Real Analysis II
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Lecture - 26.2
Riemann Integrable Functions as Upper Functions

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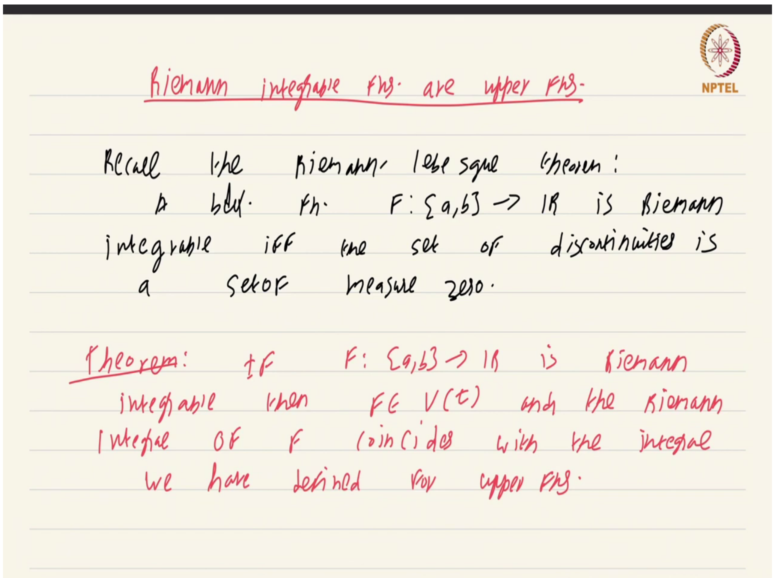

We are now going to show that the construction of this collection of upper functions and the integral on them is actually a more general thing than the Riemann integral. We are going to do this by first of all showing that all Riemann Integrable functions are automatically upper functions combined with the observation that there is an example of an upper function such that the negative of that upper function is not an upper function will clearly show that the collection of upper functions is actually a larger class.

So, we are going to show that Riemann integrable functions are upper functions. To do that, we need to recall, we need to recall the Riemann-Lebesgue; the Riemann-Lebesgue theorem that we have seen already in Real Analysis I. What does the Riemann-Lebesgue theorem say it says that a function a bounded function F from a b to R is Riemann integrable if and only if the set of discontinuities is a set of measure 0. This is a necessary and sufficient condition for Riemann integrability.

The theorem we are going to show now is the following. If F from a b to R is Riemann integrable, then F is an upper function and the Riemann integral of F coincides with the integral we have defined for upper functions, we have defined for upper functions. So, this makes the collection of upper functions a proper superset of the collection of Riemann integrable functions.

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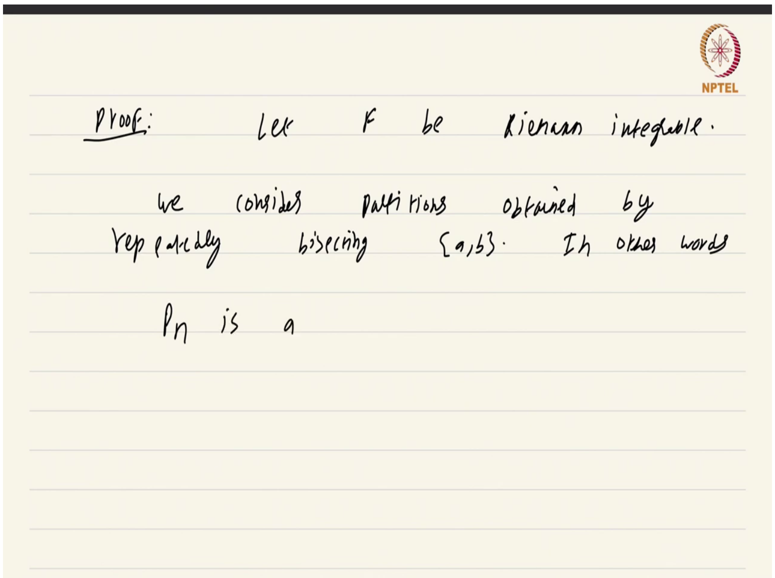

Proof, so, let F be Riemann integrable, let F be Riemann integrable. What we are going to do is we need to find a sequence of step functions that generate this function F. We know that the set of discontinuities of F is really a negligible set. We also know that the Riemann integrability of F will give that the upper sums and the lower sums of this sort of get closer and closer to each other.

Since we want step functions less than or equal to F, we will mimic what we did for the evaluation of the lower sum. What we are going to do is we are going to consider, we consider partitions obtained by repeatedly dividing a, b, repeatedly bisecting a, b.

