Real Analysis II
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Lecture - 25.2
Monotone Sequences of Functions

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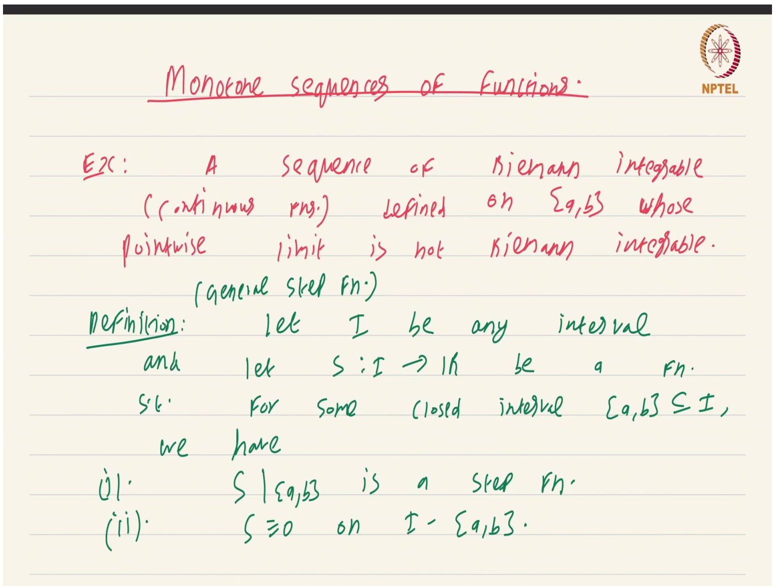

Our objective now is to construct the Lebesgue integral; we will take limits of step functions, but not the uniform limit if you take the uniform limit as we saw in the previous video we recover the Riemann integral for a sub class of functions. We will take a different sort of limit and define the integral using this limiting process.

So, essentially we are going to study Monotone Sequences of Step Functions, but before we do that I want you to think of an example of a sequence of Riemann integrable functions

Riemann integrable in fact, continuous functions I want you to think of a sequence of continuous functions defined on some a b, on a b whose pointwise limit whose pointwise limit is not Riemann integrable not Riemann integrable it is not that hard to come up with many such examples please do that now.

The goal of the Lebesgue theory is to construct an integral which behaves very nicely with respect to limits of course, in general you cannot expect with no additional condition on the type of convergence to for the limiting function to be both integrable and the limit of the integrals equal to the integral of the limit that is too much to expect, but the resultant theory that we develop will be much better than the standard Riemann integration theory where you need uniform convergence to interchange limit and integral it will be better than that.

So, what we are going to do is, we are going to now consider step functions, but for the construction it will actually help if we now can define step functions on the whole of the real line. So, the definition is as follows definition. So, we start with any interval let I be any interval open closed unbounded the whole of R I do not really care and let S from I to R be a function be a function such that for some closed interval for some closed interval a b subset of I we have number 1: S restricted to a b is a step function is a step function.

In other words we can find a partition of this closed interval a b such that on the open intervals determined by the partition the function S is constant. Number 2: S is identically 0 on I minus a b. So, outside this closed interval a b the step function is actually just 0. So, this is called a general step function, not I mean this is slightly a larger class of functions than the type of step functions we considered in the last video when we wanted to define the Riemann integral in terms of the step functions ok.

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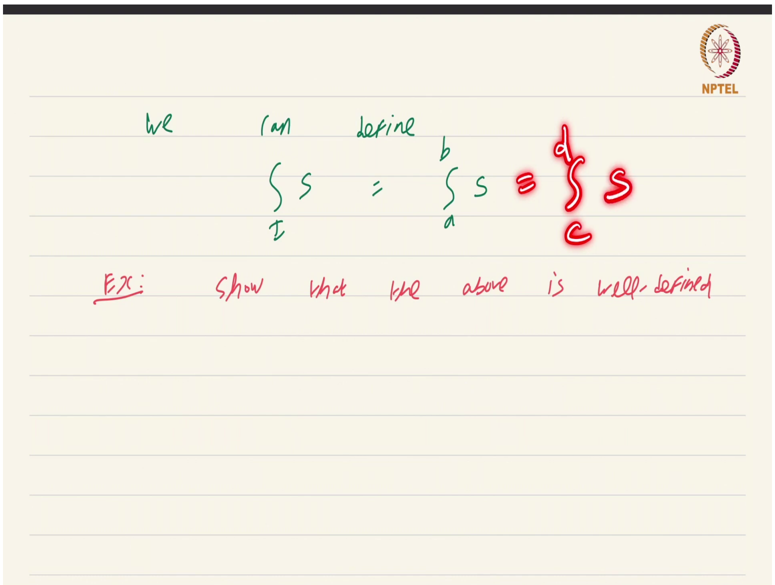

So, now that we have this we can define we can define we can define integral of a to b of the step sorry not integral of this a to b, integral of I of the step function to be nothing but integral a to b of the step function S, because as restricted a b is a step function in the usual sense that we have seen in the last video we can define this integral.

