Real Analysis II
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Lecture - 27.1
Lebesgue Integrable Functions

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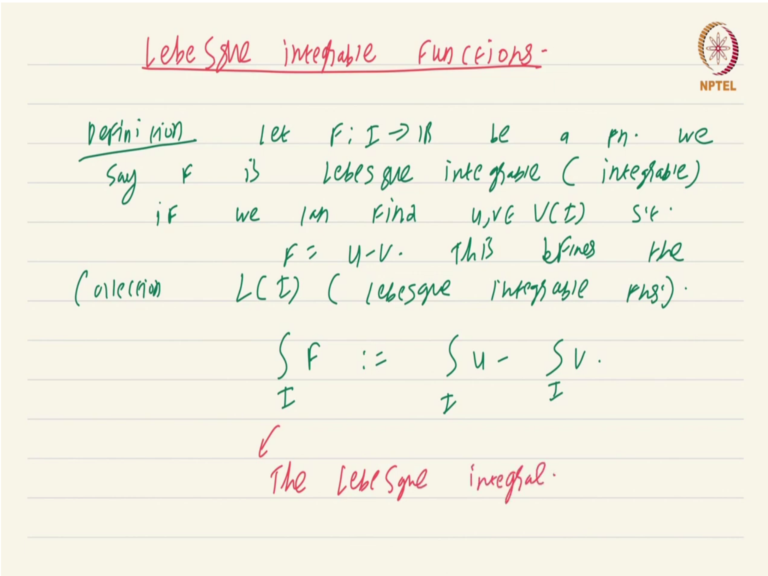

As we have remarked several times earlier, the class of upper functions has a deficiency. A negative of an upper function need not be an upper function; consequently the collection of all upper functions is not a vector space. This is not something that we would want to have in a nice theory.

We remedy this in the stupidest brain dead way possible by defining the Lebesgue Integrable Functions to be simply differences of upper integrable functions. So, without further ado let us go for the definition. Let F from I to R be a function, we say F is Lebesgue integrable sometimes just integrable because for the rest of this course we will deal only with the Lebesgue integral. So, sometimes we will just abbreviate it and say just integrable.

If we can find two upper functions u and v such that F is nothing but the difference u minus v. A moments thought will convince you that because the definition has been formulated in this way, it will be automatic that if a function F is Lebesgue integrable then so is minus of F in particular you can check it easily by considering F equal to u minus v and observing that negative of F is just v minus u.

So, this defines the collection; this defines the collection L of I. So, these are the Lebesgue integrable functions Lebesgue integrable functions. Of course, we have not yet defined what the integral of such a function is, but that is rather easy. We just define integral of I F by definition to be integral of I u minus integral of I v, ok.

So, the definition has been done in the most straightforward way to fix the obvious deficiency of upper integrable functions. Now, there is a slight problem even with this definition. This function F could be written as a difference of two upper integrable functions in a gazillion different ways. How do you know all of them are going to give the same integral? Well, that is just an easy check.

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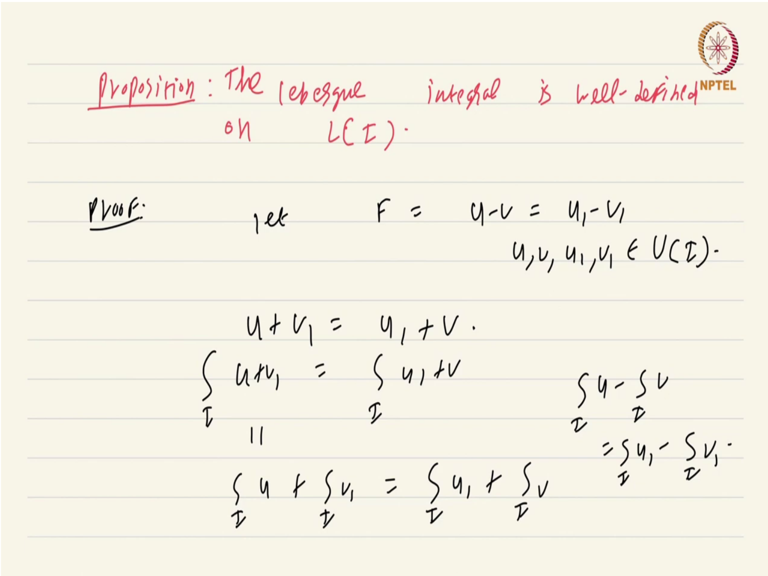