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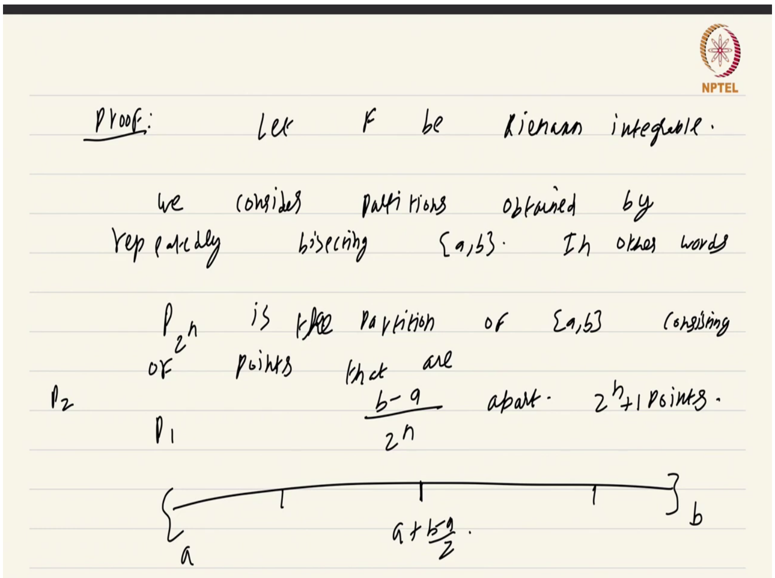

In other words, in other words P n is a partition or rather P 2 power n that is a better notation P P 2 power n is a partition or is the partition of a, b consisting of points that are b minus a by 2 power n apart ok. So, this partition will obviously have 2 power n points ok.

So, what we essentially do for instance if you take if you take the case a, b and you take the case n equal to 1, you just take b minus a by 2. This a plus b or rather this will be a plus b minus a by 2, this, these three points would be the points in the partition. So, to be precise there will be 2 power n plus 1 points ok. Similarly, when you consider P 2 which is P 2 power 4, you will have two more points you will have two more points for a total of five points.

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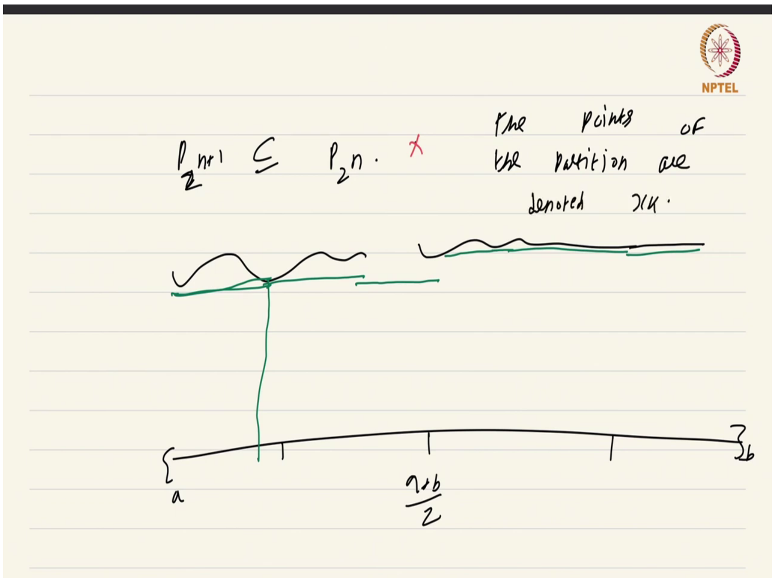

Now, the key thing that is very, very interesting about this sequence of partitions is; obviously, P 2 power n plus 1 is a subset of P 2 power n ok. So, these partitions are sort of nested. And this is going to be crucial. Now, let us redraw this picture again to get better insight of what is about to happen. So, let us just draw some this is actually a plus b by 2, I can call it that we have this we have some function. And at some points, it is not continuous. So, let us say it here. It is not continuous. We have this function.

What we are going to do is, we are going to sample the lower sums that is essentially look at the points of minima for this first interval, it is some somewhere like here. So, this will be the first piece. And for the second piece also it is almost at the same place. So, it will look like this all the way till here.

Then for the third piece, it is somewhere over here; it is sort of jumping ok. And this is the step function we are going to consider. So, we have to extend this F all the way here. And then finally, we have another piece which is like this ok.

