Real Analysis II
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Lecture - 24.1
Rectifiability and Arc-Length

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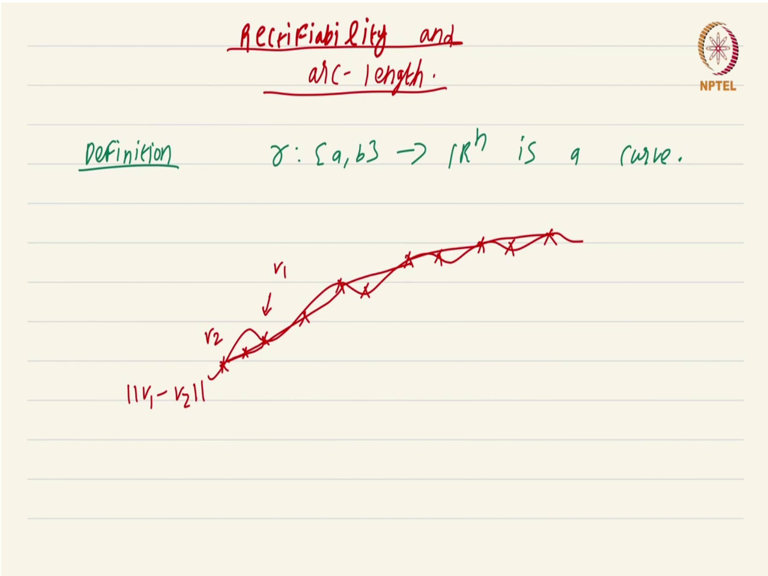

We are now interested in defining the length of a curve. The last example that we saw that of space filling curves suggest that we should approach this with a bit of care. It is unreasonable to expect that all curves will have lengths those that do are called rectifiable curves and we now give the definition of this as well as what the length of a rectifiable curve is.

So, definition as always gamma from a b to R n is a curve is a curve. What we are going to do to define rectifiability and arc length is the most obvious thing. So, let me first draw a picture to illustrate what is going to happen, then erase the picture and move on with the definition.

So, the picture is as follows. You have some curve. What you do is, you pick points on this curve, you pick points on this curve and join the line segments in between these points. So, I have not drawn it perfectly, but you will get more or less an idea of what is happening. So, you draw these line segments together.

Now, what you do is, you know how to measure the length of a line segment that is just if you call this vector v 1 and v 2, the length of this line segment from v 1 to v 2 is just norm v 1 minus v 2. So, this is what is known as a polygonal path.

You just look at the length of the polygonal path and our intuition is that this polygonal path, the length of this polygonal path will approximate the length of the curve. Then all what you do is, you just keep adding more and more points exactly analog as to what you did to define the Riemann integral as the area under the curve what we did was break it up into rectangles there.

So, here you break it up into line segments and as you make these points closer and closer to each other intuitively, the length of the polygonal path will in some sense converge to the length of the curve. That is the basic idea. Now, let us make this precise. Let us make this precise what you do is the following.

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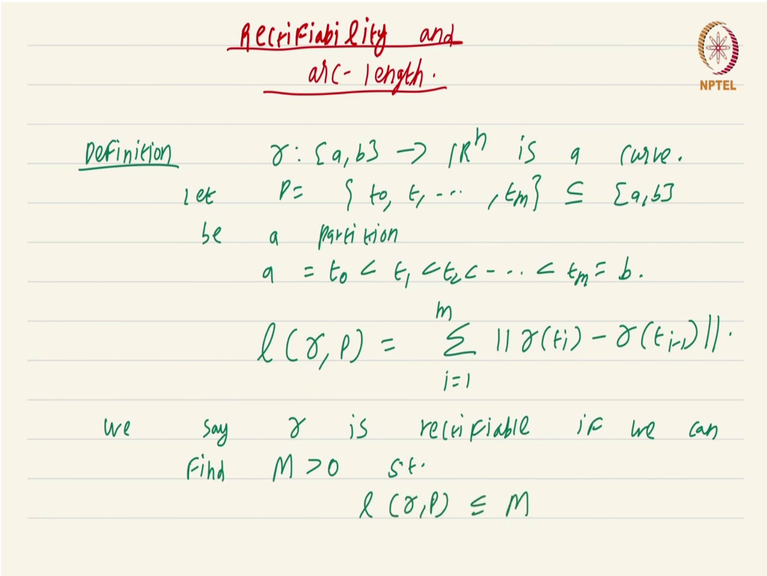

Let P equal to t naught t 1 dot dot dot t m subset of a b be a partition; be a partition and as the notation suggests, we always take t naught to be a and t naught less than t 1 less than t t 2 dot dot dot less than t m which is equal to b exactly like how you defined the Riemann integral. You consider a partition and you order the points in increasing order.

Now, we define the length of this curve gamma with respect to this partition P to be as you expect summation i running from 1 to m. The norm of gamma of t i minus gamma of t i minus 1, ok so, that this is nothing, but the length of the polygonal path whose endpoints are determined by this partition more precisely, the end points are gamma of t i ok

Now, here is the central definition we say gamma is rectifiable if we can find if we can find a number M greater than 0, such that this l gamma P is less than or equal to m irrespective of P, irrespective of p.

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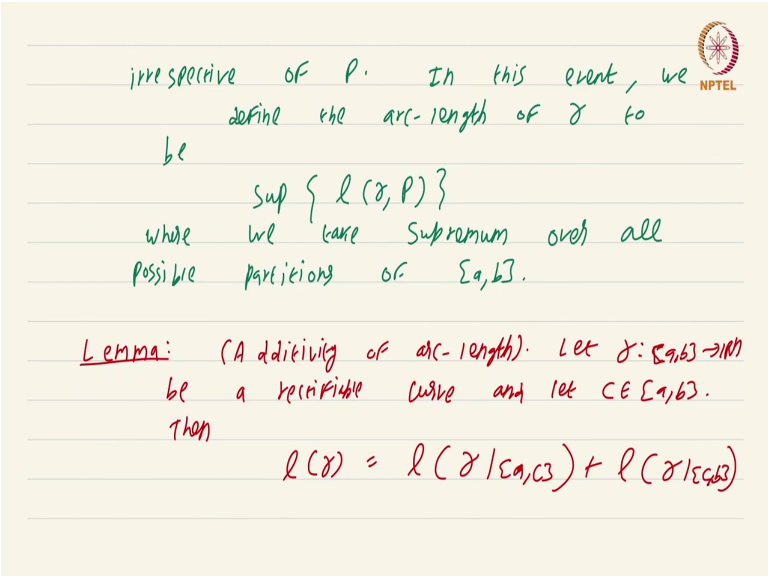

So, what this is saying is no matter what partition P you choose the length with respect to that partition will be less than or equal to M and this M is independent of the choice of partition, ok. So, in this event in this event when it happens that there is such a curve I mean there is such a constant M in this event.

