost everywhere to a function F which is in the class of upper functions it is an upper integrable function and the integral of this function f is nothing, but limit n going to infinity of integral over I S n ok.

So, notice that the crucial thing that is present in this theorem is that we automatically get a function F for free. When we defined this class of upper functions we say that S n generates F if these S n's are increasing as well as S n increasing to F almost everywhere, then and we also assume that this integral of I S n is also bounded under those circumstances we say that F is an upper function. Here we get that upper function for free just from the fact that the integrals are bounded.

So, you can already see that this is stating quite a non trivial thing. Let us go on to the proof.

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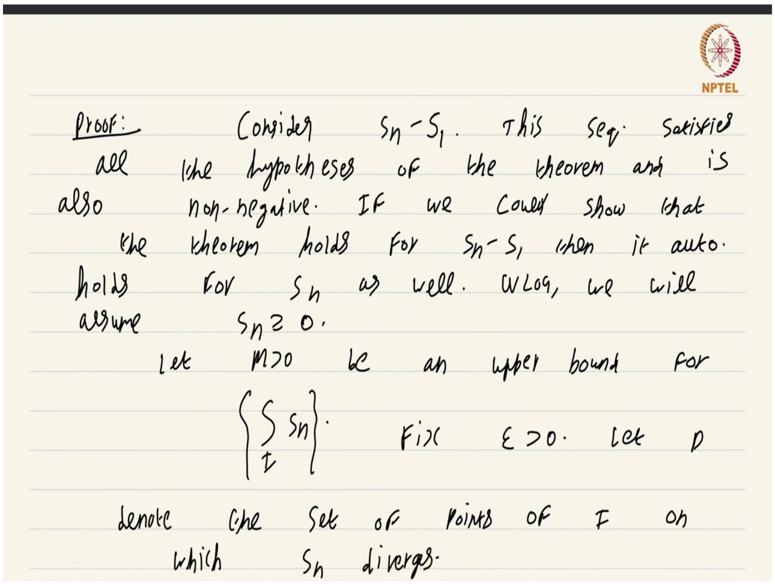

Proof: First what we are going to do is we are going to consider the case only of non negative functions. What you do is consider S n minus S 1, ok. Now, notice that if this sequence so, this sequence satisfies all the hypotheses all the hypotheses of the theorem of the theorem and is also and is also non negative which is going to prove very useful, ok.

If we could show the conclusion of the theorem if we could show that the theorem holds the theorem holds for S n minus S 1, then it automatically holds for S n as well. Well, if S n minus S 1 increases almost everywhere to a function g then the required function f is nothing, but g plus S 1 as you can see.

So, we will prove it we will henceforth so, without loss of generality we will assume we will assume each S n is itself greater than or equal to 0; just for ease of notation I do not want to introduce another sequence of step functions because I am going to do that now for the purposes of the proof anyway. So, not to deal with three different sequences of step functions to keep the notation same I am going to just call it S n again. We are going to assume S n's are non-negative ok.

Now, S n's are increasing everywhere in fact, that is our assumption how can S n fail to converge at a point? Well, the only way by which an increasing sequence can fail to converge at a point is if it diverges to plus infinity. Now, what we are going to do is we are going to control the behaviour of S n by constructing an auxiliary sequence of step functions that are special; special in the sense that they are integer valued step functions.

So, how are we going to do this? Well, let m greater than 0 be an upper bound be an upper bound for integral over I S n this collection we know that the limit exists we know that this is an increasing sequence. So, let m be an upper bound for it, ok. Now, fix epsilon greater than 0. Let D denote the set of points of I set of points of I on which S n diverges.

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Goal :- To show that D can be covered by a countable union of intervals whose net length is $< \varepsilon$.

Define $t_n := \text{int}\left(\frac{\varepsilon}{2m} S_n\right) \leq \frac{\varepsilon}{2m} S_n$

If $S_n(x)$ is convergent then $S_n(x)$ is bounded above. Then $t_n(x)$ is bdd. above and therefore convergent. On the other hand, if $S_n(x)$ diverges then so does $t_n(x)$.