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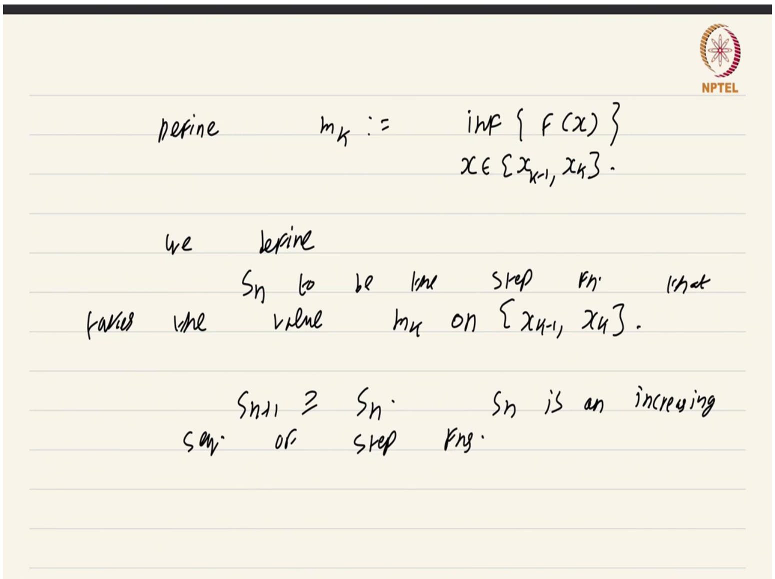

So, what we are going to do is we are going to define m k by definition to be infimum of F of x F of x as x runs through as x runs through x k minus 1 comma x k ok, where we are denoting where we are denoting the various points of the partition. So, the points of the partition of the partition there are 2 power n of n plus 1 of them are denoted x k ok. So, we just define small m k to be infimum of F of x as x runs through x k minus 1, x k.

And we define we define S n to be to be the step function to be the step function that takes the value that takes the value m k on the interval x k minus 1, x k fine. So, we have essentially the picture I drew that is going to be the step function S n. Now, clearly S n plus 1 is greater than or equal to S n.

Why is that the case? Well, look at the various sub intervals you get when you divide the partition P n to the partition P n plus 1 sorry P 2 power into P 2 power n plus 1, you are halving each interval. Therefore, when you take the infimum on the smaller intervals, the infimum can only increase, so that shows that S n plus 1 has to be greater than or equal to S n ok. So, S n is an increasing sequence of step functions.

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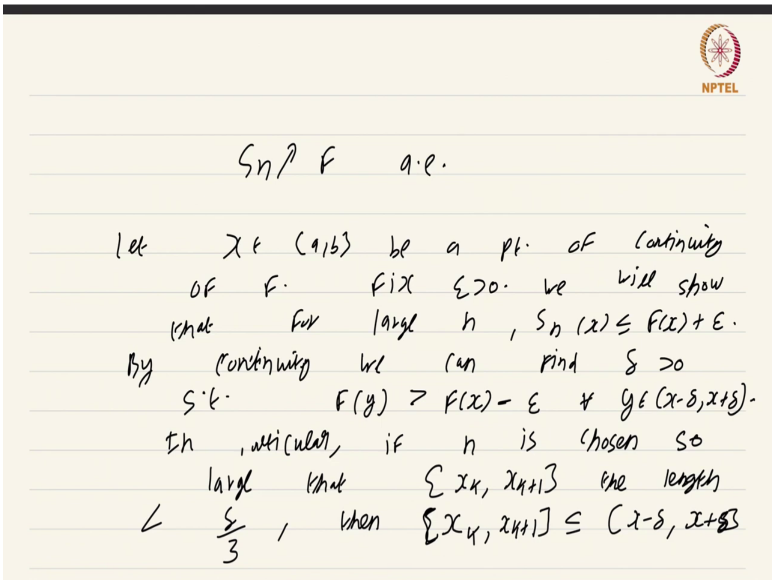

Now, what we are going to show is S n increases to F almost everywhere. And the way we do it is to use continuity. So, let x in a b be a point of continuity, point of continuity of F ok; fix epsilon greater than 0 we will show we will show that for large n, S n of x is less than or equal to F of x plus epsilon ok. And since epsilon was arbitrary, this will show that S n's increase almost everywhere to F because the set of points of discontinuities of F is a set of measures 0 ok.

Now, by continuity, we can find delta we can find delta greater than 0 such that F of y is less than F of x plus epsilon for all y in the interval x minus delta x plus delta ok. In particular, if n is chosen so large chosen so large that this entire interval x k plus 1 the length is less than delta by 3 that is the length of each sub interval that is b minus a by 2 power n is less than delta by 3, then x, x k, x k plus 1 would be a subset of x minus delta x plus delta for some k ok.

