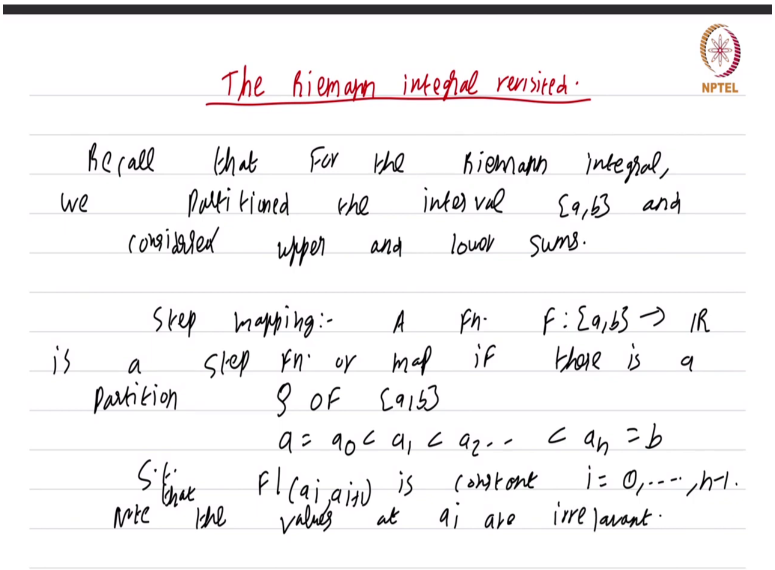
Real Analysis II
Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 25.1
The Riemann Integral Revisited

(Refer Slide Time: 00:22)



We are now in the final leg of this course on Real Analysis. In this part of the course, we shall study the Lebesgue integral. The Lebesgue integral behaves very well with respect to limits. Recall that we have already shown that if a sequence of functions that are Riemann integrable converge uniformly on an interval, then you can interchange the limit and the integral.

But, this requirement of uniform convergence makes the Riemann integrable a Riemann integral not so well suited for many many applications in analysis and outside. The Lebesgue integral remedies this deficiency of the Riemann integral. There are several approaches to the

Lebesgue integral. Henri Lebesgue created the Lebesgue integral using his concept of Lebesgue measure which was part of his PhD thesis.

Subsequently, there are several other equivalent formulations of the Lebesgue integral that bypasses measure theory. We shall use one such approach pot popularly attributed to the mathematician Rees. We shall closely follow the treatment given in Apostol's Mathematical Analysis, but before we get ahead with the actual nuts and bolts let me try to motivate how this Lebesgue integral can be viewed in light of the theory we have developed for norm vector spaces.

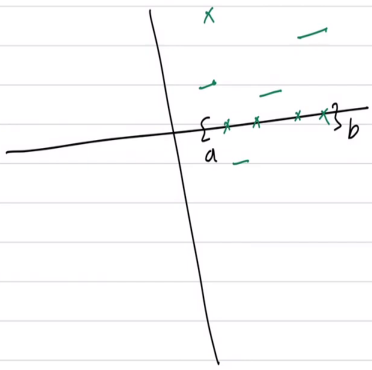
To do that, we will revisit the Riemann integral from the perspective of completion and extension of linear maps. So, recall that for the Riemann integral the Riemann integral, we partitioned the interval a, b and considered upper and lower sums.

Then, we sort of refined these partitions further and further, and we said that the Riemann integral the function is Riemann integrable on the interval a, b if the upper sums and lower sums can be made as close to each other as you desire by making the mesh of the partitions suitably small.

Now, we are going to view this Riemann integral in a slightly different way. Let us start with the step mapping let us start with the step mapping. What is a step mapping? Well, a function F from a, b to R is said to be a step mapping or function is a step function or map if there is a partition P of a, b; recall a partition is just a collection of points from the interval a, b you we usually denote the partition like this a equal to a naught less than a 1 less than a 2 dot dot dot less than a n and this a n is equal to b.

If there is a partition such that F restricted to the open interval a i, a i plus 1 is constant where i runs from 0 to n minus 1. So, you can break up the interval a, b into finitely many pieces such that on each piece the function is constant. Note that the values note that the values at a i are irrelevant, it could be anything are irrelevant, ok.