We define the arc length arc length of gamma to be the supremum of l gamma P l gamma P, where we take supremum, we take supremum over all possible partitions, all pass possible partitions. I think I wrote too many t's in partition, all possible partitions of a b.

So, the definition of the arc length of a rectifiable curve is the most natural one possible. You approximate the curve by polygonal parts and the maximum length of a polygonal path that you can obtain in this way is called the arc length. We will see maybe 7 or 8 videos down the line how this method spectacularly fails for surfaces, you would think that the exact same idea would work for surfaces, try to approximate the surface not by polygonal parts, but by triangular faces or something like that a triangulation and approximate the area of the surface by the net area of the rectangles that I mean triangles that you get that will hopelessly fail. We will see that, ok.

So, essentially what we are doing is, we are approximating the curve from the inside using polygonal paths and taking the supremum over all such lengths that you get and defining that to be the arc length, ok. Now, let us approve some basic properties of arc length none of these are particularly shocking and the proofs are more or less trivial because we have already seen the Riemann integral.

Yeah one more remark in this regard just like the Riemann integral, the moment you add an additional point or that is refined a partition, the length with respect to the new partition will be greater than or equal to the length with the original partition. I am not even going to bother writing this down. The proof is very easy. So, please do that ok.

So, under refinement the length can only increase. We will use that many times in the proofs. So, the first lemma is additivity of arc length. This is more or less geometrically obvious, but requires a proof this is states the following.

Let gamma from a b to R n be a curve be a rectifiable curve rectifiable curve and let C b a point in a b, then length of gamma is nothing, but length of gamma restricted to a C plus length of gamma restricted to C b. That is if you break a curve into two pieces and take the sum of the arc lengths, you get the sum, you get the arc length of the whole curve. Of course, this is obviously true by induction even for finitely many curves.

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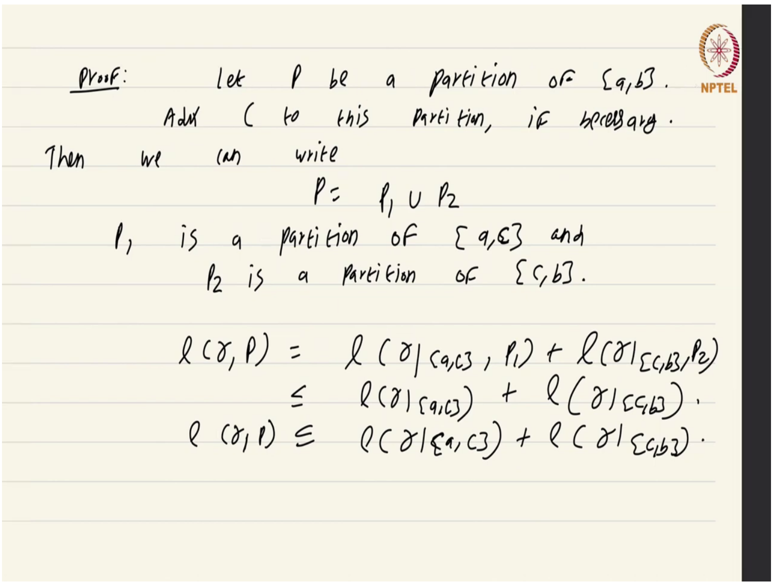

Proof, again the proof is not at all hard. What you do is, you fix a partition let P be a partition of a b, ok. Now, add C to this partition if needed this partition if necessary then we can write we can write P as P 1 union P 2, where P 1 is a partition is a partition of a C a C and P 2 is a partition of C b, right.

So, you gather together all the points in the partition P that also happen to be in the interval a C, then you gather together all the points in the partition P that also happen to be in the interval C b and call that P 1 and P 2 respectively. It is obvious that P is P 1 union P 2 and P 1 and P 2 are partitions of a C and C b respectively, ok.

Now, that you do this it is clear that the length of gamma with respect to the partition P is nothing, but length of gamma restricted to a C with respect to P 1 plus length of gamma restricted to C b with respect to P 2. The fact that you have equality here just follows by expanding out what the expression for length of gamma with respect to P is length of gamma 1 restricted to a C with respect to P 1s and length of gamma restricted to wait a second.

This is not gamma 1 that was just supposed to be gamma restricted to a C with respect to P 1 plus length of gamma restricted to C b with respect to P 2. The fact that these the left hand side and right hand side are equal just follows from the definition of what the length with respect to a partition is just write it out. You will get it, ok.

So, what this shows is that you can immediately write this is less than or equal to length of gamma restricted to a C plus length of gamma restricted to C b. I am just using the fact that the length of gamma restricted a C is a supremum over all partitions.

So, this shows one side, this shows that length of gamma with respect to P is less than or equal to length of gamma with respect to a C plus length of gamma with respect to C b. So, you can take supremum on the left over all the partitions P to conclude that length of gamma P gamma P is less than length of gamma restricted to a C plus length of gamma restricted to C b. So, that is one side, that is one side.

Now, we will have to show the other side and the other side is equally trivial.