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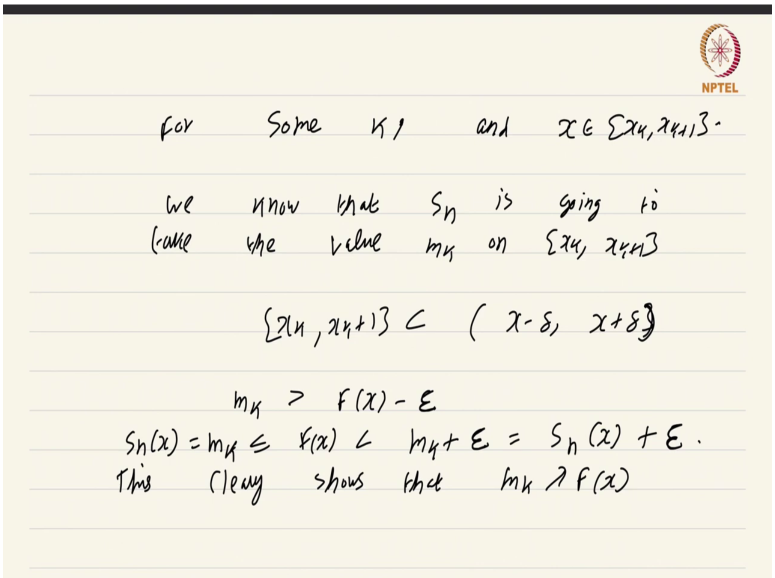

And x, x is in x k, x k plus 1. So, what is this trying to say? Well, if you choose if you choose the size of the partition to be so small, then what will happen is eventually the length of these intervals is so small that eventually this interval has to be a subset of x minus delta x plus delta that particular one that which contains x ok.

Now, we want to do the correct estimate. So, I have made a small error let me fix that such that F of y is greater than F of x minus epsilon. You will understand in a moment why I need this inequality that is also a consequence of continuity rather than the one I wrote down originally ok.

So, we know that we know that S n is going to take the value is going to take the value little m k on x k, x k plus 1. But, by the way, we have set up things x k, x k plus 1 is a subset of x minus delta x plus delta ok, which means this m k will have to be greater than F of x minus epsilon ok. So, in other words, F of x is less than m k plus epsilon. And we obviously, have that m k is less than or equal to F of x simply because m k was obtained by taking the infimum on this interval x k, x k plus 1, and x is an element of this ok.

Now, this clearly shows this clearly shows that m k is increase to F of x or rather I mean to be 100 percent precise, we have to write S n of x is equal to m k is less than or equal to F of x equal less than F of m k plus epsilon equal to S n x plus epsilon ok. So, this clearly shows that. S n of x increases to F of x ok.

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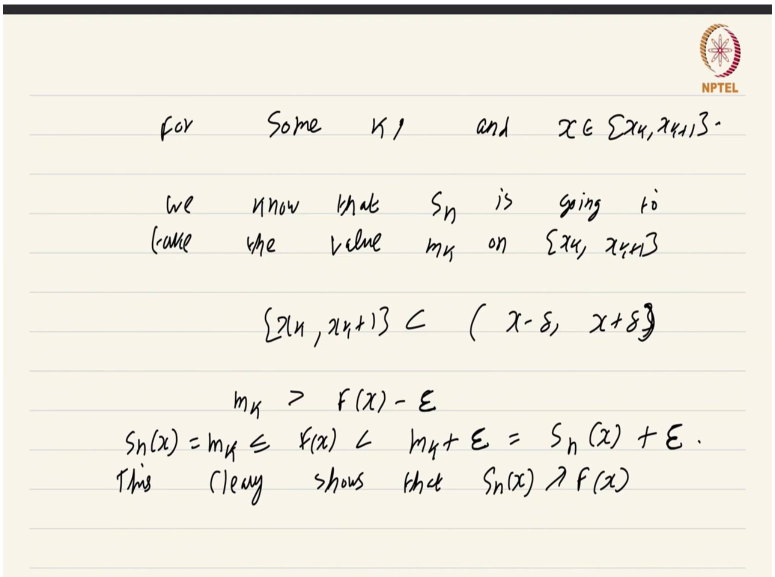