Now, goal to show that D can be covered can be covered by a countable union of intervals whose net length is less than epsilon that is the goal. Since epsilon was arbitrary that would show that the set of points where S n diverges is actually a set of measure 0 which will in fact, show that S n increases almost everywhere to some function F, ok.

Now, I said we are going to construct a sequence of auxiliary step functions which will sort of be integer value. So, what we do is we defined t n to be defined t n to be epsilon by 2 m times S n integer part integer part just means it will remove the decimal part and leave you with the integer. This is also known as the I think it is called the floor function or something it is the greatest integer that is less than or equal to that particular number ok. So, what we are doing is we are going to take the integer part of this.

Now, when dividing by 2 m what we achieve is the integral of t n is going to be quite small simply because we have already divided by the integral value and we have made it even smaller by multiplying by epsilon ok. Now, observe that just by definition this is less than or equal to epsilon by 2 m S n, that is why what I said just makes sense.

So, the integral of this t n is going to be quite small it is going to be less than epsilon by 2 in fact. And notice that if S n of x is convergent then S n of x is bounded above because it is an increasing sequence, then t n of x is bounded above and therefore, convergent and therefore, convergent. In fact, because t n is integer valued the only way by which t n could converge is after a point it becomes constant.

So, this is just a concrete version of the abstract fact that any convergence sequence in a discrete metric space must be eventually constant this is a concrete realization of a fact you are familiar with from several weeks ago ok. So, t n x being integer valued will have to be eventually constant provided S n x converges. On the other hand, on the other hand, if S n x diverges then so, does t n x that is straightforward to see just by the way t n has been defined if S n x diverges then so does t n.

Well, t n is integer valued how can in fact, positive integer non negative integer valued how can a non negative integer valued sequence diverge? Well, only way by which it can diverges it increases it is value by plus 1 infinitely often.

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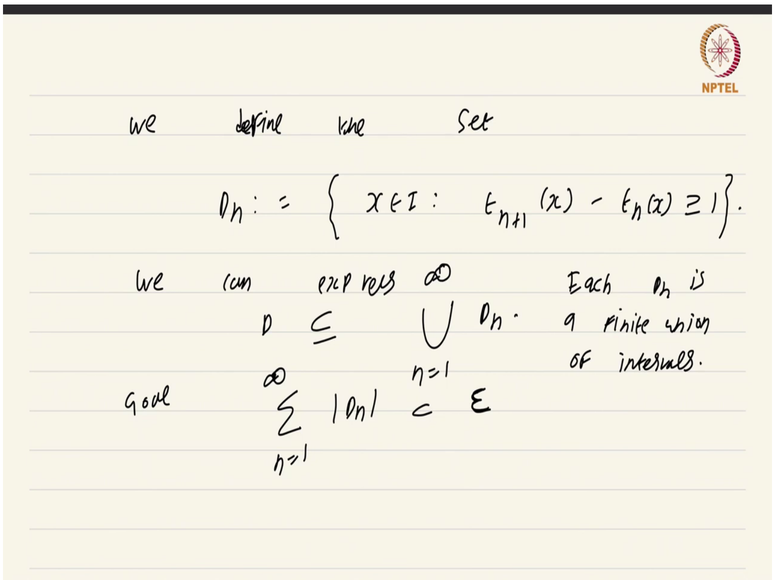

So, motivated by what I just said we define the set the set D n by definition to be the set of all x in I such that t n plus 1 x is minus t n x is greater than or equal to 1, ok. So, D n consists of those points x in the interval I where the function t n plus one exceeds t n x by at least 1. So, there is a jump ok there is a jump.

Now, the set of points where t n diverges is precisely the set of points x which on which this sequence t n jumps infinitely often; those are the points at which the sequence t n will diverge, ok. So, we can we can express we can express D as a subset of union n equal to 1 to infinity D n.

Notice that the right hand set right hand side set is far larger than what we actually want even though this set is really large the proof will still work. Why is this set really large? Well, it looks at all those points where the function the sequence of functions t n jumps at least once whereas, we want to isolate those points where there is infinitely many jumps. So, this is sort of like a very bad estimate, but surprisingly even this very bad estimate works ok.