So, we have the following proposition Lebesgue the Lebesgue integral. So, of course, I did not mention that this is called the Lebesgue integral; the Lebesgue integral. So, the Lebesgue integral is well defined is well defined on L of I. How do you prove this? Well, that is rather easy. What you do is what you do is let F be equal to u minus v which is in turn equal to u 1 minus v 1 where u, v, u 1, v 1 are all upper integrable functions ok. Then we have that u plus u 1 is v plus no, I got it wrong.

We have u plus v 1 is nothing but u 1 plus v and the sum of two upper integrable functions is thankfully upper integrable. Therefore, what you can conclude is integral over I u plus v 1 is integral over I u 1 plus v and from the results we have shown for the upper integral we have this is integral over I u plus integral over I v 1 and which in turn is equal to integral over I u 1 plus integral over I v which immediately shows us that integral of I u minus integral of I v is equal to integral of I u 1 minus integral of I v 1 which is exactly what we wanted to show.

So, the definition of the Lebesgue integral is not problematic, no matter what decomposition you choose for this function F, then the value of the integral you get is always the same. So, there is no issue. In the next video, when we discuss properties of Lebesgue integrable functions we will show that there are some nice decompositions of a Lebesgue integrable function. In particular, you can take this v to be as small as you want an upper function that is as small as you want.

So, in some sense this Lebesgue integrable functions are almost upper integrable in one sense and another decomposition you can get is you can write F as a sum of a step function plus an integrable function where that integrable function is really small. We will make these vague statements precise in the next video when we study the properties of the Lebesgue integral.

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If $F \in U(I)$, $F \in L(I)$ the upper integral coincides with the Lebesgue.
F is Riemann integrable then F is upper integrable.

$$\int_a^b F := -\int_b^a F \qquad a \leq b.$$

If $I = \{a,b\} \to a, b$ could be possibly $\pm\infty$

$$\int_a^b F := \int_I F \qquad \int_b^a F := -\int_a^b F$$

Now, before we proceed let me just make a remark. We already know that the if I mean, this should be rather obvious to you. If F is upper integrable then obviously, F is also Lebesgue integrable and I mean the upper integral coincides upper integral coincides with the Lebesgue integral. This is simply because you can just choose the trivial decomposition F minus 0 for an upper integrable function ok.

Moreover, we already know that if F is Riemann integrable if F is Riemann integrable then it is upper integrable that is what we proved in the last video then F is upper integrable and both integrals coincide. So, ultimately what this shows is the Riemann integral sort of coincides with the upper integral but, which in turn coincides with the Lebesgue integral.

But, note one thing the Riemann integral is sort of directional you have integral a to b F, you have a sort of orientation. If you reverse it and consider integral b to a of the same function we usually just define it to be negative of that ok. Now, what we are going to do is even though the Lebesgue integral that we have defined does not in any way use the structure the order structure of the real numbers, we are going to artificially impose the order structure on the Lebesgue integral.

The way we do it is if I is equal to closed interval a comma b here a comma b could also be possibly could be possibly plus minus infinity could be possibly plus minus infinity. If you take the interval like this we define integral of I F the Lebesgue integral over the interval I of F. This we also denote by integral a to b F ok.

Because the Riemann integral agrees with the Lebesgue integral this causes no confusion if F is Riemann integrable and we also define integral b to a F to be minus integral a to b F. Of course, here it is all assumed that a is less than or equal to b. So, we do the same thing that we do for the Riemann integral to define the integral when there is an opposite orientation exact same definition we do.

So, even though intrinsically the Lebesgue integral does not depend on the order structure we have artificially introduced the order structure. This will turn out to be very relevant when we study multiple integration ok.