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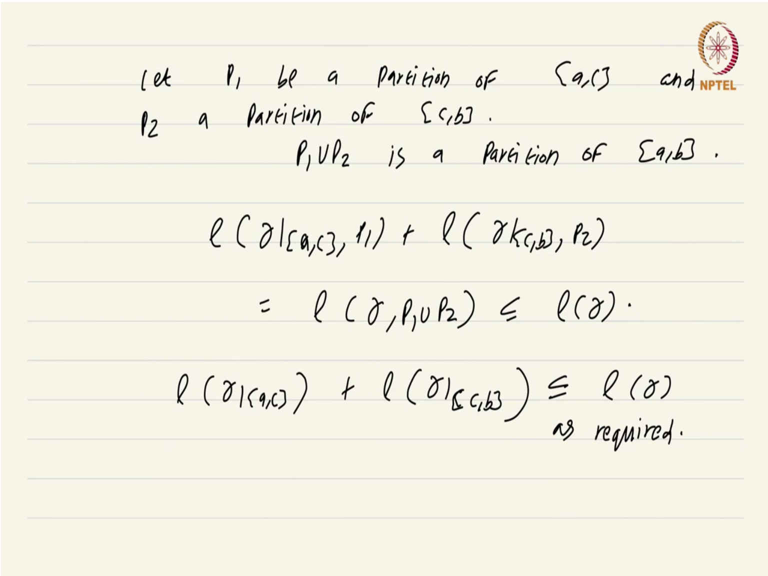

Let P 1 be a partition of a C and P 2 a partition of C b consider partitions. Then, obviously P 1 union P 2 is a partition is a partition of a b, ok and it is also clear that length of gamma restricted to a C with respect to P 1 plus length of gamma restricted to C b with respect to P 2 is equal to length of gamma P 1 union P 2 ok which is less than or equal to length of gamma.

Again taking supremum over all P 1 and all P 2 on the left hand side, we get the required result that length of gamma restricted to a C plus length of gamma restricted to C b is less than or equal to length of gamma as required. So, this concludes the proof that arc length is additive, ok.

Now, there is a monumental defect in our treatment of rectifiability and we are going to rectify it. Now, it is the fact that I have not provided you any example of rectifiable curves. I will compensate by providing uncountably many examples in one shot.

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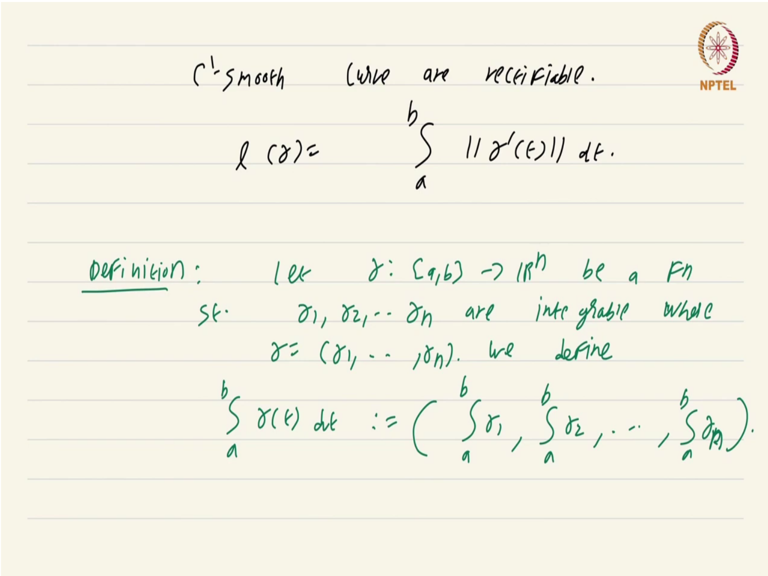

We are going to now show that a C 1 smooth curve is rectifiable, C 1 smooth curves are rectifiable. Not only that I am going to give you a formula for the arc length, now the question is what is the intuition behind this and what sort of formula can you expect? Well for a moment let us take out our physicist hats and put it on our heads and think like a physicist for a moment.

Well since this is a C 1 smooth curve this has a well defined velocity vector at all points, the norm of the velocity vector will give you the speed of the curve. This speed will be a continuous function because it is a C 1 smooth curve and the speed is nothing, but the norm of gamma prime of t gamma prime is continuous norm is continuous.

So, norm gamma prime of t is continuous. So, the speed is a continuous function of t. What could be the length of a curve other than the integral of the speed? So, you would expect that length of gamma is nothing, but integral a to b norm gamma prime of t d t.

So, if this were a physics textbook this would be the proof, but this is not a physics textbook. So, we will have to prove this rigorously. The proof is not hard. It just requires some new concepts. So, it will be convenient if I could define in an ad hoc way what the integral of a function into r and s the definition is not hard. So, let me just do it right here.

So, for the time being this is just a convenient definition. There is no do not assign any deeper meaning to this definition. Let gamma from a b to R n be a it need not actually be a curve it be a function. That is enough be a function such that gamma 1 gamma 2 dot dot dot gamma 1 gamma n are integrable are integrable there where gamma is nothing, but gamma 1 to gamma n.

So, I am taking a function gamma writing the coordinates as gamma 1 to gamma n. I am assuming that the coordinates are integrable, ok. Then we define the integral from a to b of gamma of t d t is by definition just the vector integral a to b gamma 1 integral a to b gamma 2 dot dot dot integral a to b gamma n. You just integrate coordinate wise if each coordinate function is integrable, the integral is nothing, but the integral of each coordinate and that resulting vector when you put all these coordinate integrals together, ok.

So, just I remark again there is no deeper meaning to this. This is just a convenient definition of integral of a vector valued function, ok. Now, let me just prove one basic property of this vector valued integrals which is going to be used in the proof of the fact that C 1 smooth curves are rectifiable.