However, there is one slight issue which is not that major and I am go as usual I am going to leave it as an exercise to you show that the above is well defined. The issue that can arise is there could be more than one interval closed a b outside which S is identically 0 and S restricted to a b is a step function that is very easy to see that there can be more than one interval.

I want you to check that if you choose another c d which is a subset of I such that S restricted to c d is actually going to be a step function and S is identically 0 on I minus c d that is conditions 1 and 2 satisfied for the closed interval c d also. Then you have two possible definitions you have to show that both are equal. So, you will have to show that integral a to b S is integral c to d S and that is not too hard that is rather easy and as other.

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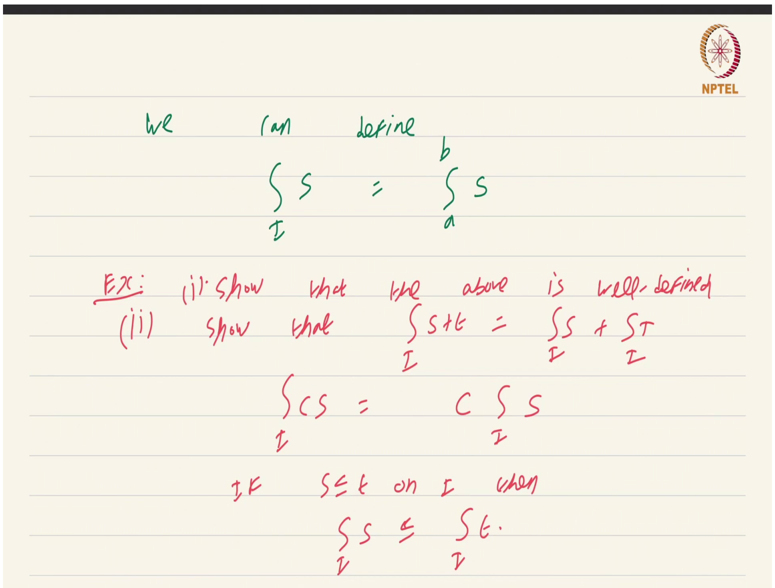

Let us call this first part of this exercise second part show that integral of S plus t over I is integral S over I plus integral T over I and integral over I of C S is C times integral of I integral of S over I and if S is less than or equal to t on I then integral of I S is less than or equal to integral of I t. All the basic properties of the integral continue to hold true for these more general types of integrals are defined for general step functions.

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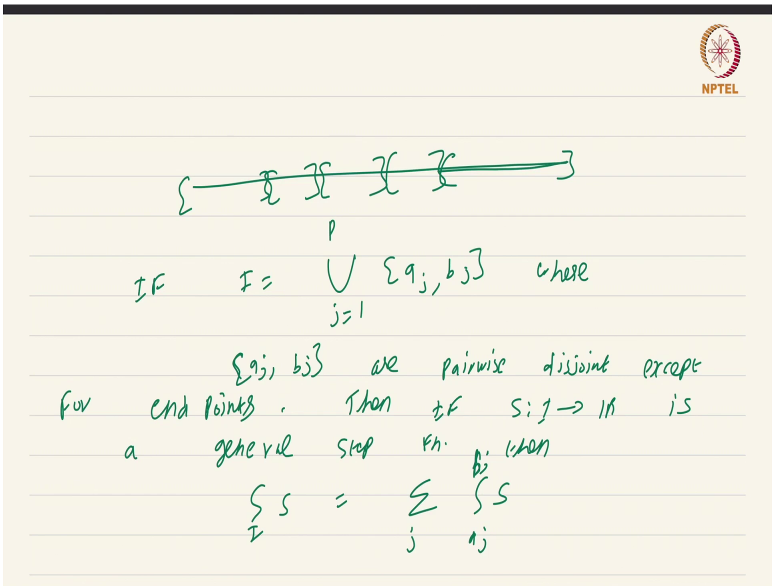

And also we have this other property that you are no doubt familiar with if you have some interval and you break it up into smaller intervals such that each piece intersects only at the end points it could be half open half close you just break it up in some way that is if I is union over let us say j running from 1 to p a j, b j ok where a j, b j are pairwise disjoint except for possibly end points except for end points that is no two intervals have an interior point in common.