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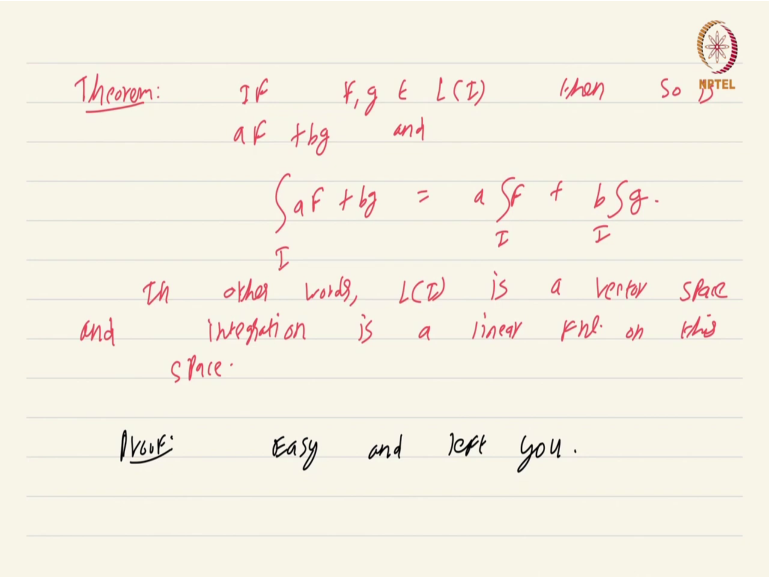

So, let us now prove some very preliminary properties of the Lebesgue integral. Theorem, if F comma g are Lebesgue integrable then so is a F plus bg and integral of aF plus bg is a integral F plus b integral g, of course, I must put I everywhere, ok. In other words in other words L of I is a vector space and integration is a linear functional on this space, ok.

So, I have written out a fancy little statement and I am going to be very devious and say the proof is easy and left to you. Really there is I mean this is not really out of laziness or anything this is really nothing to do. You just break up F into u minus v and break up g into let us say u 1 minus v 1, you already know that upper integrals behave well with respect to sums, but only the negative sign is a problem and so on.

You just manipulate this an elementary algebra and you will get the whole thing you will get that aF plus bg. First of all the fact that aF plus bg is also a Lebesgue integrable, just immediately follows from the decomposition and the fact that this integrals will coincide will follow from a similar argument to what we did for the last the for the well definedness of the Lebesgue integral. You just manipulate this algebraically and you will get it there, really there is nothing to do.

So, keep this in mind the Lebesgue integral is actually a linear functional on the vector space of Lebesgue integrable functions. This is sort of like a starting point for the study of the subject called functional analysis where you study spaces of functions. We already saw some aspects of this topic when we studied metric spaces where we studied norm vector spaces ah. Really the introduction of the Lebesgue integral into functional analysis really makes the subject extremely interesting, ok.

So, some more properties. One disadvantage of upper integrals which I have kept on repeating is that the negative of an upper integral is not upper integrable function is not necessarily upper integrable. So, what will happen is since the Lebesgue integral is defined in terms of the upper integral, at the end of the day unless you are really I mean if you want to prove some non-trivial theorems what essentially happens is you will end up in situations where you have to consider the negative of functions.

And you will not really know how to deal with it because the negative of that function might not be upper integrable. So, what you do is there is a standard device that allows us to make all the functions that we are considering positive and this sort of decomposition of a given function will turn out to be a very useful theoretical device in several proofs.

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Given $F: I \to \mathbb{R}$

$F^+ := \max(F, 0)$.

$F^- := \max(-F, 0)$.

$|F| = F^+ + F^-$.

$F = F^+ - F^-$

Exactly one of $F^+(x)$ or $F^-(x) > 0$ if $|F(x)| \neq 0$ and in this event the one that is positive has the value $|F(x)|$.

So, given a function F from I to R it need not be Lebesgue integrable or anything, you define F plus sort of to be the positive part. So, what it does is if F at a given point is greater than 0, then you just set this F plus to be F. If it is less than 0, you just remove that part so, you make it 0. So, this is just max of F comma 0.

So, this function F plus will agree with F whenever it F is greater than or equal to 0, whenever F is less than 0 this F plus will be 0. So, all those places where the function F is negative, it just pushes that negative value all the way up to 0. So, you get a non-negative function.

Similarly, you define F minus. There is a slight twist here. You take maximum of minus F comma 0. Notice what happens for F minus. Whenever F is positive whenever F is positive this function will be 0 ok, but whenever F is negative minus F will become greater than 0 therefore, this max will become minus F. So, whenever F is negative it takes the same absolute value of F, but yeah it takes I mean whenever F is negative this F minus takes the absolute value of F ok.