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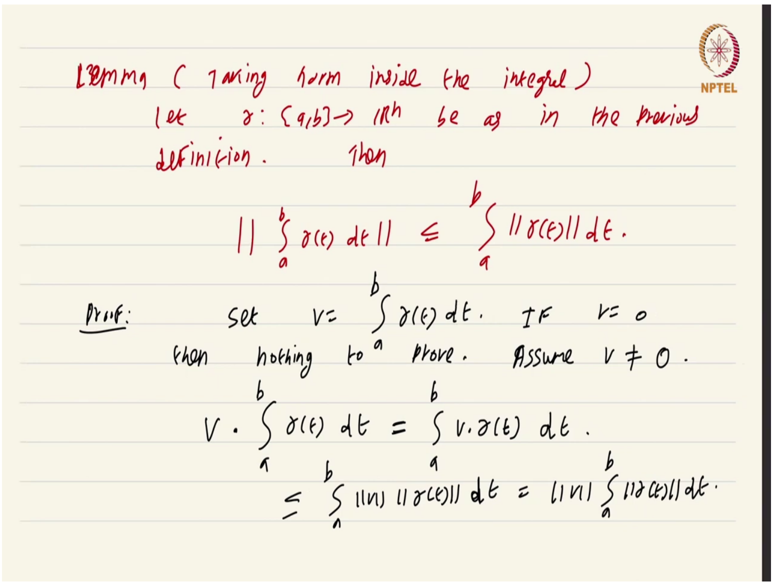

So, let me just state a lemma I will just call it taking norm inside the integral. So, this is a trick which is very commonly used in analysis. You can always in most situations take the norm inside the integral and get an inequality.

So, this is the definite I mean that lemma is as follows let gamma from a b to r n be as in the previous definition be as in the previous definition that is each coordinate function gamma 1 to gamma n is integrable, then norm of integral a to b gamma of t d t is less than or equal to integral a to b norm gamma of t d t. You can take the norm inside the integral, ok.

The proof is again not very hard. It is just a straight forward check. Now, set v to be integral a to b gamma of t d t. This is well defined simply because each function I mean gamma 1 to gamma n are assumed to be integrable set v to be this if v equal to 0, then nothing to prove, then nothing to prove ok. So, we can assume v is not 0 assume v is not 0, ok.

Now, what I am going to do is, I am going to apply the Cauchy Schwarz in equality. You first look at V the standard dot product, the standard inner product a to b gamma of t d T. Note carefully the left hand side is nothing, but v dot v, ok and just by the way this integral is defined which is component wise and just by the way the dot product is defined, this equality that v dot a integral a to b gamma of t d t is nothing, but integral a to b v dot gamma of t gamma of t d t ok. Just check this just check that both sides are equal, this just follows directly from the definition of the dot product and the definition of the integral, ok.

Now, what I do is, I apply Cauchy Schwarz inequality for the term V dot gamma of t to get this is less than or equal to integral a to b norm v norm gamma of t d t, ok. So, here I am using various basic properties of the Riemann integral, the standard Riemann integral. So, I want you to check that this is correct. This is just Cauchy Schwarz and this is just norm v integral a to b norm gamma of t d t ok. I am just taking the norm v outside.

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$$\|v\|^2 \leq \|v\| \int_a^b \|\gamma(t)\| \, dt$$

we conclude that

$$\|v\| \leq \int_a^b \|\gamma(t)\| \, dt.$$

Ex: why is $\|\gamma(t)\|$ integrable ?.

Now, observe that the left hand side what we started off with is nothing, but norm v squared. So, what we get is norm v squared is equal to norm v integral a to b norm gamma of t d t and of course, I can cancel of one of the norm.

This is not equal to this is less than or equal to sorry about that. This is less than or equal to. So, the net conclusion is we conclude that norm v is less than or equal to integral a to b non gamma of t d t, ok and here we could cancel simply because norm v is assumed to be non-zero.

So, an exercise for you if you have been carefully paying attention to this proof, this question would have cropped up into your mind. Why is norm gamma of t integrable? We have used that we have used that in this right at the statement level, we have used the fact that norm gamma of t is integrable. Why is norm gamma of t integrable, think about that.

It would be obvious if gamma were a curve because norm gamma of t would then be continuous. There will be no issues, but since we are not assuming that gamma is a curve, it could be any function which is integrable. This requires a bit of thought, ok. Now, we can finally prove that C 1 smooth curves are indeed rectifiable. So, let me state and prove the theorem.

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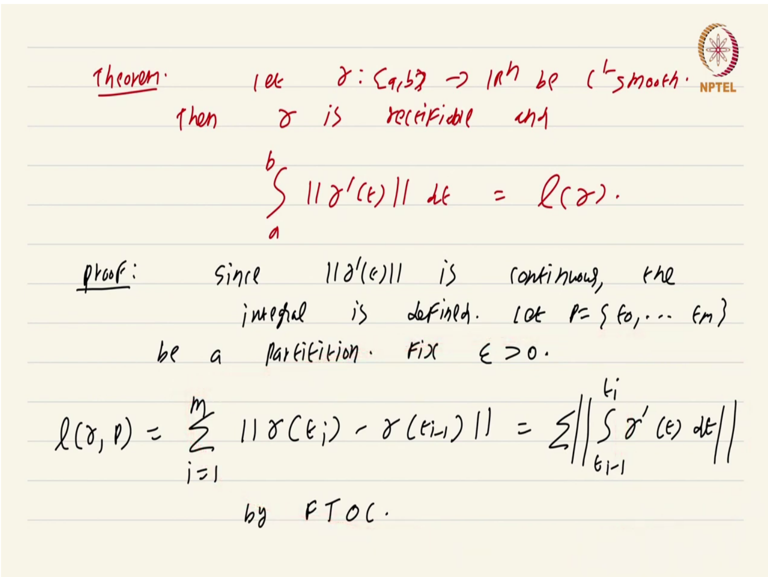

Let gamma from a b to r n b C 1 smooth, then gamma is rectifiable gamma is rectifiable and integral a to b norm gamma prime of t d t is the length of gamma, ok. So, the physicist intuition is absolutely correct in this regard, ok.

So, here the issue of well definedness of the integral does not arise since norm gamma prime of t is continuous the integral is well defined, the integral is defined. So, there is no issues here with definedness of the integral So, let us start with the partition.

Let P equal to t naught dot dot dot t m be a partition. As usual I will always write the partition out in the increasing order implicitly it is implicit in the way I have written it out t naught is less than t 1 dot dot dot less than t m fix epsilon greater than 0, ok.