(Refer Slide Time: 04:50)



So, for a picture you have let us say this is our interval a, b, then you have this partition. In each one of these sub pieces this is sort of constant. So, that is what a step function is at these end points it could be anything, it could take values that are very different at the end points it really does not matter ok.

(Refer Slide Time: 05:19)

Of course,

$$\int_a^b F := \sum_{i=0}^{n-1} w_i (a_{i+1} - a_i) \quad (*).$$

w_i is the value of $F|_{(a_i, a_{i+1})}$.

We can define $\int_a^b F$ by $(*)$.

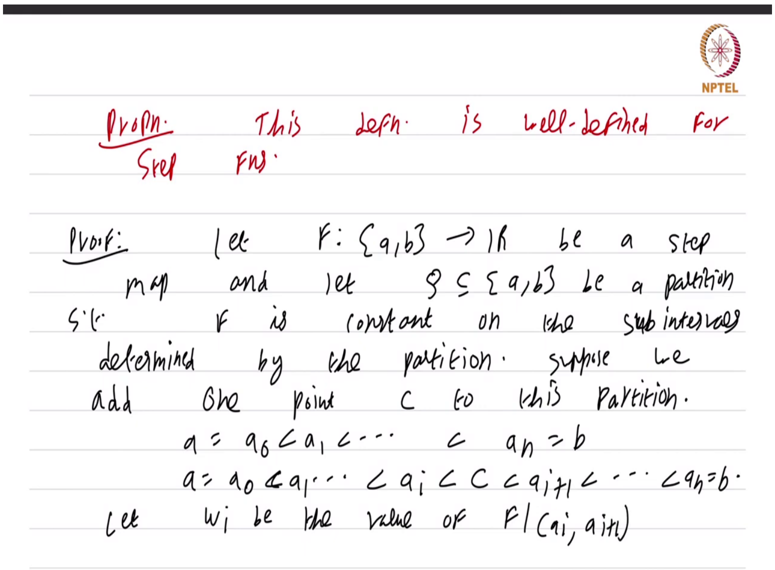
Now, of course, from the theory of Riemann integral that we are now very familiar with, it is clear that the integral of a to b of this step mapping F is nothing but summation of w i into a i plus 1 minus a i i runs from 0 to n minus 1 where w i is the value of F restricted to a i comma a i plus 1.

This is easy to see if you I mean if you are familiar with the Riemann integrable this is rather trivial. In fact, this also says that the Riemann integral of a step function of course, is the sum of the areas of the various rectangles that are formed by the values of the function f this is one rectangle this is another rectangle this is a third rectangle so on and so forth ok signed sum of course.

So, instead so, let us give this equation a name let us call this star. So, this integral a to b F being equal to this already presupposes that you are familiar with the Riemann integrable

Riemann integral, but what you can do is you can make this into a definition. That is we can define we can define we can define integral a to b F by star, right. For a step function we can define the Riemann integral to be given by this formula. Of course, there is something to check there is something to check.

(Refer Slide Time: 07:14)



So, that will be the first proposition this is well defined this definition is well defined for step functions and the proof is rather easy and should be familiar to you. I should be should remind you of several of the arguments we have done when we studied the Riemann integral in real analysis one, ok. So, let F from a, b to R be a step function step map.

And, let P subset of a, b be a partition be a partition such that F is constant on the sub intervals determined by the partition on the sub intervals determined by the partition by the partition. Now, the standard trick that we are going to use is we have to show I mean what is the meaning what possible issue could arise in this definition is F is a step function there is no unique partition P such that F is constant on the sub intervals determined by P. There could be millions of them, right.

We just know that there is one such partition how do you know that this some star is going to be the same irrespective of the choice of partition you make. Well, what we are going to do is suppose we add one point add one point C to this partition to this partition.

Now, I am going to show that when you add this one point the integral is not going to change, then by an induction argument you can prove that no matter how many points you add the integral is always going to be the same. Then if you have a step map which is constant on sub interval determined by another partition Q also then you just take the common refinement and you are done.