Then if S from I to R is a general step function general step function then integral over I S is actually summation over j integral a j to b j of S and this also follows more or less directly from the very definition of the integral of a step function. Now, notice that if you had defined the integral of step functions via the Riemann integral everything I have written down in this exercise is trivial you do not need to do anything to prove it.

But if you take the definition of the integral of a step function not using the Riemann integral, but by the way we have defined it in the last video there is a little bit of work to be done, but it is minimal ok. So, with all this said we can come to the main theorem that will allow us to define the Lebesgue integral we are going to consider sequences of functions sequences of step functions that are monotone almost everywhere.

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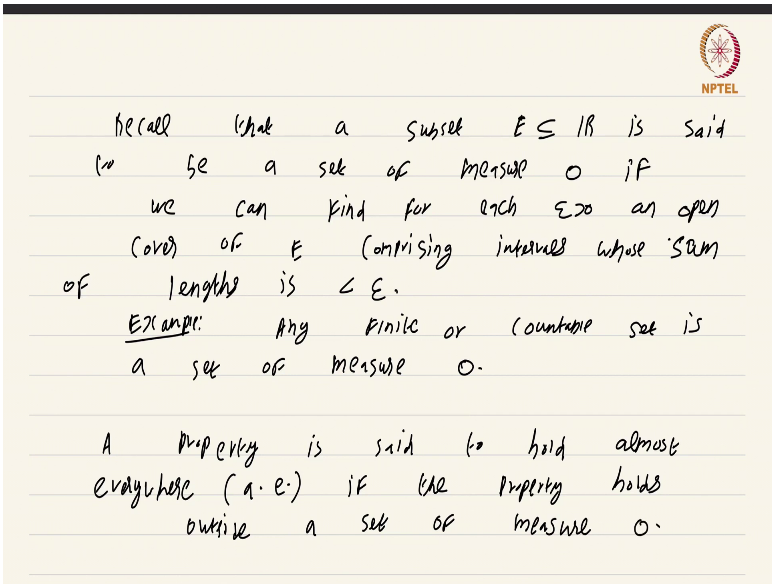

So, let me recall that a subset E of E is said to be a set of measure 0 to be a set of measure 0 if we can find for each epsilon greater than 0 an open cover open cover of E such that the sums of the lengths sums of the lengths we can for each epsilon greater than 0 an open cover of E consisting let me just make the definition slightly easier open cover of E comprising intervals whose sum of lengths sum of lengths is less than epsilon.

So, no matter what small epsilon you choose you can always find an open cover of this interval sum the sum of the lengths of this interval should be less than epsilon ok. Now, we studied this sets of measure 0 in quite some detail when we studied the Riemann Lebesgue theorem which gives a precise condition for when a function is Riemann integrable, if you have not come across this notion of sets of measure 0 that is if you have not taken real analysis 1 please go through the lecture notes provided at the very first week.

There in the chapter on Riemann integration we go at this in great detail, why these sets of measure 0 are important and what they have to do with integral, basically these are sets that are negligible in some sense ok. So, just for sake of clarity as an example you can show that any finite or countable set for countable set is a set of measure 0 and any interval of positive length that is the any interval which has an interior point will not be a set of measure 0 if you can solve these two now you are good to go for what is to come ok.

Now, one other definition a property is said to hold is said to hold almost everywhere as mathematicians are very lazy almost surely nobody is going to write this almost everywhere in full detail we will just abbreviate it to a. e. a property is said to hold almost everywhere if the property is true property holds outside a set of measure 0 outside a set of measure 0. The property that we are interested in this particular video is sequences.