Now, what I am going to do is, I am going to apply fundamental theorem of calculus in a slightly different way than usual. Look at summation norm gamma of t i minus gamma of t i minus 1. Look at this. This is i running from 1 to m. This is nothing, but l gamma of P ok, the length with respect to the partition P.

You can check at that a trivial application of fundamental theorem of calculus will tell you that this is summation integral a to b gamma prime of t d t and I have to put a norm outside, ok. This just follows by applying the fundamental theorem of calculus to each coordinate separately, ok. So, this is by fundamental theorem of calculus I will just abbreviate it to FTOC if you do not mind, ok. Now, we can take the norm inside the integral, we just proved a lemma and we are duty bound to use a lemma that appears just before theorem.

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$$\leq \sum \int_{t_{i-1}}^{t_i} \|\gamma'(t)\| \, dt = \int_a^b \|\gamma'(t)\| \, dt$$

γ is certainly rectifiable and

$$\ell(\gamma) \leq \int_a^b \|\gamma'(t)\| \, dt.$$

Define $\quad s(t) := \ell(\gamma|_{[a,t]}) \quad t \in [a,b].$

For $h > 0$, by additivity of arc-length

$$\ell\left(\gamma|_{[t,t+h]}\right) = s(t+h) - s(t).$$

So, this is this quantity is less than or equal to summation integral t i minus 1 to t i. So, one second. Yeah that is correct. This is nothing, but summation integral t i minus 1 to t i norm gamma prime of t d t ok which is in fact nothing, but integral a to b norm gamma prime of t d t, ok.

So, I have just used I have just used the basic properties of the Riemann integral to go from this sum to this equality, ok. Going from here going from here to here is just taking the norm inside the integral and applying again some basic properties of the Riemann integral, there is nothing much happening here ok.

One second I think I have made a typo and that is why I am sounding a bit confused. This is not a to b. Of course this is not a to b, this is nonsense. This is not a to b, this is t i minus 1 to t i right where did I mean of course if I do a to b, I will get a different expression, but not the one that I want, ok.

So, here I am applying the fundamental theorem of calculus on each one of these intervals t i minus 1 to t i. Sorry about that. So, yeah now I am a bit happy. I was wondering what happened, how did I split that a b into t i minus 1 t i. Suddenly it was done in the prior step actually, ok.

Anyway we got this. So, what this shows is that gamma is certainly rectifiable and length of gamma is less than or equal to integral a to b norm gamma prime of t d t. So, one side we have shown, we have shown rectifiability and we have also shown that the integral of the speed at least is greater than the length of the curve, ok.

Now, we have to show that this inequality is actually an equality to do that what we do is, we define the arc length function. So, define S of t by definition to be integral a to t not integral a to t. Sorry about that. Define it to be, define it to be length of gamma restricted to a t, ok. So, t is in a b. So, we have now defined the function S of t where t runs from a to b. If you at the point t, it just gives you the arc length up until t, ok.

Now, for h greater than 0 by additivity of arc length by additivity of arc length by additivity of arc length, it is immediate that gamma restricted to t t plus h. The length of this is nothing, but S of t plus h minus S of t. I have just used the additivity of arc length and taken terms to one side. I mean just shuffled around the terms and you get this, ok.

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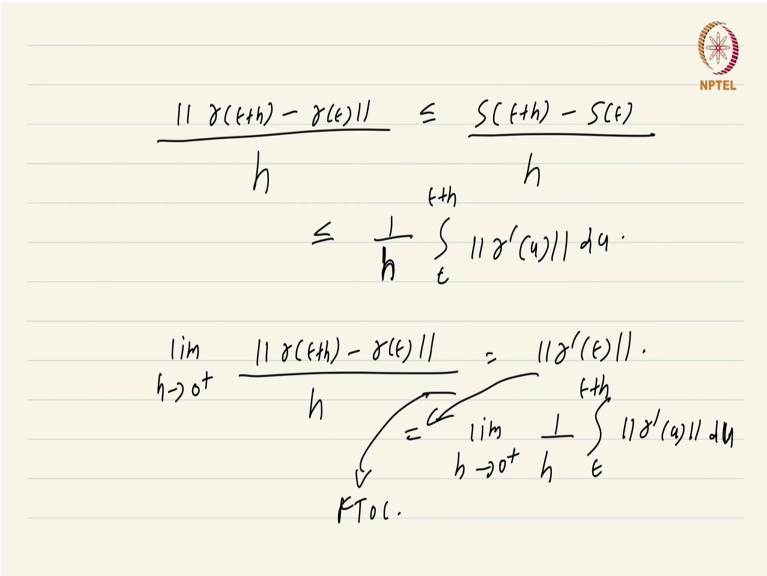

Now, once I have this I can now try to sort of take the difference quotient, the Newton quotient and see what happens, look at norm gamma of t plus h minus gamma of t by h, ok. Now, this is less than or equal to S of t plus h minus S of t by h.

Why is that the case? Well if you I mean I just about a minute ago said that this S of t plus h minus S of t is nothing, but the length of gamma restricted to t t plus h and this is just if I take the trivial partition t t plus h and find out the length of that curve with respect to this partition, this single term is what I would get. So, this single term will be less than or equal to the length which is equal to this. So, that is how we get this inequality.

And this again by the fundamental theorem of calculus, this will be less than or equal to 1 by h 1 by h integral t to t plus h norm gamma prime of u d u, ok. So, this part this inequality, this was origin. I mean this is due to the fundamental theorem of calculus, but actually this we just proved this via the fundamental theorem of calculus.

We have just shown that the C 1 smooth curves are rectifiable and their arc lengths are less than or equal to the speed integral. So, we get this when applied to the curve gamma restricted to t t plus h. So, I should not really say by the fundamental theorem of calculus. Ultimately it is by the fundamental theorem of calculus, but this particular inequality is via what we have just proved, ok.

Now, taking limits taking limit h going to 0 plus of this norm gamma of t plus h minus gamma of t by norm h by h, this is clearly just non-gamma prime of t check that this is a trivial check. This is just follows almost immediately from the definition and this now is the place where I apply the fundamental theorem of calculus.