So, this discussion should have already made what I am about to write obvious F is nothing but F plus minus F minus ok and exactly one of F plus x or F minus x is greater than 0 if mod F x is not equal to 0 and in this event; in this event; in this event the one that is positive one that is positive has the value mod F has the value mod F of x.

So, at a given point where the absolute value does not vanish one of F plus or F minus will be that absolute value and the other will be 0. So, the second identity will also be obvious to you mod F is nothing but F plus plus F minus ok. So, these functions F plus and F minus will play an important role in what is about to follow. When you essentially want to reduce everything to upper integrals this will become very useful.

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Let us illustrate this with a simple fact. Theorem, yeah, this is too basic to call it theorem.

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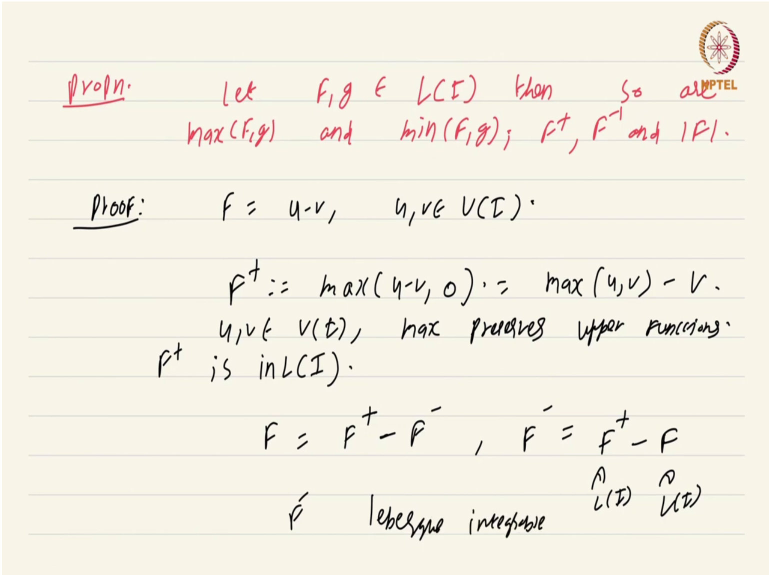

Let me just call it a proposition. Let F comma g be Lebesgue integrable a Lebesgue integrable functions then so are max F comma g and minimum of F comma g. Both these functions are Lebesgue integrable and so, and also F plus F minus and mod F. So, all these associated functions are Lebesgue integrable.

Proof. So, we let us first deal with how to show that F plus, F minus and mod F are Lebesgue integrable. We know that F is u minus v, where u and v are upper integrable functions. This is just the definition of the class. Now, notice that F plus would be by definition it will be just minimum of u minus v sorry, not minimum maximum; maximum of u minus v comma 0.

Now, let us stare at this and think for a moment. Let us stare at this max of u minus v comma 0 and let us see whether we can simplify it in some way. There are two possibilities. Either the max of u and v is u or it is v. If the max of u and v is v then this whole thing will have to be 0 because u minus v in that event will be negative, right. So, if max of u comma v is v, then this whole expression will be 0.

So, in that case I can write this as max of u comma v minus v. Why is this correct? Because in the event that v is greater than or equal to u then max of u comma v is v and the second expression and the first expression will just cancel off; max u, v minus v will be just 0 which is the expected value. But, hold on a second, if max of u comma v is u, then this whole thing is going to be just u minus v. So, in that event max of u comma v is just u and you just get u minus v. So, in both scenarios this expression is correct.

So, the max of u minus v comma 0 is just max of u comma v minus v ok, but u and v are upper functions and we have already shown that max preserves upper functions, that is, if you take the maximum of two upper functions you still end up with an upper function. So, which shows that F plus is Lebesgue is in L of I simply because we have exhibited it as the difference of two upper integrable functions ok.

Now, notice that F is F plus minus F minus right that is the very definition of F plus which immediately gives that F minus sorry, I should not write inverse F minus is F plus minus F, but both of these are Lebesgue integrable and therefore, F minus is also Lebesgue integrable; is also Lebesgue integrable.