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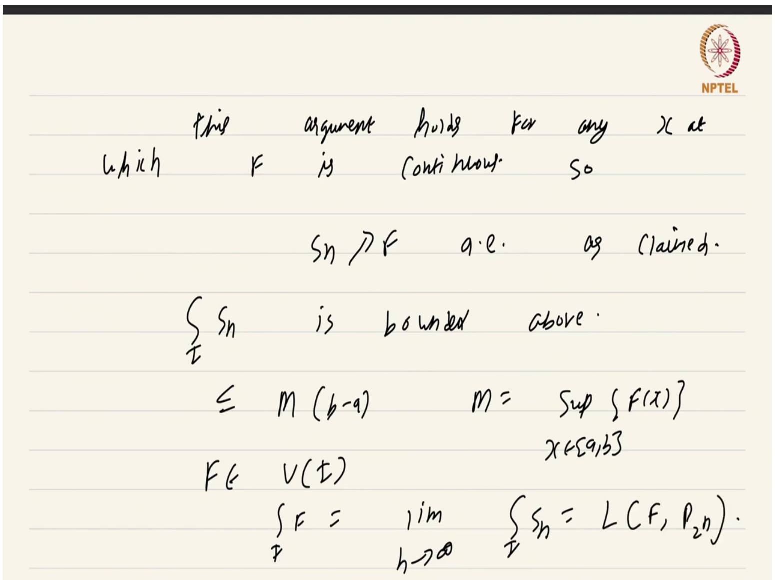

And since the set of I mean this argument holds this argument holds for any x any x at which at which F is continuous. So, S n increases to F almost everywhere as claimed ok. So, we still have to show that integral over I S n is bounded above the sequence is bounded above. Well, that is obvious because integral over I S n is going to be less than m times b minus a, where m is equal to supremum of F x as x ranges through closed interval a, b. Recall that F is a bounded function being Riemann integrable, therefore, there will be a supremum for this function on the whole interval close interval a, b.

Now, what does this allow us to show? Well, this shows that F is an upper function and the integral over I of F is nothing but limit n going to infinity integral over I of S n, but this limit n going to infinity of integral over I S n is nothing but the lower sum of F with respect to P 2 power n right.

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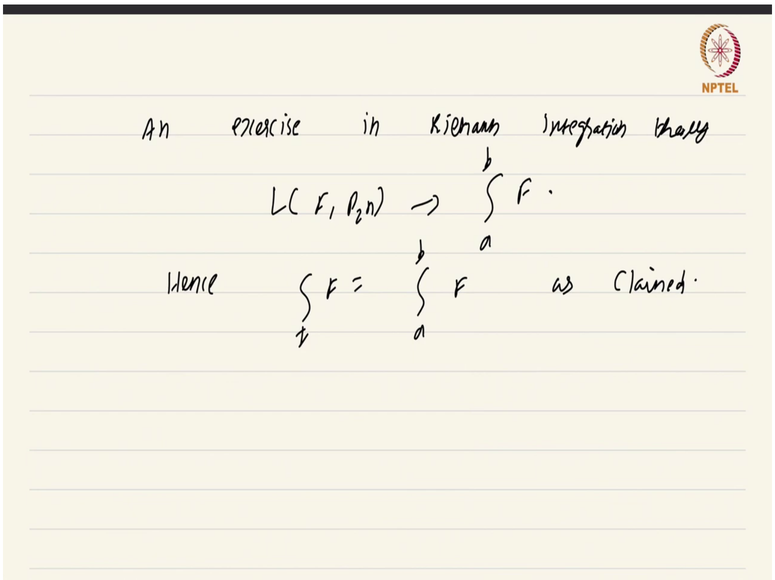

And an exercise in Real Analysis-I, if you have not solved that exercise, solve it now. An exercise in Riemann integration theory Riemann integration theory will tell you that L F P 2 power n actually converges to integral a to b F ok. Since this the mesh of the partition is becoming smaller and smaller, this will actually show that this will have to converge to integral a to b F.

Hence, integral of I F is equal to integral a to b F as claimed. So, both notions of integral, both the Riemann integral as well as the upper integral that we have defined thankfully coincide for Riemann integrable functions.

As I remarked earlier there are functions you should definitely solve the exercise in a postal where you construct a function f whose negative is not an upper is not an element of is not an upper function that is an upper function whose negative is not an upper function. This will show that the set of upper functions is a proper superset of the set of Riemann integrable functions.

In the next set of videos, we will define and study the basic properties of the Lebesgue integrable functions. Lebesgue integrable functions are those which do not have this annoying property that the negative is not integrable. We will get rid of this unnatural and not nice thing in the next video.

This is a course on Real Analysis, and you have just watched the video on Riemann Integrable Functions are Upper Functions.