You have shown that the integral defined with respect to this formula for the first partition P is same as the answer you get when you use the same definition for the partition Q, ok. So, what happens if you add one point? Well, suppose the original partition was a naught or rather a equal to a naught less than a 1 less than dot dot dot less than a n equal to b.

Well, let us say between a i and a i plus 1 we add this point C. So, we have a equal to a naught less than a 1 dot dot dot less than a i we add a C in between here less than a i plus 1 less than dot dot dot less than a n equal to b; in one of the sub intervals we have added a single point.

(Refer Slide Time: 10:36)

What is $(*)$ evaluate to?

$$w_0(a_1-a_0) + \cdots + w_i(C-a_i) + w_i(a_{i+1}-C) + w_{i+1}(a_{i+2}-a_i) \cdots + w_{n-1}(a_n-a_{n-1}).$$

$$= \sum w_i(a_{i+1}-a_i) = (*) \text{ w.r.t to } \wp.$$

By induction and taking common refinement, it is clear $(*)$ is well-defined.

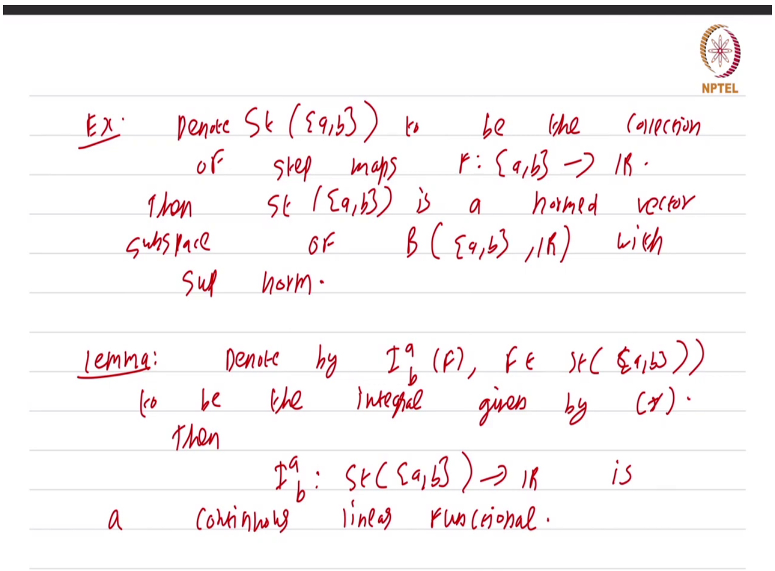
Well, what is star with respect to this new partition what is star evaluate to? Well, if I choose the value. So, let w i be the value of F restricted to a i comma a i plus 1 then what a star evaluate to? Well, it is the same as summation let us not write a summation let us just keep it as w 1 into a 1 minus a naught plus dot dot dot then what will happen is when you reach this interval a i a i plus 1 the crucial part where a point has been added. In fact, the only place where something has changed what will happen is you will get two terms.

You will get w i into C minus a i plus w i into a i plus 1 minus C plus w i i plus 1 a i plus 2 minus a i plus dot dot dot w n sorry, w n minus 1 a n minus a n minus 1, ok. So, technically this should be w naught this should be w naught by our notation ok excellent this is nothing but summation w i into a i plus 1 minus a i.

These two terms just combine together and this is just equal to star with respect to the partition P, ok. So, adding one point to the partition does not change the value of the integral we have defined for step mappings. By induction by induction and taking common refinement taking common refinement it is clear that star is well defined.

I will not belabor this proof anymore, it should be clear to you what is happening and you have done such proofs quite a lot when you studied the Riemann integral as part of real analysis one or elsewhere wherever you picked up the Riemann integral from. Excellent, so, we have shown that this star is a good definition for the Riemann integral for step functions. Now, here is the crucial point.

(Refer Slide Time: 13:18)



And, this is sort of like an exercise at easy exercise. Denote St a, b to be the collection of step maps F from closed interval a b to R. Then S t a, b is a normed vector subspace of B a, b R, that is the set of all bounded functions on the closed interval a, b to the real numbers with sup norm.