This is nothing, but limit h going to 0 plus 1 by h integral t to t plus h norm gamma prime of u d u. This part is the fundamental theorem of calculus, this equality this is FTOC, ok. So, this part this equality is FTOC. So, check that.

Now, because this term limit h going to 0 plus blah blah blah is mod gamma prime of t, this is nothing but the left hand side ok and taking limit h going to 0 plus of the right hand side also gives the same thing. It also gives mod gamma prime of t.

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By the squeeze theorem

$$\lim_{h\to 0^+} \frac{S(t+h)-S(t)}{h} = \|\gamma'(t)\|.$$

similarly, we can deal with case $h<0$.

$$S'(t) = \frac{d}{dt} S(t) = \lim_{h\to 0} \frac{S(t+h)-S(t)}{h} = \|\gamma'(t)\|$$

$$S(b) = \int_a^b S'(t)\,dt = \int_a^b \|\gamma'(t)\|\,dt.$$

So, the net upshot is the net upshot is by squeeze theorem by the squeeze or sandwich theorem whichever you prefer. By the squeeze theorem limit h going to 0 plus of S of t plus h minus S of t by h this is nothing, but norm gamma prime of t. This is just the squeeze theorem.

Similarly, we can deal with the case similarly we can deal we can deal with the case h less than 0. So, net up short is limit h going to 0 of S of t plus h minus S of t by h is norm gamma prime of t ok, but this is nothing, but d by d t of S of t ok which is nothing, but S prime, the derivative of t.

So, again the fundamental theorem of calculus will now tell you that S of b is nothing, but integral a to b, S prime of t d t. This is because S of a is just 0 as you can see and this is integral a to b norm gamma prime of t d t, ok.

So, this is the proof it is not too hard. It is just slightly different way by which we apply the fundamental theorem of calculus. We can conclude that the C 1 smooth curves are rectifiable and their lengths are nothing, but the speed integrals ok.

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Example: $$F(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Check that F is continuous.

$x \mapsto (x, F(x))$ is not rectifiable on $[0,1]$.

$$P_m := \left\{\frac{1}{k} : 1 \le k \le 2m\right\} \cup \{0\}$$

So, I have now given you uncountably many examples of rectifiable curves. Let me give you one example of a non-rectifiable curve. One might be under the impression that all curves are rectifiable as the space filling curve suggests this might probably not be true. This example is significantly simpler than the space filling curve example, ok.

Look at the function F of x equal to x cos pi by 2 pi by 2 x if x is not 0 and 0 if x is equal to 0, ok. Now, first exercise for you check that F is continuous check that F is continuous ok. I am going to claim that the function x going to x, F of x is not rectifiable on 0 1.

I am going to treat it as a curve defined on close 0 1. This is essentially the graph of the function F as we have remarked in when we discussed examples, the graph of a function from an interval can be naturally considered as a curve.

I am claiming that this curve x going to x, F of x is not a rectifiable curve. To do this consider the partition P m which is by definition 1 by k, such that 1 is less than or equal to k is less than or equal to t 2 m along with the 0.2 ok. This will be a partition of this will be a partition of 0 1 and I am going to try to find out how F behaves at the part at these at these partition points, ok.

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$$\sum_{i=1}^{2m} |F(t_i) - F(t_{i-1})| = \sum_{i=1}^{2m} \frac{1}{i}$$

You can now show that $x \mapsto (x, F(x))$ is not rectifiable.

Theorem (Continuity of arc-length). Let $\gamma : [a,b] \to \mathbb{R}^n$ be a rectifiable curve. Let $s(t) := \ell(\gamma|_{[a,t]})$, $t \in [a,b]$. Then

Observe that we have summation i equals 1 to 2 m mod F of t i minus F of t i minus 1. This is nothing, but summation i equals 1 to 2 m 1 by i as you can check just by substituting the function here which in this situation is nothing, but x cos pi by 2 x. You will see that this is true, ok.

Now, you can show you can now show that x going to x, F of x is not rectifiable. I leave it to you as an interesting exercise. Essentially you have to use the fact that summation 1 by n diverges that is what is going to be used. So, this shows that you cannot expect all curves to be rectifiable. There are quite simple curves that we can define which are not rectifiable. Now, I am going to show one more interesting property about arc length.

Arc length is now defined for a continuous function. We can now consider the arc length function S of t that we already considered in the proof of the fact that C 1 smooth curves are rectifiable. Now, it is intuitively obvious that this function S of t itself would be continuous. In fact, that was obtained as a part of a proof at least for the smooth C 1 smooth case. We in fact showed that the arc length is differentiable and the derivative of the arc length is the speed.

So, these are intuitively physical facts that we were able to prove rigorously, but what is nice is it is true in general that the arc length is a continuous function even when the function gamma is not assumed to be C 1 smooth, but just rectifiable So, that is the next theorem, that is the next theorem. This is the continuity of arc length, this is the continuity of arc length.

So, the statement runs as follows. Let gamma from a b to just a second from a b to R n be a rectifiable curve be a rectifiable curve, let S of t be by definition, the length of gamma restricted to a t ok. So, t is also in a b ok.

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1. S is an increasing fn.
2. If γ is non-constant on any non-singleton subinterval of $[a,b]$ then S is strictly increasing.
3. S is continuous.

Proof. 1 is obvious.
For 2, assume that γ is non-constant on non-singleton subintervals. Let $c \in [a,b]$ and take $t > c$. Because $\gamma|_{[c,t]}$ is non-constant, we will be able to find $t_1 \in (c,t)$ s.t. $\gamma(t_1) \neq \gamma(c)$.

Then, we have the following fact. Number 1, S is an increasing function. I am not claiming strictly increasing or anything. It is an increasing function. 2, if gamma is non-constant on any non-singleton subinterval of a b, then S is strictly increasing S is strictly increasing 3 gamma naught gamma S is continuous. So, this theorem shows the continuity of arc length.

So, let us see a proof 1 is obvious I am not even going to dignify 1 with any more time. 1 is obvious for 2 assume that gamma is non-constant or non-singleton sub intervals, ok. Now, let C be a point in a b and take t greater than C, ok.