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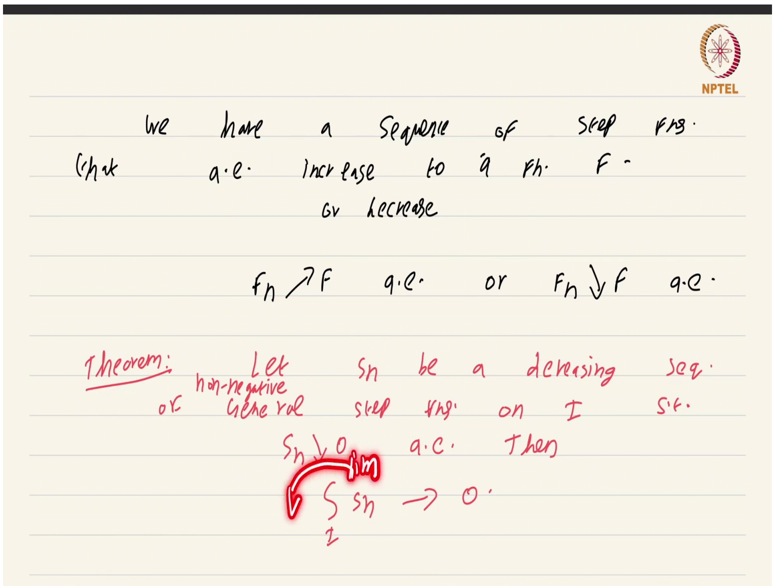

So, we have a sequence we have a sequence of step functions that almost everywhere increase to a function f ok that increase or decrease, that is these are functions that are monotone except for a small negligible set outside of that small negligible set these functions are either increasing or decreasing to a given function.

We are going to consider a sequence of such functions and if those sequence of functions satisfy some conditions then actually you can interchange limit and integral this is the starting point of the Lebesgue theory. We will denote this by f n increasing to f a. e. or f n decreasing to f a. e. we will use this slant wing upward slanting arrow for increasing and downward slanting arrow for decreasing.

So, the precise theorem which is the main theorem of this particular video and the theorem that is going to allow us to define the Lebesgue integral is as follows. Let S n be a decreasing sequence of general step functions general step functions on I such that these are non negative ok decreasing sequence of non-negative that is crucial non negative general step functions on I such that S n decreases to 0 almost everywhere ok.

So, we are considering functions which are decreasing number 1 and this decreasing is everywhere ok not just almost everywhere and these S n's are decreasing to 0 almost everywhere then integral over I of S n converges to 0. Inside if you notice the integrand the limit of this is of course, 0 because these sequences are decreasing almost everywhere to 0 we what is essentially happening is if the limit were inside you will get 0 that limit can be taken outside that is essentially what is being asserted.

So, at least for these very nice sequence of non negative step functions defined on the interval I we can interchange the limit and the integral and we are going to use this as the starting point of our theory. Those who took real analysis 1 and studied the proof of the Riemann Lebesgue theorem in some detail will find this proof very very similar to the proof that we saw there.

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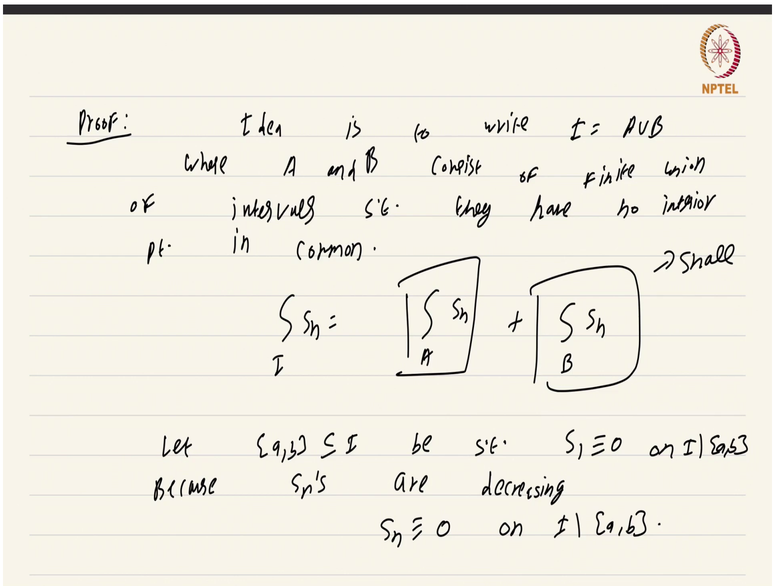

The ideas in this particular proof are typical of the proofs in analysis. You want to estimate this integral over I S n what is going to happen to that and the goal the way you estimate that is you break up I into two pieces in one piece you can control S n in the other piece you do not have control over S n, but you can control the size of that other piece.

So, what you essentially do is since this function behaves well except on a set of measure 0 because we have decreasing to 0 almost everywhere outside of the points where it is outside of the points where it is not decreasing from 0 the function can be made really small and it those points where it is failing to decrease to 0 it is actually a negligible set because those points are only going to be a set of measure 0 by hypothesis there you can control the size of the intervals though you cannot control S n there together will yield the result.