This is a rather trivial exercise. This is a rather trivial exercise. All you have to show is sum of two step functions. This is a step function a scalar times step function is a step function and the fact that any step mapping is bounded and all that ok. So, this is a rather trivial exercise. So, the collection of step mappings is in fact, a normed vector subspace of the set of all bounded mappings given the sup norm, ok.

Now, another lemma which is almost immediate once you prove this exercise is it will be rather easy to define I mean easy to prove. So, denote by I a b of F, F in S t a b to be the integral given by star the integral given by star. Then I a b from St a b to R is a continuous linear map linear functional that is a bounded linear functional in the terminology that we have developed in the first part of the course on metric spaces.

(Refer Slide Time: 15:38)

Proof: Linearity of I^a_b is trivial.

If $||f|| = M$

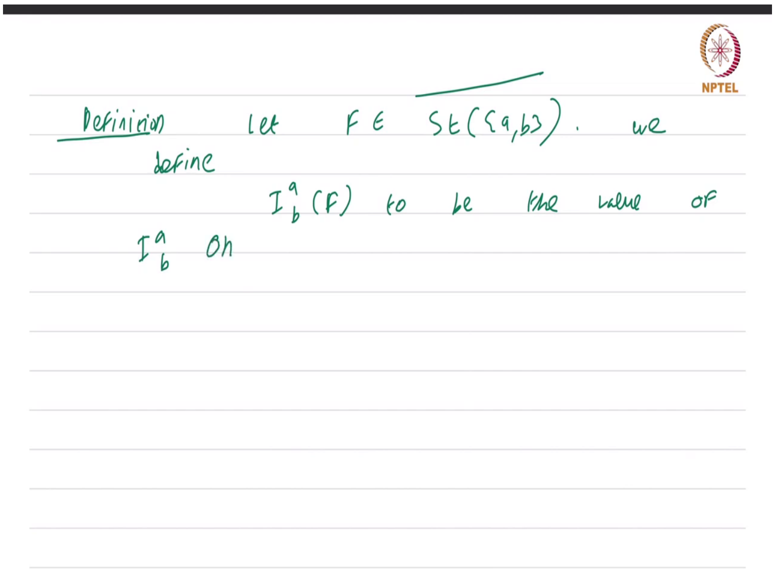
Then $(*)$ immediately gives

$I^a_b(f) \leq M(b-a)$, which gives continuity.

Well, the proof of this is not so hard proof linearity is trivial that is just follows from the definition and taking refinements linearity of star or linearity of I a b; how did I put it? I a b. I a b is trivial. As for continuity, well, observe that if supremum of or rather norm F under the sup norm is some M, then star immediately gives I a b of F is less than or equal to M times b minus a, ok. So, which immediately proves continuity, which gives continuity, from the theorems that we have developed about continuity of linear maps on a normed vector space.

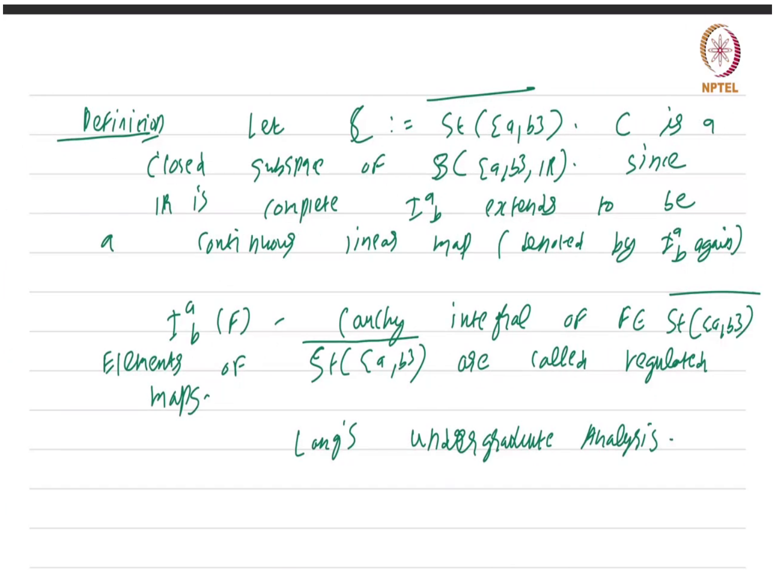
So, we have defined an integral not on all continuous functions or all Riemann integrable functions, but on a very simple set of functions the set of all step mappings, on that we have defined an integral that obviously, agrees with the classical Riemann integral that we have defined in real analysis one. But, this I a b is linear on the space of all of space of all step mappings.