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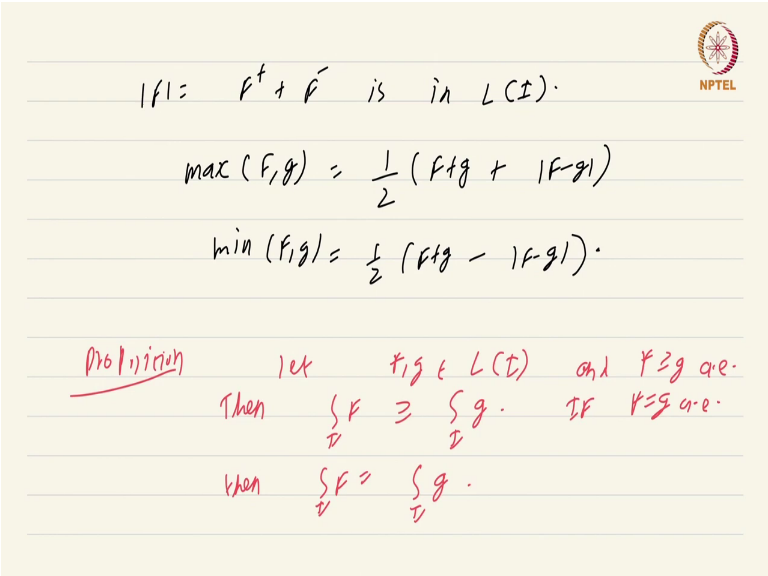

Finally, mod F being just F plus plus F minus is Lebesgue integrable. So, this was quick. We still have to deal with max F comma g and minimum of F comma g, we completely ignored g and it is feeling very bad. Well, that just follows from the following two basic formulas that you would have probably seen at least once in your life, if not just prove it is not difficult.

Max of F comma g is just half of F plus g plus modulus of F minus g and minimum of F comma g is nothing but half of F plus g minus modulus of F minus g. So, you can exhibit the maximum and minimum of two functions as an algebraic combination of the two functions along with the absolute value. So, you can these expressions are going to be very useful in many other scenarios as well. So, if you have never seen it or never proved it in your life please do it now ok.

So, we have got some useful properties of the Lebesgue integral. Let us proceed and try to see the properties of the Lebesgue integral under ordering. So, that is also another proposition. Let F comma g be Lebesgue integrable, then ok and F greater than or equal to g almost everywhere. Then integral of I F greater than or equal to integral of I g. If F equal to g almost everywhere, then integral of I F is equal to integral of I g. Well, the second part is rather trivial you can just show it by using the first part twice.

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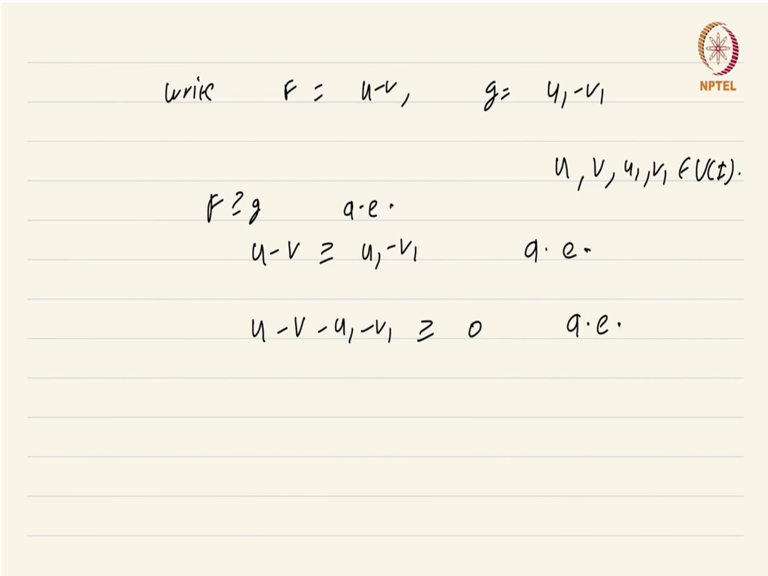

So, let us just deal with the first part. So, the first part asks us to consider F and g with F greater than or equal to g almost everywhere. So, of course, write F as u minus v and g as u 1 minus v 1 with u comma v comma u 1 comma v 1 all upper functions upper integrable functions ok.