So, let us see the proof idea is to write I equal to A union B where A and B; A and B are intervals close A and B sorry A and B consist of finite union of intervals such that they have no interior point in common. Then we can use the last property of the integrals that I stated in the exercise when you can break up I into a union of intervals then there is a nice summation property we will use that they have no interior point in common ok.

Then; obviously, you will have integral over I S n is sort of integral over S n A plus integral over S and B and each one of these will go to 0 both this as well as this will go to 0 as n increases that is the basic idea that is we can make both of these small both are small ok.

So, what are we going to do well let a, b subset of I be such that S 1 is identically 0 on I minus a b outside of this closed interval a b S 1 is identically 0 because S n's are decreasing S n's are decreasing S n is identically 0 on I minus a b that is why we assume that S n's are globally decreasing and not just almost everywhere it will make our life easy ok.

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Each S_n is constant in the open intervals determined by some partition of $[a,b]$. Let D_n be the end points of such a partition.

Notice that D_n is finite

$$D := \bigcup_{n=1}^{\infty} D_n \text{ is a countable set and}$$

is therefore a set of measure 0.

Furthermore, we have a set of measure 0, say $E \subseteq [a,b]$ s.t.

$$S_n \to 0 \text{ on } [a,b] \setminus E.$$

Now, each S n is constant in some partition in the sorry in the open intervals determined by some partition of a b, note this partition will depend on n ok. Let D n be the end points of such a partition end points of such a partition. Now why we want to focus on the end points is because the way we defined a step function we sort of said that we do not care what happens at the end points it could take whatever value the function pleases ok.

So, we have really no control over the behavior of the functions S n on the end points, but notice that notice that D n is finite. Therefore, the union of D n the union of D n as n runs from 1 to infinity which we just called D is a countable set is a countable set and is therefore, a set of measure 0 therefore, a set of measure 0.

So, we have isolated one set of bad points those points where there is an end point of the interval of constancy of one of the step functions all such end points of partitions we have put together in the set D and that set is thankfully a small set because it is countable. Furthermore we have a set of measure 0 say E subset of a b such that S n decreases to 0 converges to 0 essentially on a b minus E, outside of a set of measure 0 we already know that S n converges to 0 because we are assuming S n decreases to 0 almost everywhere ok.

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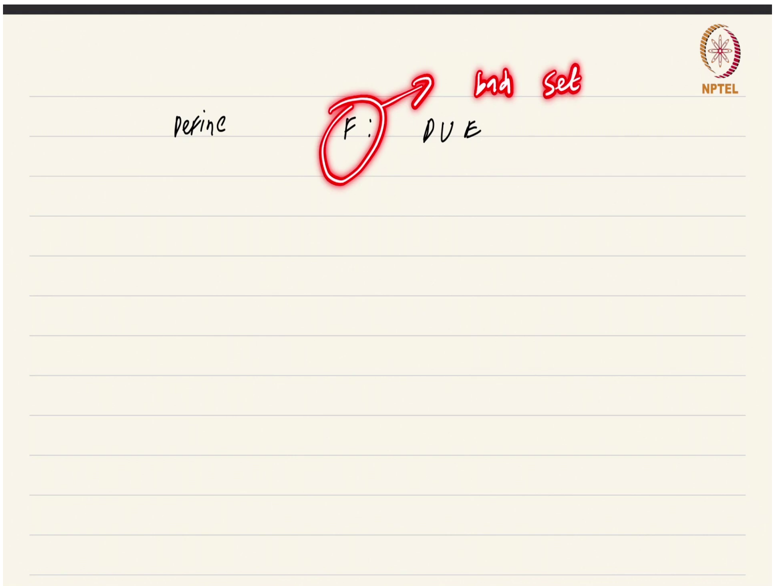

What we are going to do is, we are going to define our bad set f to be D union E. So, this f is the bad set, outside of this bad set we can completely control the behavior of the functions S n by using the fact that S n decreases to 0 there, how do we do that, well what you do is the following.