Now, because gamma restricted to t C not t C, C t because t is greater than C because gamma restricted the C t is non-constant is non-constant because of this we will be able to find we will be able to find t 1 which is there in this interval C t, such that gamma of t 1 is not equal to gamma of C, ok. So, in fact I want this to be in the open interval, not just the closed interval. I want it to be in the open interval C t.

So, if this were not true, then for all t 1 in this interval C t, then we have gamma of t 1 equal to gamma of C which contradicts the assumption that on non-singleton intervals, it is non-constant gamma is non-constant, ok.

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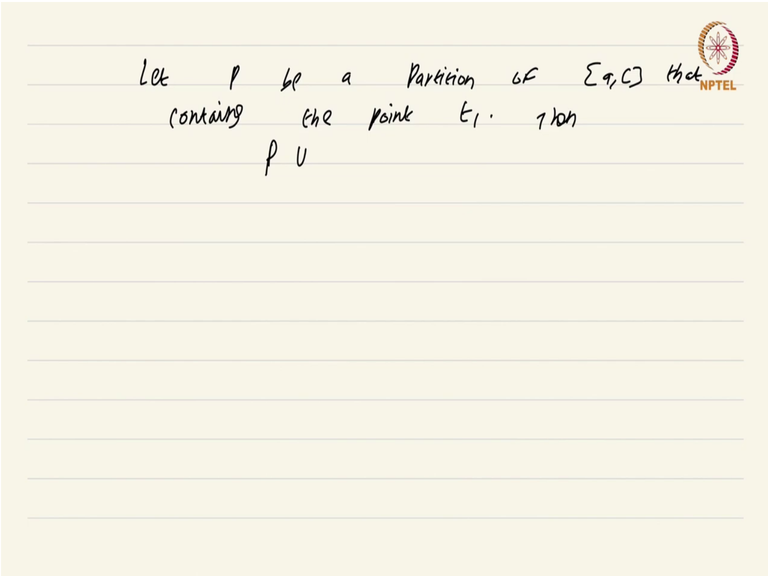

Having done this let P be a partition, let P be a partition of a C of a C that contains that contains the point t 1, ok. Take a partition of a C that contains this point at t 1, ok then it is clear that P union no let me make a slight change, let me make a slight change. Ultimately what is it that I want to prove? I want to show that gamma of t is greater than gamma of C, right.

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Let P be a partition of $[a, c]$.
Set $P_1 = P \cup \{t, t_1\}$.

It is obvious that P_1 is a partition of $[a, t]$. Furthermore,

$$\ell(\gamma|_{[a,t]}, P_1) = \ell(\gamma|_{[a,c]}, P) + \|\gamma(t) - \gamma(t_1)\| + \|\gamma(t_1) - \gamma(c)\|$$

$$\ell(\gamma|_{[a,t]}) \geq \ell(\gamma|_{[a,c]}, P) + \text{Positive constant.}$$

$$\ell(\gamma|_{[a,t]}) > \ell(\gamma|_{[a,c]}).$$

So, let P be a partition of a C set P 1 to be P union t, t 1, ok then it is obvious it is obvious that P 1 is a partition of a t, ok. It is obvious that P 1 is a partition of a t. Furthermore we have this pi definition. The way we have set up things, the length of gamma restricted to a C with respect to this partition P is going to be less than or equal to not less than or equal to this let me write the inequality the other way which should be better and more transparent.

Length of gamma restricted to a t with respect to P 1 is nothing, but length of gamma restricted to a C with respect to P plus these additional points that we have considered t and t 1 which is norm of gamma of t minus gamma of t 1 plus norm gamma of t 1 minus gamma of C, ok.

So, what we can conclude is, we can conclude that length of gamma restricted to a t is itself this is always is greater than or equal to length of gamma restricted to a C, P plus some positive constant, some plus some positive constant independent of the choice of partitions ok plus some positive constant.

Now, taking the supremum taking the supremum on the right hand side with respect to P, we conclude that length of gamma with respect restricted to a t is strictly greater than length of gamma a C, ok.

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Let P be a partition of $[a, c]$.
Set $P_1 := P \cup \{t, t_1\}$.

It is obvious that P_1 is a partition of $[a, t]$. Furthermore,

$$\ell(\gamma|_{[a,t]}, P_1) = \ell(\gamma|_{[a,c]}, P) + \|\gamma(t) - \gamma(t_1)\| + \|\gamma(t_1) - \gamma(c)\|$$

$$\ell(\gamma|_{[a,t]}) \geq \ell(\gamma|_{[a,t]}, P_1) \geq \ell(\gamma|_{[a,c]}, P) + \text{positive constant.}$$

$$\ell(\gamma|_{[a,t]}) > \ell(\gamma|_{[a,c]}).$$

So, first we took supremum over I mean actually it is better to just combine both steps. There is no we can write this gamma of a t with respect to P 1 is this I can write this is greater than or equal to length of gamma restricted to a t. Yes now this is perfect, ok.

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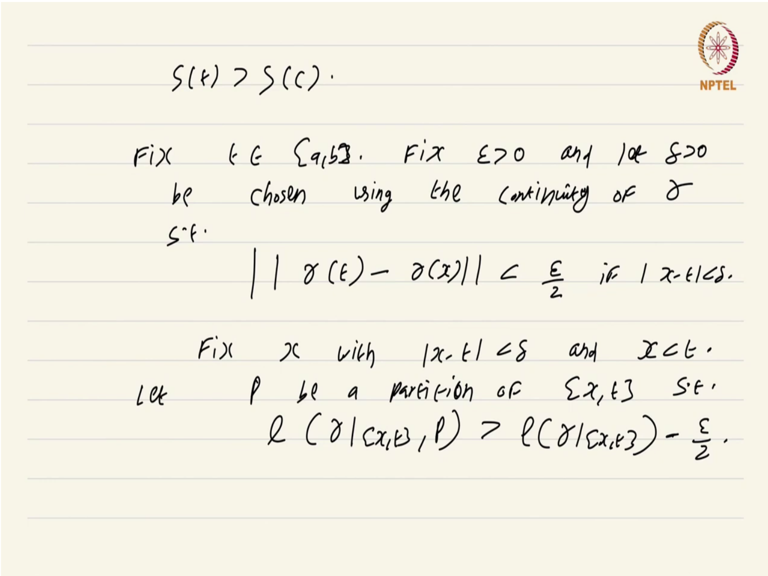

So, net upshot is S of t is greater than S of C. So, this concludes the second part that the arc length is strictly increasing if gamma is non-constant on any non-singleton interval. Now, the final and the most challenging part is to show continuity.