(Refer Slide Time: 17:23)



Well, the next step is the definition let F be an element of St a, b closure. We define I a b of F to be the value of I a b on a or rather let us let me just state it in a slightly different way.

(Refer Slide Time: 18:14)



Let me define it in a slightly different way. This is a bit complicated. Let the space let us call it C, at the space C be by definition S t a, b closure ok. Now, C is a closed subspace of the bounded functions from a, b to R. Since R is complete R is complete I a b extends to be a continuous linear map continuous linear map denoted again by I a b denoted by I a b again that is we do not introduce another notation just to burden ourselves.

I a b extends to be a continuous linear map denoted by I a b again, then the definition is I a b of F is said to be the Cauchy integral of F which is there in St a, b closure elements of this St a, b closure a, b closure are called regulated maps called regulated maps ok.

Let us take a deep breath. What have we done? Well, defining the Riemann integral as the area under the curve is rather straightforward and trivial to do for step mappings, there is no complication that arises. So, we do that that was stage 1. Then we exploit the advanced machinery of non-vector spaces and linear mappings on them and extensions on them to immediately extend the definition of that integral to a much larger class of St a, b closure.

Now, it will turn out that every single continuous function is actually there in St a, b closure. I will not be proving that in this course. You can check Lang's undergraduate analysis for details undergraduate analysis for details, since we have already spent the time developing the Riemann integral in the classical way I will not do it once again using these step mappings and taking the closure and linear extensions. This is just to prepare ourself for what is about to come as part of the Lebesgue integral.

So, we using this abstract machinery of linear extension we have managed to get an integral defined on a substantially good class of functions St a, b closure; it will contain all the continuous functions for instance. It will also contain plenty of non continuous functions. In fact, most step mappings are not continuous and what is so special about this approach is proving the basic properties of the integral for step mappings is rather trivial.

In fact, I just left it as an exercise show that it is linear and all that, it is rather easy to do. What you can do is exploit this fact that this I a b comes from a linear extension to get many of the properties of the Riemann integral almost for free. We did quite a bit of hard work when we proved the basic properties of the Riemann integral, you can sort of get all of those for almost free if you take this approach.

One disadvantage of this approach is it is quite non-constructive in the sense that you are appealing to quite an abstract theorem or normed vector spaces to even define the integral I a b of F. But, nevertheless as a theoretical reformulation of the Riemann integrable; Riemann integral this is quite useful, and one can justifiably define this integral as the standard integral and forget about that more technical construction using upper sums and lower sums.

It is purely a matter of popularity the older construction is still more popular, but many people do believe that this abstract construction is the correct one but, nevertheless leaving aside the controversial points regarding pedagogy. Coming back to what we are about to do we are going to be inspired by this approach define the same integral for step functions but this time take a different sort of limit.

Here we are sort of taking when you take the closure under the sup norm that same as uniform convergence. So, here we are taking sort of the uniform limit of functions and calling those the regulated functions. Instead, we are going to consider as different sort of limit and define a new integral and that new integral will be much more powerful. The collection of all such integral integrable functions will be a much bigger class and it will behave well with respect to taking limits, ok.

So, for the interested student you can refer to Lang's undergraduate analysis for all the details of how to recover all the basic theorems that we approve for the Riemann integrable Riemann integral for the Cauchy integral and also C cannot contrast the two approaches. But, that is up to you, that is not part of the syllabus of this course.

In the next video, we shall talk more about step functions and monotone sequences of step functions and prove a certain limit theorem for such sequences that will pave the way for eventually defining the Lebesgue integral, which as I have repeatedly said is a substantially more powerful integral from the perspective of analysis. This is a course on Real Analysis and you have just watched the video on the Riemann integral revisited.