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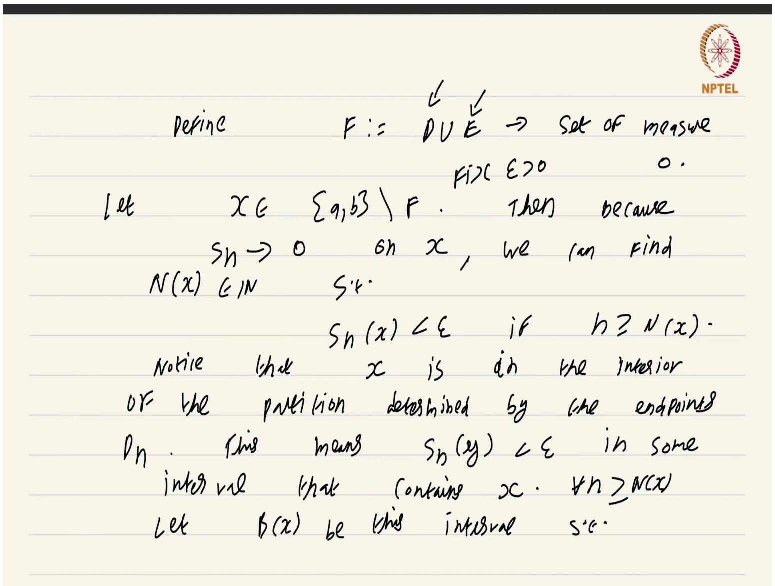

Let x be an element of a b minus f oh before that simple observation this is a set of measure 0 union of set of measure 0 countable union of sets of measure 0 is also a set of measure 0 this is just a union of two sets of measure 0 the set D is a set of measure 0 because it is countable, E is a set of measure 0 by hypothesis.

So, let x be in an element in a b minus f ok. Then because S n converges to 0 converges to 0 on x because we are choosing x outside the bad set we can find we can find N x some natural number such that S n x is less than epsilon if n is greater than or equal to capital N x ok.

So, we must write fix epsilon greater than 0 of course so, once you have fixed an epsilon we can find a large enough natural number such that S n x is less than epsilon if n is greater than or equal to N x. Of course this happens because S n is decreasing to 0 on a b minus f ok.

Notice that x is in the interior of the partition determined by the end points D n right, we started off with the partition all the end points we called D n and put them separately in this set D which we eventually put in the set f, this x is coming from a b minus f; that means, in the partition determined by D n this x has got to be one of the interior points it cannot be the end point of one of the sub intervals.

That means this means S n x is less than sorry S n y; S n y is less than epsilon in some interval that contains x interval that contains x ok and this is true for this is true for all n greater than N x greater than or equal to N x. Why is it true for all, well because the moment S n x is less than epsilon then because it is constant on the sub interval at some small piece it is going to be constant and that constant is less than epsilon by a hypothesis, but S n's are decreasing.

So, if you take an even larger n that constant can only decrease ok. So, what happens is sorry I said that that constant can decrease it can happen that if you take a larger n the function is no longer constant on the whole interval, but nevertheless it has to be less than S n x which is constant ok. So, apologies for that mistake, but I hope you catch my drift ok.

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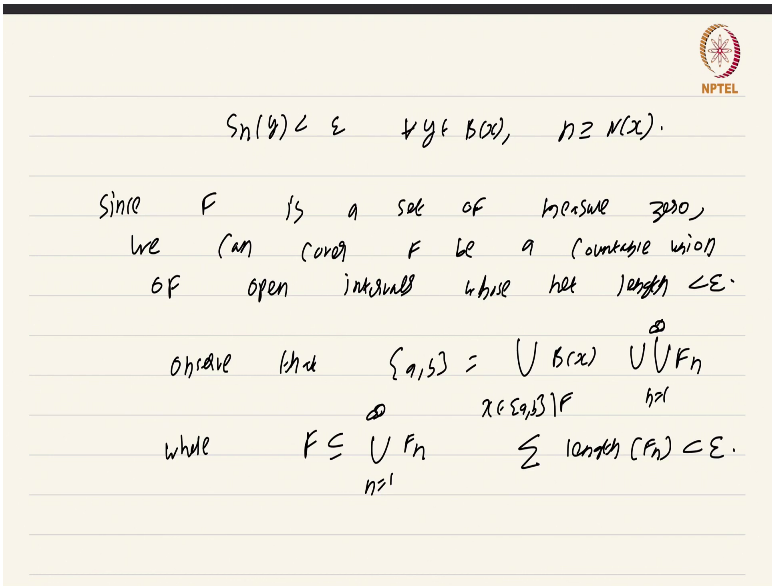

Now, what you do is let B of x be this interval such that S n y is less than epsilon for all y in B x and greater than or equal to N x ok. Now here is the crux since that other set bad set f is a set of measure 0 is a set of measure 0 we can cover; we can cover f by a countable union of open intervals whose net length whose net length is less than epsilon.