So, fix t in a b our goal is to show that gamma is continuous at this point. So, fix epsilon greater than 0 and let delta greater than 0 be chosen using the continuity of gamma using the continuity of gamma, such that mod gamma of t minus gamma of x or rather I should put norms here norm gamma of t minus gamma of x is less than epsilon by 2. If mod x minus t or here it is just mod. If mod x minus t is less than delta, this is just the continuity of the function gamma, ok.

Now, fix x with mod x minus t less than delta and x less than t, ok. Let P be a partition of this interval x t, such that the length of gamma restricted to x t with respect to this partition is greater than or equal to the length of gamma restricted to x t plus epsilon by 2.

Now, what is happening here things might seem a bit confusing. What is essentially happening is the following I am going to choose a partition of P, such that the partition is so fine that the length with respect to this partition is not that different from the actual length since this quantity is nothing, but the supremum over all partitions of the quantity on the left given any epsilon given any epsilon. One second I made a massive goof up. This is minus epsilon by 2, this is minus epsilon by 2.

So, given any epsilon or in this case epsilon by 2, I can always choose the partition. So, fine such that this is at the max epsilon by 2 distance away from the actual value of the length, this is just the way supreme is defined. This follows just from a basic real analysis basic properties of the supremum, ok.

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Let t_{m-1} be the last but one point in P. Then

$$\|\gamma(t) - \gamma(t_{m-1})\| < \frac{\varepsilon}{2}.$$

$$\ell(\gamma|[x, t_{m-1}], P\setminus\{t\}) > \ell(\gamma|[x,t]) - \varepsilon$$

By additivity of arclength

$$\ell(\gamma|[x, t_{m-1}], P\setminus\{t\}) \leq \ell(\gamma|[x, t_{m-1}])$$

$$\ell(\gamma|[t_{m-1}, t]) < \varepsilon.$$

Now, what I am going to do is, I am going to choose a special partition not a special partition. I am going to choose a point on this partition let t m minus 1 be the last, but one point, but one point in P ok, then because we have chosen that delta originally. So, small and x is within the delta neighborhood, we have that non-gamma of t minus gamma of t m minus 1 is less than epsilon by 2, ok.

Now, what this means is that if I consider the length of the curve gamma up until the point x up until the point t minus t m minus 1 up until the last, but one point and I take this special partition P minus the final point P, this will obviously give a partition of x t m minus 1. With respect to this partition we know that this is going to be greater than length of gamma x t minus epsilon, ok.

That is because the partition P was so chosen in such a manner that the length of gamma restricted x t with respect to P is at the max epsilon by two distance of a, but this last point the contribution from the last term arising in the sum can be at the max epsilon by 2. In fact, it is strictly less than epsilon by 2, therefore the contribution from the rest which is just this must be at least this must be at least length of gamma restricted to x t minus epsilon, ok.

Now, by additivity of arc length by additivity of arc length by additivity of arc length and the fact that l of gamma x restricted to t m minus 1 is, P of course with respect to this partition P minus t is less than or equal to length of gamma of x restricted to t m minus 1 combining identity of arc length. With this fact, we can conclude that length of gamma restricted to t m minus 1 t is at the max epsilon is at the max epsilon.

So, this just follows from writing length of gamma restricted to x t as the sum of length of gamma restricted to x t m minus 1 plus length of gamma restricted to t m minus 1 t and this previous equation immediately gives us that length of gamma with restricted to t m minus 1 t has to be less than epsilon, ok.

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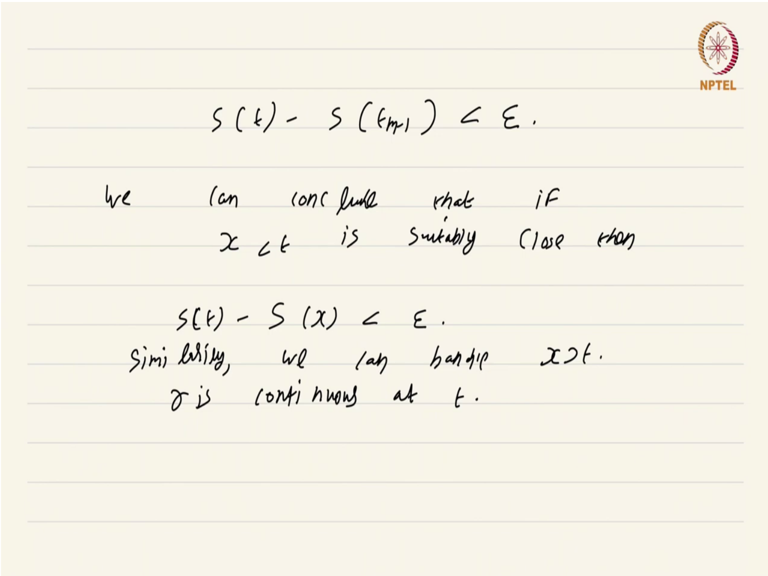

Now, what does this show? Well again you apply additivity of arc length. This shows that S of t minus S of t m minus 1 is less than epsilon because this is nothing, but the length of gamma restricted to t m minus 1 t.

So, we can conclude, we can conclude that if x less than t is suitably close is suitably close, then S of x S of t minus S of x is less than epsilon. Similarly, we can handle we can handle x greater than t analogous argument we can give. So, net up short is gamma is continuous at t.

So, this concludes the proof that the arc length function S is actually continuous. So, the last proof is a bit tricky. So, I would suggest that you watch it again and read the notes and make sure you understood the ideas behind the proof.

This is a course on Real Analysis and you have just watched the video on Rectifiability and Arc Length.