Now, goal remember the goal was to show that D can be covered by a countable union of intervals whose net length is less than epsilon. Well, turns out that the net length of this D n n equals 1 to infinity is itself less than epsilon ok also because t n's are step functions and D n's measure the places where there is there is a jump it is clear that each D n each D n is a finite union finite union of intervals, ok.

Simply because t n plus 1 and t n are both step functions and we are simply measuring those I mean we are simply putting together in a set those points where t n plus 1 jumps ahead of t n. What you do is you look at a partition common refinement of the partition with respect to which t n plus 1 and t n are step functions you look at a common refinement and you will notice that the jumps will always have to occur on a finite union of intervals, ok. Now, how do we find out the length of D n?

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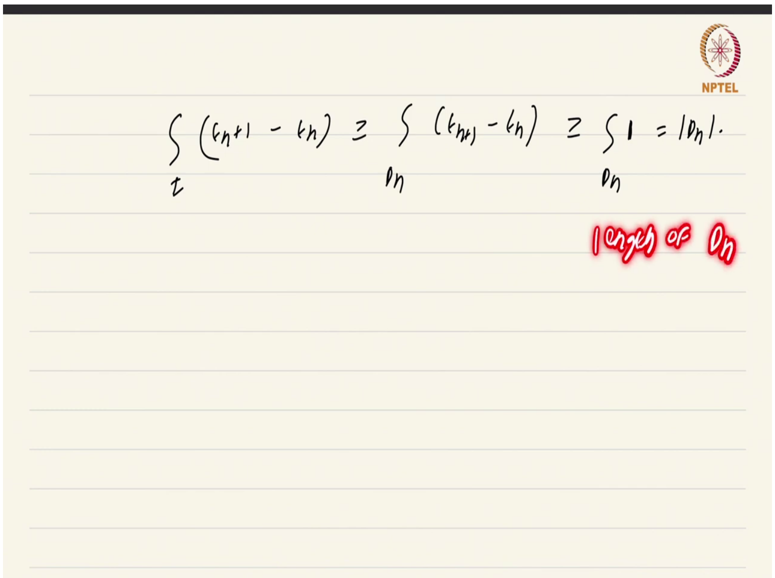

Well, observe that integral over I t n plus 1 minus t n well, what will this measure? This will be nonzero precisely at those places where there is a jump, ok and because the set D n is a subset of y we first have this obvious in trivial estimate. Note here I am using the fact that these functions are t n plus 1 minus t n is a non negative and this is of course, greater than or equal to integral over D n 1 right because on D n t n plus 1 minus t n is at least 1 and this is just length of D n oh ok.

So, note that I maybe I did not introduce this notation. This is just the length of D n. Since D n is a finite unit of intervals you can find out the length by just summing up the lengths of the intervals ok, excellent.

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$$\int_I (t_{n+1} - t_n) \geq \int_{D_n} (t_{n+1} - t_n) \geq \int_{D_n} 1 = |D_n|.$$

$$\left|\bigcup_{n=1}^{\infty} D_n\right| \leq \lim_{m \to \infty} \sum_{n=1}^{m} |D_n|.$$

$$\leq \lim_{m \to \infty} \sum_{n=1}^{m} \int_I (t_{n+1} - t_n)$$

$$= \lim_{m \to \infty} \int_I t_{m+1} - t_1$$

Now, our goal is to estimate the length of the union of D n's. Well, of course, you can do that by saying union of D n; n equal to 1 to infinity, the length of this is certainly less than or equal to limit m going to infinity summation n equals 1 to m of mod D n, ok. And from what we have established this is less than or equal to limit m going to infinity of sum n equal to 1 to m integral over I t n plus 1 minus t n by exactly the previous line by this line we immediately get this.

But, this is sort of like a cascading sum this is equal to limit m going to infinity integral over I t n plus 1 minus t n or rather t m plus 1 sorry, this is t m plus 1 minus t 1, ok.