The set f is a set of measure 0 therefore, we can find countably many open intervals whose union both cover f and whose net length is less than epsilon excellent. Now, observe that a b is union of these B x's as x ranges through a b minus f and union these f n's ok union of f n equals 1 to infinity, where f is a subset of union n equal to one to infinity f n and summation of length of f n is less than epsilon. These f ns are open intervals whose net length is less than epsilon and they cover f, a b is compact a b is compact.

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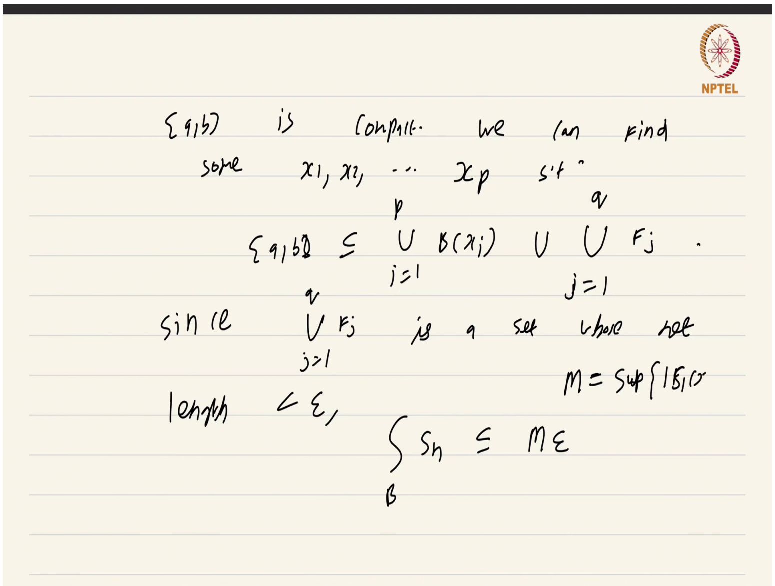

This means we can find we can find some x 1, x 2 dot dot dot x p such that a b is a subset of union i equals 1 to p, B x i and union sum sub collection of these f js. So, I can say j running from 1 to some q of f j ok. So, you can find as finite sub collection of these intervals B x is and a finite sub collection of these f j's such that the union is going to be a b. Note the Heine Borel theorem will just given sub cover please think over why we can write this ok, it is direct from I mean it is one step away from Heine Borel this is not a direct conclusion of Heine Borel theorem think about why you get this.

Now, we are going to define a and b the original goal was to break up the integral into two pieces on one piece A we can control the behavior of S n and on the other piece S n on the other piece B the set B can be made really small that is the idea and as you can guess this part is going to be the small length intervals and this part we have a good control of the functions S n ok.

So, what are we going to do, well since the union j equal to 1 to q f j is a set whose net length whose net length is less than epsilon we have integral over B of S n is less than or equal to M times epsilon where M is the supremum of mod S 1 of x or rather I can just write it in shortcut notation as the norm sup norm of S 1 right.

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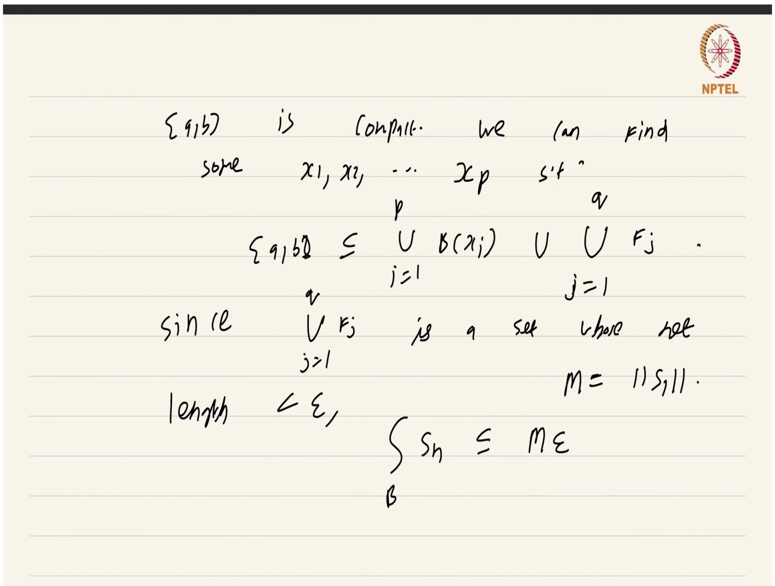

Since the function S 1 is a step function which vanishes outside the interval a b this is going to have a supremum, it is going to have a sup norm, M is that sup norm and integral over B S n is M epsilon ok less than or equal to M epsilon, this is simply because the net length of these intervals f j has got to be less than epsilon excellent.