Now, what we do is the following. We know that F is greater than or equal to g almost everywhere. So, that will give us that u minus v is greater than or equal to u 1 minus v 1 almost everywhere right which would in turn give that u minus v minus u 1 minus v 1 is greater than or equal to 0 almost everywhere ok.

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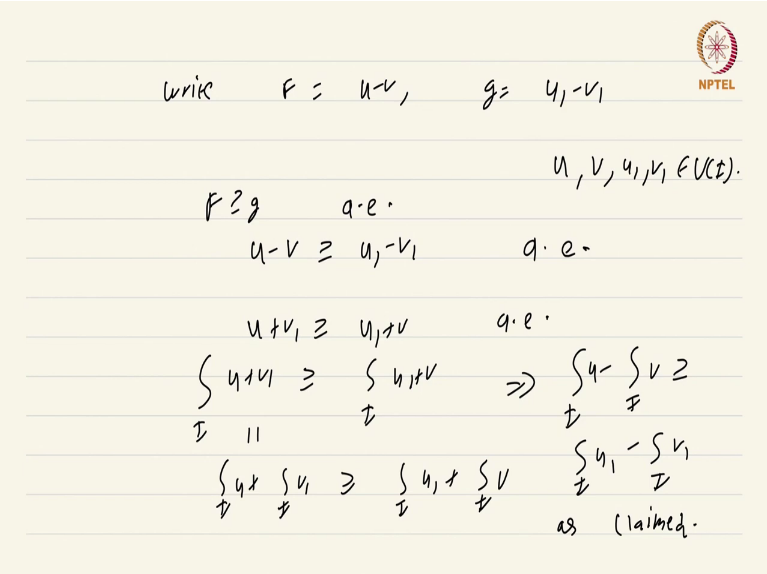

So, in other words actually I do not want to rewrite it like this. I want to rewrite it in a better way, so that taking into account that the negative of an upper function is not necessarily an upper function. So, you just write it as u plus v 1 is greater than or equal to u 1 plus v almost everywhere which in turn gives that integral of u plus v 1 I is greater than or equal to integral u 1 plus v. This simply follows because the upper functions behave will well with respect to ordering.

Now, you break this up integral of I u plus integral of I v 1 is greater than or equal to integral of I u 1 plus integral of I v which immediately gives that integral of I u minus integral of I v is greater than or equal to integral of I u 1 minus integral of I v 1 as claimed. So, the Lebesgue integral also behaves well with respect to ordering. The second part as I remarked is quite easy.

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Proposition: Let $F \in L(I)$ then

$$\left| \int_I F \right| \leq \int_I |F| .$$

$$-|F(x)| \leq F(x) \leq |F(x)| .$$

$$-\int_I |F| \leq \int_I F \leq \int_I |F| .$$

$$\left| \int_I F \right| \leq \int_I |F| .$$

Now, one more proposition one more proposition and this is also not so hard, what this proposition says is let F be a Lebesgue integrable function, then the absolute value of the integral is less than or equal to the integral of the absolute value ok. How does one prove this? Well, observe that we have minus mod F of x is less than or equal to F of x is less than or equal to mod F of x and since F is Lebesgue integrable, both minus F minus mod F and plus mod F are both Lebesgue integrable.

So, you get integral minus mod F over I is less than integral over I F is less than or equal to integral over I mod F ok. And, you can take the minus sign outside from the first expression you can take the minus sign outside and you will get this. This is the same as saying that integral of I mod F is less than or equal to integral I mod F exactly I mean it is just rephrasing the same thing. So, these are some nice properties of the Lebesgue integral.

There are some more properties that are there in the exercises that is to do with the invariance of the Lebesgue integral under translation and the behavior of the Lebesgue integral under expansion or contraction. Those exercises are all straightforward, you just have to use the same algorithm for proving those exercises. You have to first show these things for step functions and then show them for upper functions, then show them for Lebesgue integrable functions.

The proofs are there no real idea involved in the proofs, you just have to successively increase the class of functions for which the result is true and each step is really trivial. So, please solve that exercise, it will be very useful. In the next video, we shall see some more properties of the Lebesgue integral this is a course on Real Analysis and you have just watched the video on Lebesgue Integrable Functions.