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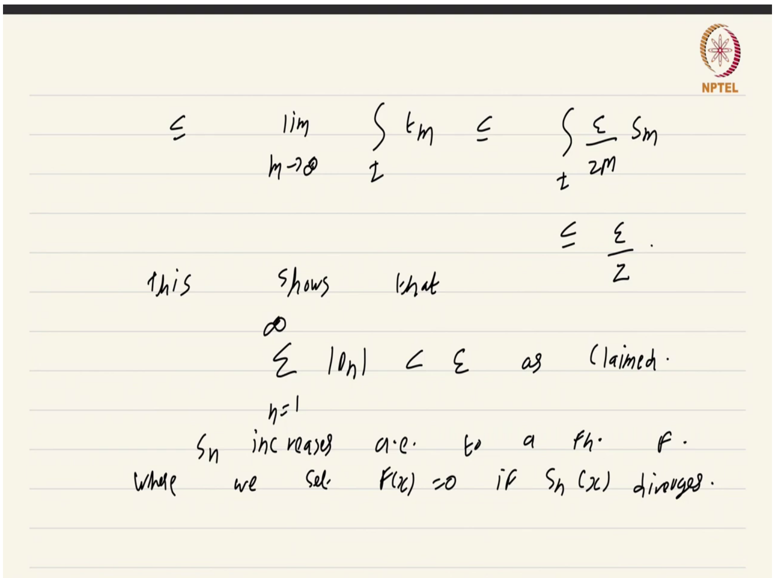

And, since t m plus 1 minus t 1 and this t 1 is a nonnegative function we can just get rid of this in an estimate and write that this is less than or equal to limit m going to infinity integral over I of t m which is going to be less than or equal to epsilon by 2. Why is this going to be less? So, let us elaborate that well, this is less than or equal to integral over I epsilon by 2 m S m which is going to be less than or equal to epsilon by 2 ok.

So, this shows that this shows that the length summation n equals 1 to infinity of mod D n is less than epsilon as claimed. So, conclusion is S n increases almost everywhere to a function to a function F, ok. Of course, this function F is not defined everywhere so, where we set F of x to be 0 if S n x diverges. We do not get a function F just from the convergence because at those points where S n x diverges we do not know what the value of F of x is. So, we just set it to be 0 there, rest of the points we just take the limit of S n x, ok.

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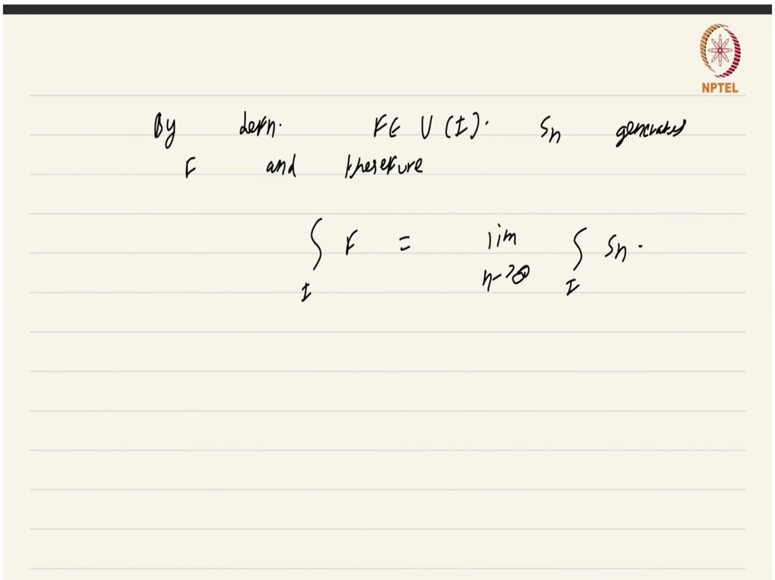

Then by definition F is an upper function and S n generates F and therefore, therefore, integral over I F is actually just by definition limit n going to infinity integral over I S n. So, this concludes the proof of Levi’s theorem Levi’s monotone convergence theorem for step functions. In the next video, we will extend this to upper functions and in the video after that we will extend it to the Lebesgue integrable functions.

This is a course on real analysis and you have just watched the video on the Levi monotone convergence theorem for step functions.