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$\int_A s_n$ But for all points $x \in A$, we have

$s_n(x) < \varepsilon$ if $n > \max\{N(x_1), N(x_2), \ldots N(x_p)\}$.

$\int_A s_n \leq \varepsilon(b-a)$.

$\int_I s_n \leq m_\varepsilon + \varepsilon(b-a)$.

We are done.

Now, we are going to deal with this integral of S n over A right, but for all points for all points x in A we have we have S n x, S n x is less than epsilon if n is greater than maximum of N x 1 comma N x 2 comma dot dot dot N x p. Recall how were these intervals B x is chosen well where did that definition go we chose we first chose this N x such that if n is greater than or equal to N x, S n x is less than epsilon and B x was the corresponding interval on which this S n is constant and therefore, S n x will be less than epsilon for all y in B sorry S n y will be less than epsilon for all y in B x whenever n is greater than or equal to N x.

This is the way this intervals B x and capital N x were chosen now what I am saying is if you choose n to be larger than N x 1 dot dot dot N x p, then the corresponding intervals B x 1, B x 2 dot dot dot B x p will cover a b minus f and moreover whatever point x you choose from a b minus f that is precisely not a b minus f a b, yes correct actually a b minus f whatever point you choose from a b minus f S n x will have to be less than epsilon if you choose n to be suitably large ok.

So, we have S n x less than epsilon which means that integral of S n over A is less than or equal to epsilon times b minus a, because the net length of the intervals A will be less than b minus a. So, what this shows is that integral of S n over I is less than or equal to M epsilon plus epsilon b minus a and by the k epsilon principle which we are invoking after many lectures we are done; we are done.

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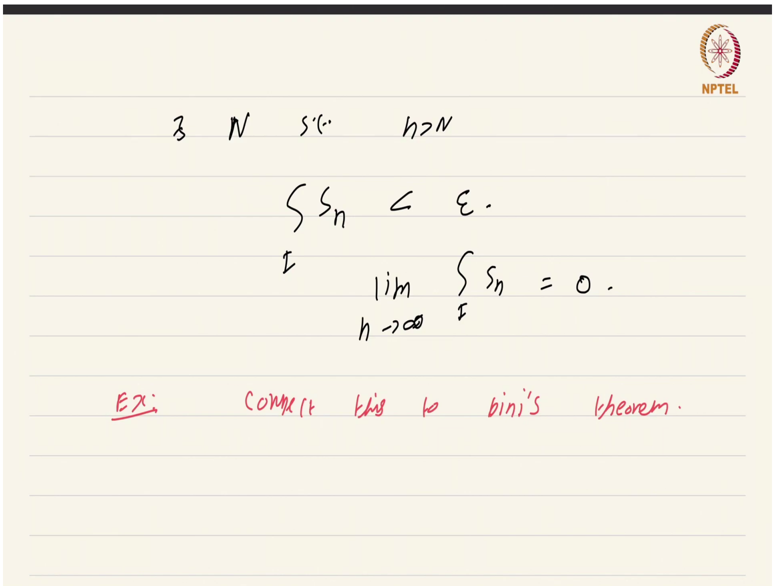

What more elaborately what we have shown is there x is capital N such that if n is greater than capital N integral of I over S can be made less than epsilon. In other words limit n going to infinity sorry this should be S n limit n going to infinity integral over I S n is 0. So, when you have a decreasing sequence of functions that almost everywhere decrease to 0 then what happens is the integral of those step functions will also converge to 0.

Now, as an exercise since this proof looks really complicated, but the basic idea is simple as an exercise connect this to Dini's theorem. If you do not remember what Dini's theorem is or you have not come across it please control F on the PDF notes for real analysis 1 and search Dini there is an exercise where this has been done, check the proof of that exercise hints are given, solve that exercise and try to connect this particular theorem to Dini's theorem that is a very interesting exercise to work out.

So, in the next video we will define Riemann integrals for upper functions, these are functions that these are step functions I mean upper functions are functions that arise as limits of step functions that are increasing almost everywhere and the fact that this integral is sort of well defined will be a consequence of this theorem. So, in the next video we will define that. This is a course on Real Analysis and you have just watched the video on Monotone sequences of functions.